The ABC's of Calculus

Angelo B. Mingarelli
School of Mathematics and Statistics,
Carleton University
Canada

http://people.math.carleton.ca/∼angelo/calculus/cal104.html

June 22, 2016

ISBN 978-0-9698889-5-6

This book is dedicated
to the immutable memory of my father,
Giosafat Mingarelli,
and
to my mother, Oliviana Lopez,
who showed a young child of 10
how to perform long division ...

Contents

List of Tables

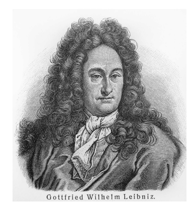
This page intentionally left blank

This page intentionally left blank

Chapter 1

Functions and Their Properties

The Big Picture



Gottfried Wilhelm Leibniz

1646 - 1716

This chapter deals with the definition and properties of things we call *functions*. They are used all the time in the world around us although we don't recognize them right away. Functions are a mathematical device for describing an inter-dependence between things or objects, whether real or imaginary. With this notion, the original creators of Calculus, namely, the English mathematician and physicist, Sir Isaac Newton, and the German philosopher and mathematician, Gottfried Wilhelm Leibniz (see inset), were able to quantify and express relationships between real things in a mathematical way. Most of you will have seen the famous Einstein equation

$$E = mc^2.$$

This expression defines a dependence of the quantity, E, called the *energy* on m, called the *mass*. The number c is the *speed of light* in a vacuum, some 300,000 kilometers per second. In this simple example, E is a *function* of m. Almost all naturally occuring phenomena in the universe may be quantified in terms of functions and their relationships to each other. A complete understanding of the material in this chapter will enable you to gain a foothold into the fundamental vocabulary of Calculus.

Review

You'll need to remember or learn the following material before you get a thorough understanding of this chapter. Look over your notes on functions and be familiar with all the **basic algebra** and **geometry** you learned and also don't forget to review your **basic trigonometry**. Although this seems like a lot, it is necessary as mathematics is a sort of language, and before you learn any language you need to be familiar with its vocabulary and its grammar and so it is with mathematics. Okay, let's start ...

1.1 The Meaning of a Function

You realize how important it is for you to remember your social security/insurance number when you want to get a real job! That's because the employer will associate

You can think of a function f as an I/O device, much like a computer CPU; it takes input, x, works on x, and produces only one output, which we call $f(x)$.

you with this number on the payroll. This is what a **function** does ... a function is a rule that associates to each element in some set that we like to call the **domain** (in our case, the name of anyone eligible to work) only one element of another set, called the **range** (in our case, the set of all social security/insurance numbers of these people eligible to work). In other words the function here associates to each person his/her social security/insurance number. Each person can have only one such number and this lies at the heart of the definition of a function.

Example 1. In general everybody as an *age* counted historically from the moment he/she is born. Consider the rule that associates to each person, that person's age. You can see this depicted graphically in Figure 1. Here A has age a, person B has age c while persons C, D both have the same age, that is, b. So, by definition, this rule is a function. On the other hand, consider the rule that associates to each automobile driver the car he/she owns. In Figure 2, both persons B and C share the automobile c and this is alright, however note that person A owns two automobiles. Thus, by definition, this rule cannot be a function.

This association between the domain and the range is depicted graphically in Figure 1 using arrows, called an **arrow diagram**. Such arrows are useful because they start in the domain and point to the corresponding element of the range.

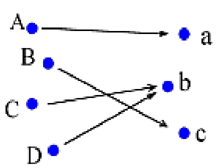


Figure 1.

Example 2. Let's say that Jennifer Black has social security number 124124124. The arrow would start at a point which we label "Jennifer Black" (in the domain) and end at a point labeled "124124124" (in the range).

Okay, so here's the formal definition of a function ...

NOTATION

Dom (f) = Domain of f

Ran (f) = Range of f

Definition 1. *A function f is a rule which associates with each object (say, x) from one set named the domain, a single object (say, $f(x)$) from a second set called the range. (See Figure 1)*

Objects in the domain of f are referred to as independent variables, while objects in the range are dependent variables.

Rather than replace every person by their photograph, the objects of the domain of a function are replaced by **symbols** and mathematicians like to use the symbol "x" to mark some unknown quantity in the domain (this symbol is also called an **independent variable**), because it can be *any* object in the domain. If you don't like this symbol, you can use any other symbol and this won't change the function. The symbol, $f(x)$, is also called a **dependent variable** because its value generally *depends* on the value of x. Below, we'll use the "box" symbol, $\square$, in many cases instead of the more standard symbol, x.

Example 3. Let f be the (name of the) function which associates a person with their height. Using a little shorthand we can write this rule as $h = f(p)$ where p is a particular person, (p is the *independent variable*) and h is that person's height (h is the *dependent variable*). The domain of this function f is the set of all persons, right? Moreover, the range of this function is a set of numbers (their height, with some units of measurement attached to each one). Once again, let's notice that many people can have the same height, and this is okay for a function, but clearly there is no one having two different heights!

A rule which is NOT a function

LOOK OUT! When an arrow "splits" in an arrow diagram (as the arrow starting from A does in Figure 2) the resulting rule is *never* a function.

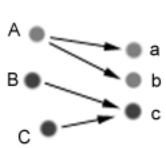


Figure 2.

In applications of calculus to the physical and natural sciences the domain and the range of a function are both sets of real (sometimes complex) numbers. The symbols

used to represent objects within these sets may vary though ... E may denote energy, p the price of a commodity, x distance, t time, etc. The domain and the range of a function may be the same set or they may be totally unrelated sets of numbers, people, aardvarks, aliens, etc. Now, functions have to be identified somehow, so rules have been devised to *name them*. Usually, we use the lower case letters $f, g, h, k, \ldots$ to name the function itself, but you are allowed to use any other symbols too, but try not to use x as this might cause some confusion ... we already decided to name objects of the domain of the function by this symbol, x, remember?

Quick Summary Let's recapitulate. A function has a *name*, a *domain* and a *range*. It also has a *rule* which associates to every object of its domain only one object in its range. So the rule (whose name is) g which associates to a given number its square as a number, can be denoted quickly by $g(x) = x^2$ (Figure 3). You can also represent this rule by using the symbols $g(\square) = \square^2$ where $\square$ is a "box"... something that has nothing to do with its shape as a symbol. It's just as good a symbol as "x" and both equations represent the *same* function.

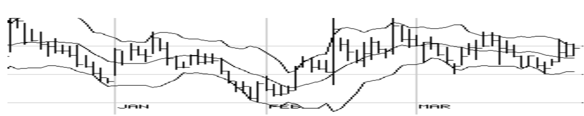
You need to think beyond the *shape* of an independent variable and just keep your mind on a generic "variable", something that has nothing to do with its shape.

Remember ... x is just a symbol for what we call an independent variable, that's all. We can read off a rule like $g(x) = x^2$ in many ways: The purist would say "The value of g at x is x^2" while some might say, "g of x is x^2". What's really important though is that you *understand the rule*... in this case we would say that the function associates a symbol with its square *regardless of the shape of the symbol itself*, whether it be an x, $\square$, $\triangle$, $\heartsuit$, t, etc.

Example 4. *Generally speaking,*

- The association between a one-dollar bill and its serial number is a function (unless the bill is counterfeit!). Its domain is the collection of all one-dollar bills while its range is a subset of the natural numbers along with some 26 letters of the alphabet.

- The association between a CD-ROM and its own serial number is also a function (unless the CD was copied!).
- Associating a fingerprint with a specific human being is another example of a function, as is ...
- Associating a human being with the person-specific DNA (although this may be a debatable issue).
- The association between the monetary value of a stock, say, x, at time t is also function but this time it is a function of *two* variables, namely, x, t. It could be denoted by $f(x, t)$ meaning that this symbol describes the value of the stock x at time t. Its graph may look like the one below.



- The correspondence between a patent number and a given (patented) invention is a function
- If the ranges of two functions are subsets of the real numbers then the difference between these two functions is also a function. For example, if $R(x) = px$

denotes the *total revenue* function, that is, the product of the number of units, x, sold at price p, and $C(x)$ denotes the total cost of producing these x units, then the difference, $P(x) = R(x) - C(x)$ is the *profit* acquired after the sale of these x units.

Composition of Functions:

There is a fundamental operation that we can perform on two functions called their "composition".

Let's describe this notion by way of an example. So, consider the domain of all houses in a certain neighborhood. To each house we associate its owner (we'll assume that to each given house there is only one owner). Then the rule that associates to a given house its owner is a function and we call it "f". Next, take the rule that associates to a given owner his/her annual income from all sources, and call this rule "g". Then the new rule that associates with each house the annual income of its owner is called the composition of g and f and is denoted mathematically by the symbols $g(f(x))$. Think of it . . . if x denotes a house then $f(x)$ denotes its owner (some name, or social insurance number or some other unique way of identifying that person). Then $g(f(x))$ must be the annual income of the owner, $f(x)$.

Once can continue this exercise a little further so as to define compositions of more than just two functions . . . like, maybe three or more functions. Thus, if h is a (hypothetical) rule that associates to each annual income figure the total number of years of education of the corresponding person, then the composition of the three functions defined by the symbol $h(g(f(x)))$, associates to each given house in the neighborhood the total number of years of education of its owner.

In the next section we show how to calculate the values of a composition of two given functions using symbols that we can put in "boxes"...so we call this the "box method" for calculating compositions. Basically you should always look at what a function does to a generic "symbol", rather than looking at what a function does to a specific symbol like "x".

NOTES:

1.2 Function Values and the Box Method

Now look at the function g defined on the domain of real numbers by the rule $g(x) = x^2$. Let's say we want to know the value of the mysterious looking symbols, $g(3x+4)$, which is really the same as asking for the composition $g(f(x))$ where $f(x) = 3x + 4$. How do we get this?

> **The Box Method**
>
> To find the value of $g(3x+4)$ when $g(x) = x^2$: We place all the symbols "$3x+4$" (i.e., all the stuff between the parentheses) in the symbol "$g(...)$" inside a box, say, $\Box$, and let the function g take the box $\Box$ to $\Box^2$ (because this is what a function *does* to a symbol, regardless of what it looks like, right?). Then we "remove the box" , replace its sides by parentheses, and there you are ... what's left is the value of $g(3x + 4)$.

We call this procedure the **Box Method**.

Example 5. So, if $g(x) = x^2$, then $g(\Box) = \Box^2$. So, according to our rule, $g(3x + 4) = g(\boxed{3x + 4}) = \boxed{3x + 4}^{\,2} = (3x + 4)^2$. This last quantity, when simplified, gives us $9x^2 + 24x + 16$. We have found that $g(3x + 4) = 9x^2 + 24x + 16$.

Example 6. If f is a new function defined by the rule $f(x) = x^3 - 4$ then $f(\Box) = \Box^3 - 4$ (regardless of *what's in the box!*), and

$$f(a+h) = f(\boxed{\;a+h\;}) = \boxed{\;a+h\;}^{\,3} - 4 = (a+h)^3 - 4 = a^3 + 3a^2h + 3ah^2 + h^3 - 4.$$

Also,

$$f(2) = 2^3 - 4 = 4,$$

and

$$f(-1) = (-1)^3 - 4 = -5,$$
$$f(a) = a^3 - 4,$$

where a is another symbol for any object in the domain of f.

Example 7. Let $f(x) = \dfrac{1.24x^2}{\sqrt{2.63x - 1}}$. Find the value of $f(n+6)$ where n is a positive integer.

Solution The Box Method gives

$$\begin{aligned}
f(x) &= \frac{1.24x^2}{\sqrt{2.63x - 1}} \\
f(\Box) &= \frac{1.24\Box^2}{\sqrt{2.63\Box - 1}} \\
f(\boxed{\text{n+6}}) &= \frac{1.24\boxed{\text{n+6}}^{\,2}}{\sqrt{2.63\boxed{\text{n+6}} - 1}} \\
f(n+6) &= \frac{1.24(n+6)^2}{\sqrt{2.63(n+6) - 1}} \\
&= \frac{1.24(n^2 + 12n + 36)}{\sqrt{2.63n + 15.78 - 1}} \\
&= \frac{1.24n^2 + 14.88n + 44.64}{\sqrt{2.63n + 14.78}}.
\end{aligned}$$

The function $g(x) = x^2$ and some of its values.

x	$g(x)$
-2	4
-1	1
0.5	0.25
1.5	2.25
3	9
0.1	0.01
-2.5	6.25
5	25
10	100
-3	9

Figure 3.

NOTATION for Intervals.

$(a, b) = \{x : a < x < b\}$, and this is called an **open interval**. $[a, b] = \{x : a \le x \le b\}$, is called a **closed interval**. $(a, b], [a, b)$ each denote the sets $\{x : a < x \le b\}$ and $\{x : a \le x < b\}$, respectively (either one of these is called a **semi-open interval**).

"Formerly, when one invented a new function, it was to further some practical purpose; today one invents them in order to make incorrect the reasoning of our fathers, and nothing more will ever be accomplished by these inventions."

Henri Poincaré, 1854 - 1912

French mathematician

Example 8. On the other hand, if $f(x) = 2x^2 - x + 1$, and $h \neq 0$ is some real number, how do we find the value of the quotient

$$\frac{f(x+h) - f(x)}{h} \ ?$$

Solution Well, we know that $f(\square) = 2\square^2 - \square + 1$. So, the idea is to put the symbols $x + h$ inside the box, use the rule for f on the *box* symbol, then expand the whole thing and subtract the quantity $f(x)$ (and, finally, divide this result by h). Now, the value of f evaluated at $x + h$, that is, $f(x + h)$, is given by

$$\begin{aligned} f(\boxed{x+h}) &= 2\,\boxed{x+h}^{\,2} - \boxed{x+h} + 1, \\ &= 2(x+h)^2 - (x+h) + 1 \\ &= 2(x^2 + 2xh + h^2) - x - h + 1 \\ &= 2x^2 + 4xh + 2h^2 - x - h + 1. \end{aligned}$$

From this, provided $h \neq 0$, we get

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{2x^2 + 4xh + 2h^2 - x - h + 1 - (2x^2 - x + 1)}{h} \\ &= \frac{4xh + 2h^2 - h}{h} \\ &= \frac{h(4x + 2h - 1)}{h} \\ &= 4x + 2h - 1. \end{aligned}$$

Example 9. Let $f(x) = 6x^2 - 0.5x$. Write the values of f at an integer by $f(n)$, where the symbol "n" is used to denote an integer. Thus $f(1) = 5.5$. Now write $f(n) = a_n$. Calculate the quantity $\frac{a_{n+1}}{a_n}$.

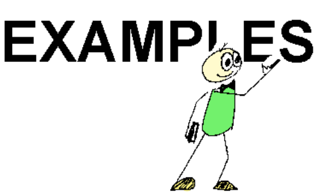
Solution The Box method tells us that since $f(n) = a_n$, we must have $f(\square) = a_\square$. Thus, $a_{n+1} = f(n + 1)$. Furthermore, another application of the Box Method gives $a_{n+1} = f(n + 1) = f(\boxed{\text{n+1}}) = 6\boxed{\text{n+1}}^2 - 0.5\boxed{\text{n+1}}$. So,

$$\frac{a_{n+1}}{a_n} = \frac{6(n+1)^2 - 0.5(n+1)}{6n^2 - 0.5n} = \frac{6n^2 + 11.5n + 5.5}{6n^2 - 0.5n}.$$

Example 10. Given that

$$f(x) = \frac{3x + 2}{3x - 2}$$

determine $f(x - 2)$.



Solution Here, $f(\square) = \frac{3\square+2}{3\square-2}$. Placing the symbol "$x - 2$" into the box, collecting terms and simplifying, we get,

$$f(x-2) = f(\boxed{\text{x-2}}) = \frac{3\boxed{\text{x-2}} + 2}{3\boxed{\text{x-2}} - 2} = \frac{3(x-2) + 2}{3(x-2) - 2} = \frac{3x - 4}{3x - 8}.$$

Example 11. If $g(x) = x^2 + 1$ find the value of $g(\sqrt{x-1})$.

Solution Since $g(\square) = \square^2 + 1$ it follows that

$$g(\sqrt{x-1}) = [\sqrt{x-1}]^2 + 1 = [x-1] + 1 = x$$

on account of the fact that $\sqrt[2]{\square}^{\,2} = \square$, regardless of "what's in the box".

Example 12. If $f(x) = 3x^2 - 2x + 1$ and $h \neq 0$, find the value of

$$\frac{f(x+h) - f(x-h)}{2h}.$$

Solution We know that since $f(x) = 3x^2 - 2x + 1$ then $f(\square) = 3\square^2 - 2\square + 1$. It follows that

$$\begin{aligned} f(x+h) - f(x-h) &= \left\{3\boxed{\text{x+h}}^2 - 2\boxed{\text{x+h}} + 1\right\} - \left\{3\boxed{\text{x-h}}^2 - 2\boxed{\text{x-h}} + 1\right\} \\ &= 3\left\{(x+h)^2 - (x-h)^2\right\} - 2\left\{(x+h) - (x-h)\right\} \\ &= 3(4xh) - 2(2h) = 12xh - 4h. \end{aligned}$$

It follows that for $h \neq 0$,

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{12xh - 4h}{2h} = 6x - 2.$$

Example 13. Let f be defined by

$$f(x) = \begin{cases} x+1, & \text{if} \quad -1 \leq x \leq 0, \\ x^2, & \text{if} \quad 0 < x \leq 3, \end{cases}$$

This type of function is said to be "**defined in pieces**", because it takes on different values depending on where the "x" is...

a) What is $f(-1)$?
b) Evaluate $f(0.70714)$.
c) Given that $0 < x < 1$ evaluate $f(2x+1)$.

Solution a) Since $f(x) = x + 1$ for any x in the interval $-1 \leq x \leq 0$ and $x = -1$ is in this interval, it follows that $f(-1) = (-1) + 1 = 0$.

b) Since $f(x) = x^2$ for any x in the interval $0 < x \leq 3$ and $x = 0.70714$ is in this interval, it follows that $f(0.70714) = (0.70714)^2 = 0.50005$

c) First we need to know what f does to the symbol $2x + 1$, that is, what is the value of $f(2x + 1)$? But this means that we have to know where the values of $2x + 1$ are when $0 < x < 1$, right? So, for $0 < x < 1$ we know that $0 < 2x < 2$ and so once we add 1 to each of the terms in the inequality we see that $1 = 0 + 1 < 2x + 1 < 2 + 1 = 3$. In other words, whenever $0 < x < 1$, the values of the expression $2x + 1$ must lie in the interval $1 < 2x + 1 < 3$. We now use the Box method: Since f takes a symbol to its square whenever the symbol is in the interval $(0, 3]$, we can write by definition $f(\square) = \square^2$ whenever $0 < \square \leq 3$. Putting $2x + 1$ in the box, (and using the fact that $1 < \boxed{\text{2x+1}} < 3$) we find that $f(\boxed{\text{2x+1}}) = \boxed{\text{2x+1}}^2$ from which we deduce $f(2x+1) = (2x+1)^2$ for $0 < x < 1$.

Some useful angles expressed in radians

degrees	*radians*
0	0
30^o	$\pi/6$
45^o	$\pi/4$
60^o	$\pi/3$
90^o	$\pi/2$
180^o	π
270^o	$3\pi/2$
360^o	2π

Figure 4.

We'll need to recall some notions from geometry in the next section.

Don't forget that, in Calculus, we always assume that angles are described in **radians** and not degrees. The conversion is given by

$$Radians \;=\; \frac{(Degrees) \times (\pi)}{180}$$

For example, 45 degrees $= 45\ \pi/180 = \pi/4 \approx 0.7853981633974$ radians.

NOTES:

SNAPSHOTS

Example 14. This example requires knowledge of trigonometry. Given that $h(t) = t^2 \cos(t)$ and $Dom(h) = (-\infty, \infty)$. Determine the value of $h(\sin x)$.

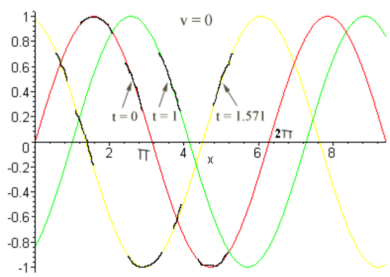
Solution We know that $h(\square) = \square^2 \cos(\square)$. So, $h(\boxed{\sin x}) = \boxed{\sin x}^2 \cos(\boxed{\sin x})$. Removing the box we get, $h(\sin x) = (\sin x)^2 \cos(\sin x)$, or, equivalently, $h(\sin x) = (\sin x)^2 \cos(\sin x) = \sin^2(x) \cos(\sin x)$.

The function $f(x) = \sin x$ and some of its values.

x *(in radians)*	$\sin x$
0	0
$\pi/6$	$1/2 = 0.5$
$-\pi/6$	$-1/2 = -0.5$
$\pi/3$	$\sqrt{3}/2 \approx 0.8660$
$-\pi/3$	$-\sqrt{3}/2 \approx -0.8660$
$\pi/2$	1
$-\pi/2$	-1
$\pi/4$	$\sqrt{2}/2 \approx 0.7071$
$-\pi/4$	$-\sqrt{2}/2 \approx -0.7071$
$3\pi/2$	-1
$-3\pi/2$	1
2π	0

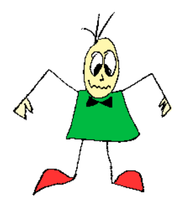
Figure 5.

Example 15. Let f be defined by the rule $f(x) = \sin x$. Then the function whose values are defined by $f(x - vt) = \sin(x - vt)$ can be thought of as representing a travelling wave moving to the right with velocity $v \geq 0$. Here t represents *time* and we take it that $t \geq 0$. You can get a feel for this motion from the graph below where we assume that $v = 0$ and use three increasing times to simulate the motion of the wave to the right.



Example 16. On the surface of our moon, an object P falling from rest will fall a distance, $f(t)$, of approximately $5.3t^2$ feet in t seconds. Let's take it for granted that its, so-called, **instantaneous velocity**, denoted by the symbol $f'(t)$, at time $t = t_0 \geq 0$ is given by the expression

$$\textit{Instantaneous velocity at time } t = f'(t) = 10.6\ t.$$



Determine its (instantaneous) velocity after 1 second (at $t = 1$) and after 2.6 seconds ($t = 2.6$).

Solution We calculate its instantaneous velocity, at $t = t_0 = 1$ second. Since, in this case, $f(t) = 5.3t^2$, it follows that its instantaneous velocity at $t = 1$ second is given by $10.6(1) = 10.6$ feet per second, obtained by setting $t = 1$ in the formula for $f'(t)$. Similarly, $f'(2.6) = 10.6(2.6) = 27.56$ feet per second. The observation here is that one can conclude that an object falling from rest on the surface of the moon will fall at approximately *one-third* the rate it does on earth (neglecting air resistance, here).

Example 17. Now let's say that f is defined by

$$f(x) = \begin{cases} x^2 + 1, & \text{if} \quad -1 \leq x \leq 0, \\ \cos x, & \text{if} \quad 0 < x \leq \pi, \\ x - \pi, & \text{if} \quad \pi < x \leq 2\pi. \end{cases}$$

Find an expression for $f(x + 1)$.

Solution Note that this function is defined *in pieces* (see Example 13) ... Its domain is obtained by taking the union of all the intervals on the right side making up one big interval, which, in our case is the interval $-1 \leq x \leq 2\pi$. So we see that $f(2) = \cos(2)$

because the number 2 is in the interval $0 < x \leq \pi$. On the other hand, $f(8)$ is not defined because 8 is not within the domain of definition of our function, since $2\pi \approx 6.28$.

Now, the value of $f(x+1)$, say, will be different depending on where the symbol "$x+1$" is. We can still use the "box" method to write down the values $f(x+1)$. In fact, we replace every occurence of the symbol x by our standard "box", insert the symbols "$x+1$" inside the box, and then remove the boxes... We'll find

$$f(\boxed{x+1}) = \begin{cases} \boxed{x+1}^2 + 1, & \text{if} \quad -1 \leq \boxed{\text{x + 1}} \leq 0, \\ \cos \boxed{x+1}, & \text{if} \quad 0 < \boxed{\text{x + 1}} \leq \pi, \\ \boxed{x+1} - \pi, & \text{if} \quad \pi < \boxed{x+1} \leq 2\pi. \end{cases}$$

or

$$f(x+1) = \begin{cases} (x+1)^2 + 1, & \text{if} \quad -1 \leq x+1 \leq 0, \\ \cos(x+1), & \text{if} \quad 0 < x+1 \leq \pi, \\ (x+1) - \pi, & \text{if} \quad \pi < x+1 \leq 2\pi. \end{cases}$$

We now solve the inequalities on the right for the symbol x (by subtracting 1 from each side of the inequality). This gives us the values

$$f(x+1) = \begin{cases} x^2 + 2x + 2, & \text{if} \quad -2 \leq x \leq -1, \\ \cos(x+1), & \text{if} \quad -1 < x \leq \pi - 1, \\ x + 1 - \pi, & \text{if} \quad \pi - 1 < x \leq 2\pi - 1. \end{cases}$$

Note that the graph of the function $f(x+1)$ is really the graph of the function $f(x)$ shifted to the left by 1 unit. We call this a "translate" of f.

Exercise Set 1.

Use the method of this section to evaluate the following functions at the indicated point(s) or symbol.

1. $f(x) = x^2 + 2x - 1$. What is $f(-1)$? $f(0)$? $f(+1)$? $f(1/2)$?
2. $g(t) = t^3 \ \sin t$. Evaluate $g(x+1)$.
3. $h(z) = z + 2 \ \sin z - \cos(z+2)$. Evaluate $h(z-2)$.
4. $k(x) = -2 \ \cos(x - ct)$. Evaluate $k(x + 2ct)$.
5. $f(x) = \sin(\cos x)$. Find the value of $f(\pi/2)$. [*Hint:* $\cos(\pi/2) = 0$]
6. $f(x) = x^2 + 1$. Find the value of
$$\frac{f(x+h) - f(x)}{h}$$
whenever $h \neq 0$. Simplify this expression as much as you can!
7. $g(t) = \sin(t+3)$. Evaluate
$$\frac{g(t+h) - g(t)}{h}$$
whenever $h \neq 0$ and simplify this as much as possible.
Hint: Use the trigonometric identity
$$\sin(A+B) = \sin A \ \cos B + \cos A \ \sin B$$
valid for any two angles A, B where we can set $A = t + 3$ and $B = h$ (just two more **symbols**, right?)

8. Let x_0, x_1 be two symbols which denote real numbers. In addition, for any real number x let $f(x) = 2x^2 \cos x$.
   a) If $x_0 = 0$ and $x_1 = \pi$, evaluate the expression
$$\frac{f(x_0) + f(x_1)}{2}.$$
   Hint: $\cos(\pi) = -1$ b) What is the value of the expression
$$f(x_0) + 2\ f(x_1) + f(x_2)$$
   if we are given that $x_0 = 0$, $x_1 = \pi$, and $x_2 = 2\ \pi$.?

   Hint: $\cos(2\pi) = 1$

If you want to use your calculator for this question (you don't have to...) don't forget to change your angle settings to **radians**!

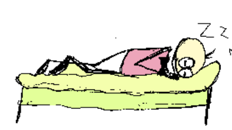
9. Let f be defined by
$$f(x) = \begin{cases} x+1, & \text{if} \quad -1 \le x \le 0, \\ -x+1, & \text{if} \quad 0 < x \le 2, \\ x^2, & \text{if} \quad 2 < x \le 6. \end{cases}$$
   a) What is $f(0)$?
   b) Evaluate $f(0.142857)$.
   c) Given that $0 < x < 1$ evaluate $f(3x+2)$.

   Hint Use the ideas in Example 17.
10. Let $f(x) = 2\ x^2 - 2$ and $F(x) = \sqrt{\frac{x}{2} + 1}$. Calculate the values $f(F(x))$ and $F(f(x))$ using the *box method* of this section. Don't forget to expand completely and simplify your answers as much as possible.
11. $g(x) = x^2 - 2x + 1$. Show that $g(x+1) = x^2$ for every value of x.
12. $h(x) = \dfrac{2x+1}{1+x}$. Show that $h\left(\dfrac{x-1}{2-x}\right) = x$, for $x \neq 2$.
13. Let $f(x) = 4x^2 - 5x + 1$, and $h \neq 0$ a real number. Evaluate the expression
$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$
14. Let f be defined by
$$f(x) = \begin{cases} x-1, & \text{if} \quad 0 \le x \le 2, \\ 2x, & \text{if} \quad 2 < x \le 4, \end{cases}$$
   Find an expression for $f(x+1)$ when $1 < x \le 2$.

The **Binomial Theorem** states that, in particular,
$$(\Box + \triangle)^2 = \Box^2 + 2\ \Box\triangle + \triangle^2$$
for any two symbols $\Box, \triangle$ representing real numbers, functions, etc. **Don't forget the middle terms**, namely, " $2\ \Box\triangle$ " in this formula.

Suggested Homework Set 1. *Go into a library or the World Wide Web and come up with five (5) functions that appear in the literature. Maybe they have names associated with them? Are they useful in science, engineering or in commerce? Once you have your functions, identify the dependent and independent variables and try to evaluate those functions at various points in their domain.*

NOTES



1.3 The Absolute Value of a Function

One of the most important functions in the study of calculus is the absolute value function.

Definition of the absolute value function

> **Definition 2.** *The function whose rule is defined by setting*
>
> $$|x| = \begin{cases} x, & \textit{if} \quad x \geq 0, \\ -x, & \textit{if} \quad x < 0. \end{cases}$$
>
> *is called the* **absolute value** *function.*

For example,$|-5| = -(-5) = +5$, and $|6.1| = 6.1$. You see from this Definition that the absolute value of a number is either that same number (if it is positive) or the original *unsigned* number (dropping the minus sign completely). Thus, $|-5| = -(-5) = 5$, since $-5 < 0$ while $|3.45| = 3.45$ since $3.45 > 0$. Now the inequality

$$(0 \leq) \quad |\Box| \leq \triangle \tag{1.1}$$

between the symbols $\Box$ and $\triangle$ is equivalent to (i.e., exactly the same as) the inequality

$$-\triangle \ \leq \ \Box \ \leq \ \triangle. \tag{1.2}$$

where $\Box$ and $\triangle$ are any two symbols whatsoever (x, t, or any function of x, etc). Why is this true? Well, there are only only two cases. That is, $\Box \geq 0$ and $\Box \leq 0$, right? Let's say $\Box \geq 0$. In this case $\Box = |\Box|$ and so (1.1) implies (1.2) immediately since the left side of (1.2) is already negative. On the other hand if $\Box \leq 0$ then, by (1.1), $|\Box| = -\Box \leq \triangle$ which implies $\Box \geq -\triangle$. Furthermore, since $\Box \leq 0$ we have that $\Box \leq -\Box \leq \triangle$ and this gives (1.2). For example, **the inequality** $|x - a| < 1$ **means that** *the distance from x to a is at most* 1 and, in terms of an inequality, this can be written as

$$|x - a| < 1 \quad \text{is equivalent to} \quad -1 < x - a < +1.$$

Why? Well, put $x - a$ in the box of (1.1) and the number 1 in the triangle. Move these symbols to (1.2) and remove the box and triangle, then what's left is what you want. That's all. Now, adding a to both ends and the middle term of this latest inequality we find the equivalent statement

$$|x - a| < 1 \quad \text{is also equivalent to} \quad a - 1 < x < a + 1.$$

This business of passing from (1.1) to (1.2) is **really important in Calculus** and you should be able to do this without thinking (after lots of practice you will, don't worry).

Example 18. Write down the values of the function f defined by the rule $f(x) = |1 - x^2|$ as a function defined in pieces. That is "remove the absolute value" around the $1 - x^2$.

Solution Looks tough? Those absolute values can be very frustrating sometimes! Just use the Box Method of Section 1.2. That is, use Definition 2 and replace all the symbols between the vertical bars by a $\Box$. This really makes life easy, let's try it out. Put the symbols between the vertical bars, namely, the "$1 - x^2$" inside a box, $\Box$. Since, by definition,

$$|\ \Box\ | = \begin{cases} \Box, & \text{if} \quad \Box \geq 0, \\ -\Box, & \text{if} \quad \Box < 0, \end{cases} \tag{1.3}$$

we also have

$$|1-x^2| = |\ \boxed{1-x^2}\ | = \begin{cases} \boxed{1-x^2}, & \text{if } \boxed{1-x^2} \geq 0, \\ -\boxed{1-x^2}, & \text{if } \boxed{1-x^2} < 0. \end{cases}$$

Removing the boxes and replacing them by parentheses we find

$$|1-x^2| = |\ (1-x^2)\ | = \begin{cases} (1-x^2), & \text{if } (1-x^2) \geq 0, \\ -(1-x^2), & \text{if } (1-x^2) < 0. \end{cases}$$

Adding x^2 to both sides of the inequality on the right we see that the above display is equivalent to the display

$$|1-x^2| = \begin{cases} (1-x^2), & \text{if } 1 \geq x^2, \\ -(1-x^2), & \text{if } 1 < x^2. \end{cases}$$

Almost done! We just need to solve for x on the right, above. To do this, we're going to use the results in Figure 6 with $A = 1$. So, if $x^2 \leq 1$ then $|x| \leq 1$ too. Similarly, if $1 < x^2$ then $1 < |x|$, too. Finally, we find

$$|\ 1-x^2\ | = \begin{cases} (1-x^2), & \text{if } 1 \geq |x|, \\ -(1-x^2), & \text{if } 1 < |x|. \end{cases}$$

Now by (1.1)-(1.2) the inequality $|x| \leq 1$ is equivalent to the inequality $-1 \leq x \leq 1$. In addition, $1 < |x|$ is equivalent to the double statement "either $x > 1$ or $x < -1$". Hence the last display for $|1-x^2|$ may be rewritten as

$$|\ 1-x^2\ | = \begin{cases} 1-x^2, & \text{if } -1 \leq x \leq +1, \\ x^2-1, & \text{if } x > 1 \text{ or } x < -1. \end{cases}$$

A glance at this latest result shows that the natural domain of f (see the Appendix) is the set of all real numbers.

NOTE The procedure described in Example 18 will be referred to as the process of **removing the absolute value**. You just can't leave out those vertical bars because you feel like it! Other functions defined by absolute values are handled in the same way.

Example 19. Remove the absolute value in the expression $f(x) = \left|x^2+2x\right|$.

Solution We note that since x^2+2x is a polynomial it is defined for every value of x, that is, its natural domain is the set of all real numbers, $(-\infty, +\infty)$. Let's use the Box Method. Since for any symbol say, $\square$, we have by definition,

$$|\ \square\ | = \begin{cases} \square, & \text{if } \square \geq 0, \\ -\square, & \text{if } \square < 0. \end{cases}$$

we see that, upon inserting the symbols x^2+2x inside the box and then removing

Solving a square root inequality!

If for some real numbers A and x, we have

$$x^2 < A,$$

then, it follows that

$$|\ x\ | < \sqrt{A}$$

More generally, this result is true if x is replaced by any other symbol (including functions!), say, $\square$. That is, if for some real numbers A and $\square$, we have

$$\square^2 < A,$$

then, it follows that

$$|\square| < \sqrt{A}$$

These results are still true if we replace "$<$" by "$\leq$" or if we reverse the inequality and $A > 0$.

Figure 6.

Steps in removing absolute values in a function f

- Look at that part of f with the absolute values,
- Put all the stuff between the vertical bars in a $\boxed{\text{box}}$,
- Use the definition of the absolute value, equation 1.3.
- Remove the boxes, and replace them by parentheses,
- Solve the inequalities involving $x's$ for the symbol x.
- Rewrite f *in pieces*

  (See Examples 18 and 20)

Figure 7.

its sides, we get

$$\left|\ \boxed{x^2+2x}\ \right| = \begin{cases} \boxed{x^2+2x}, & \text{if } \boxed{x^2+2x} \geq 0, \\ -(\boxed{x^2+2x}), & \text{if } \boxed{x^2+2x} < 0. \end{cases}$$

So the required function *defined in pieces* is given by

$$|\ x^2+2x\ | = \begin{cases} x^2+2x, & \text{if } \ x^2+2x \geq 0, \\ -(x^2+2x), & \text{if } \ x^2+2x < 0. \end{cases}$$

where we need to solve the inequalities $x^2+2x \geq 0$ and $x^2+2x < 0$, for x. But $x^2+2x = x(x+2)$. Since we want $x(x+2) \geq 0$, there are now *two cases.* Either *both* quantities $x, x+2$ must be greater than or equal to zero, OR *both* quantities $x, x+2$ must be less than or equal to zero (so that $x(x+2) \geq 0$ once again). The other case, the one where $x(x+2) < 0$, will be considered separately.

Sir Isaac Newton

1642 - 1727

Solving $x^2 + 2x \geq 0$:

Case 1: $x \geq 0$, $(x+2) \geq 0$. In this case, it is clear that $x \geq 0$ (since if $x \geq 0$ then $x+2 \geq 0$ too). This means that the polynomial inequality $x^2+2x \geq 0$ has among its solutions the set of real numbers $\{x : x \geq 0\}$.

Case 2: $x \leq 0$, $(x+2) \leq 0$. In this case, we see that $x+2 \leq 0$ implies that $x \leq -2$. On the other hand, for such x we also have $x \leq 0$, (since if $x \leq -2$ then $x \leq 0$ too). This means that the polynomial inequality $x^2+2x \geq 0$ has for its *solution* the set of real numbers $\{x : x \leq -2 \text{ or } x \geq 0\}$.

A similar argument applies to the case where we need to solve $x(x+2) < 0$. Once again there are two cases, namely, the case where $x > 0$ and $x+2 < 0$ and the separate case where $x < 0$ and $x+2 > 0$. Hence,

Solving $x^2 + 2x < 0$:

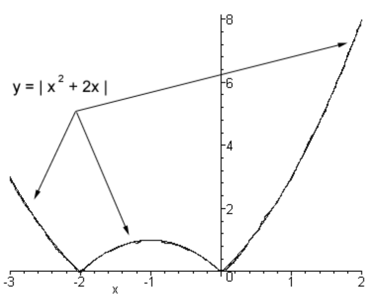
Case 1: $x > 0$ and $(x+2) < 0$. This case is impossible since, if $x > 0$ then $x+2 > 2$ and so $x+2 < 0$ is impossible. This means that there are no x such that $x > 0$ and $x+2 < 0$.

Case 2: $x < 0$ and $(x+2) > 0$. This implies that $x < 0$ and $x > -2$, which gives the inequality $x^2+2x < 0$. So, the the solution set is $\{x : -2 < x < 0\}$.

Combining the conclusions of each of these cases, our function takes the form,

$$|\ x^2+2x\ | = \begin{cases} x^2+2x, & \text{if } \ x \leq -2 \text{ or } x \geq 0, \\ -(x^2+2x), & \text{if } \ -2 < x < 0. \end{cases}$$

Its graph appears in the margin.



The graph of $f(x) = |x^2 + 2x|$.

$\boxed{\textbf{Example 20.}}$ Rewrite the function f defined by

$$f(x) = |x-1| + |x+1|$$

for $-\infty < x < +\infty$, as a function defined in pieces.

Solution How do we start this? Basically we need to understand the definition of the absolute value and apply it here. In other words, there are really 4 cases to consider when we want to remove these absolute values, the cases in question being:

1. $x-1 \geq 0$ and $x+1 \geq 0$. These two together imply that $x \geq 1$ and $x \geq -1$, that is, $x \geq 1$.
2. $x-1 \geq 0$ and $x+1 \leq 0$. These two together imply that $x \geq 1$ and $x \leq -1$ which is impossible.
3. $x-1 \leq 0$ and $x+1 \geq 0$. These two together imply that $x \leq 1$ and $x \geq -1$, or $-1 \leq x \leq 1$.
4. $x-1 \leq 0$ and $x+1 \leq 0$. These two together imply that $x \leq 1$ and $x \leq -1$, or $x \leq -1$.

Combining these four cases we see that we only need to consider the three cases where $x \leq -1$, $x \geq 1$ and $-1 \leq x \leq 1$ separately.

In the first instance, if $x \leq -1$, then $x+1 \leq 0$ and so $|x+1| = -(x+1)$. What about $|x-1|$? Well, since $x \leq -1$ it follows that $x-1 \leq -2 < 0$. Hence $|x-1| = -(x-1)$ for such x. Combining these two results about the absolute values we get that

$$f(x) = |x+1| + |x-1| = -(x+1) - (x-1) = -2x,$$

for $x \leq -1$.

In the second instance, if $x \geq 1$, then $x-1 \geq 0$ so that $|x-1| = x-1$. In addition, since $x+1 \geq 2$ we see that $|x+1| = x+1$. Combining these two we get

$$f(x) = |x+1| + |x-1| = (x+1) + (x-1) = 2x,$$

for $x \geq 1$.

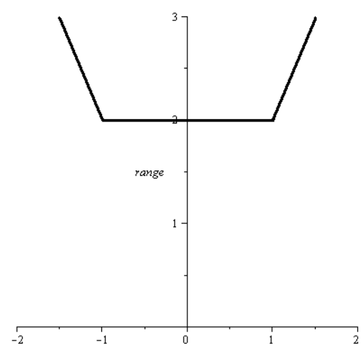
In the third and final instance, if $-1 \leq x \leq 1$ then $x+1 \geq 0$ and so $|x+1| = x+1$. Furthermore, $x-1 \leq 0$ implies $|x-1| = -(x-1)$. Hence we conclude that

$$f(x) = |x+1| + |x-1| = (x+1) - (x-1) = 2,$$

for $-1 \leq x \leq 1$. Combining these three displays for the pieces that make up f we can write f as follows:

$$|x+1| + |x-1| = \begin{cases} -2x, & \text{if} \quad x \leq -1, \\ 2, & \text{if} \quad -1 \leq x \leq 1. \\ 2x, & \text{if} \quad x \geq 1. \end{cases}$$

and this completes the description of the function f as required. Its graph consists of the darkened lines in the adjoining Figure.



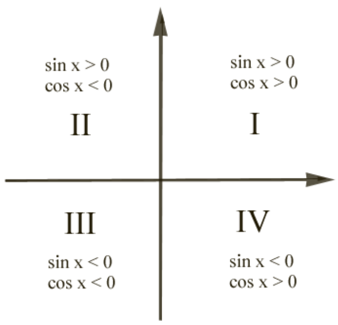
The graph of $f(x) = |x+1| + |x-1|$.

Example 21. Remove the absolute value in the function f defined by $f(x) = \sqrt{|\cos x|}$ for $-\infty < x < +\infty$.

Solution First, we notice that $|\cos x| \geq 0$ regardless of the value of x, right? So, the square root of this absolute value is defined for every value of x too, and this explains the fact that its natural domain is the open interval $-\infty < x < +\infty$. We use the method in Figure 7.

- Let's look at that part of f which has the absolute value signs in it...
  In this case, it's the part with the $|\cos x|$ term in it.
- Then, take all the stuff between the vertical bars of the absolute value and stick them in a box ...
  Using Definition 2 in disguise namely, Equation 1.3, we see that

$$|\ \cos x\ | = |\ \boxed{\cos x}\ | = \begin{cases} \boxed{\cos x}, & \text{if} \quad \boxed{\cos x} \geq 0, \\ \\ -\boxed{\cos x}, & \text{if} \quad \boxed{\cos x} < 0. \end{cases}$$



- Now, remove the boxes, and replace them by parentheses if need be ...

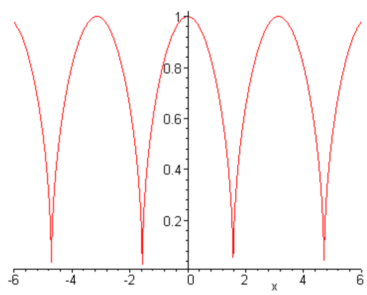
$$| \cos x | = \begin{cases} \cos x, & \text{if} \quad \cos x \geq 0, \\ -\cos x, & \text{if} \quad \cos x < 0. \end{cases}$$

- Next, solve the inequalities on the right of the last display above for x.
  In this case, this means that we have to figure out when $\cos x \geq 0$ and when $\cos x < 0$, okay? There's a few ways of doing this... One way is to look at the graphs of each one of these functions and just find those intervals where the graph lies above the x-axis.

  Another way involves remembering the trigonometric fact that the cosine function is positive in Quadrants I and IV (see the margin for a quick recall). Turning this last statement into symbols means that if x is between $-\pi/2$ and $\pi/2$, then $\cos x \geq 0$. Putting it another way, if x is in the interval $[-\pi/2, +\pi/2]$ then $\cos x \geq 0$.
  But we can always add positive and negative multiples of 2π to this and get more and more intervals where the cosine function is positive ... why?. Either way, we get that $\cos x \geq 0$ whenever x is in the closed intervals $[-\pi/2, +\pi/2]$, $[3\pi/2, +5\pi/2]$, $[7\pi/2, +9\pi/2], \ldots$ or if x is in the closed intervals $[-5\pi/2, -3\pi/2]$, $[-9\pi/2, -7\pi/2]$, ... (Each one of these intervals is obtained by adding multiples of 2π to the endpoints of the basic interval $[-\pi/2, +\pi/2]$ and rearranging the numbers in increasing order).

  Combining these results we can write

$$| \cos x | = \begin{cases} \cos x, & \text{if } x \text{ is in } [-\pi/2, +\pi/2], [3\pi/2, +5\pi/2], \\ & [7\pi/2, +9\pi/2], \ldots, \text{ or if } x \text{ is in} \\ & [-5\pi/2, -3\pi/2], [-9\pi/2, -7\pi/2], \ldots, \\ & \\ -\cos x, & \text{if } x \text{ is NOT IN ANY ONE} \\ & \text{of the above intervals.} \end{cases}$$



The graph of $y = \sqrt{|\cos x|}$

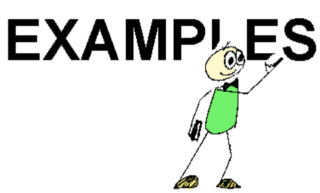
Figure 8.

- Feed all this information back into the original function to get it "in pieces"
  Taking the square root of all the cosine terms above we get

$$\sqrt{| \cos x |} = \begin{cases} \sqrt{\cos x}, & \text{if } x \text{ is in } [-\pi/2, +\pi/2], [3\pi/2, +5\pi/2], \\ & [7\pi/2, +9\pi/2], \ldots, \text{ or if } x \text{ is in} \\ & [-5\pi/2, -3\pi/2], [-9\pi/2, -7\pi/2], \ldots, \\ & \\ \sqrt{-\cos x}, & \text{if } x \text{ is NOT IN ANY ONE} \\ & \text{of the above intervals.} \end{cases}$$

  Phew, that's it! Look at Fig. 8 to see what this function looks like.

You shouldn't worry about the minus sign appearing inside the square root sign above because, inside those intervals, the cosine is negative, so the negative of the cosine is positive, and so we can take its square root without any problem! Try to understand this example completely; then you'll be on your way to mastering one of the most useful concepts in Calculus, handling absolute values!



SNAPSHOTS

Example 22. *The natural domain of the function h defined by $h(x) = \sqrt{x^2 - 9}$ is the set of all real numbers x such that $x^2 - 9 \geq 0$ or, equivalently, $|x| \geq 3$ (See Table 12.1 and Figure 6).*

Example 23. *The function f defined by $f(x) = x^{2/3} - 9$ has its natural domain given by the set of all real numbers, $(-\infty, \infty)$! No exceptions! All of them...why?*

Solution Look at Table 12.1 and notice that, by algebra, $x^{2/3} = (\sqrt[3]{x})^{\ 2}$. Since the natural domain of the "cube root" function is $(-\infty, \infty)$, the same is true of its "square". Subtracting "9" doesn't change the domain, that's all!

Example 24. *Find the natural domain of the the function f defined by*

$$f(x) = \frac{x}{\sin x \ \cos x}.$$

Solution The natural domain of f is given by the set of all real numbers with the property that $\sin x \ \cos x \neq 0$, (cf., Table 12.1), that is, the set of all real numbers x with $x \neq \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \pm 7\pi/2, ..., 0, \pm \pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, ...$ (as these are the values where the denominator is zero).

Sofya Kovalevskaya (1850 - 1891)

Celebrated immortal mathematician, writer, and revolutionary, she was appointed Professor of Mathematics at the University of Stockholm, in Sweden, in 1889, the first woman ever to be so honored in her time and the second such in Europe (Sophie Germain, being the other one). At one point, she was apparently courted by Alfred Nobel and brothers (of Nobel Prize fame). Author of more than 10 mathematical papers, she proved what is now known as the **Cauchy-Kovalevski Theorem**, one of the first deep results in a field called *Partial Differential Equations*, an area used in the study of airplane wings, satellite motion, wavefronts, fluid flow, among many other applications. A. L. Cauchy's name appears because he had proved a more basic version of this result earlier.

Example 25. *The natural domain of the function f given by*

$$f(x) = \frac{|\sin x|}{\sqrt{1 - x^2}}$$

is given by the set of all real numbers x with the property that $1 - x^2 > 0$ or, equivalently, the open interval $|x| < 1$, or, $-1 < x < +1$ (See Equations 1.1 and 1.2).

Example 26. *Find the natural domain of the function f defined by $f(x) = \sqrt{|\sin x|}$*

Solution The natural domain is given by the set of all real numbers x in the interval $(-\infty, \infty)$, that's right, *all* real numbers! Looks weird right, because of the square root business! But the absolute value will turn any negative number inside the root into a positive one (or zero), so the square root is always defined, and, as a consequence, f is defined everywhere too.

List of Important Trigonometric Identities

Recall that an **identity** is an equation which is true for any value of the variable for which the expressions are defined. So, this means that the identities are true regardless of whether or not the variable looks like an x, y, [], $\diamond$, $f(x)$, etc.

YOU'VE GOT TO KNOW THESE!

Odd-even identities

$$\begin{aligned} \sin(-x) &= -\sin x, \\ \cos(-x) &= \cos x. \end{aligned}$$

Pythagorean identities

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1, \\ \sec^2 x - \tan^2 x &= 1. \\ \csc^2 x - \cot^2 x &= 1, \end{aligned}$$

Addition identities

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y,\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y.\end{aligned}$$

Double-angle identities

$$\begin{aligned}\sin(2x) &= 2\sin x\cos x,\\ \cos(2x) &= \cos^2 x - \sin^2 x,\\ \sin^2 x &= \frac{1-\cos(2x)}{2},\\ \cos^2 x &= \frac{1+\cos(2x)}{2}.\end{aligned}$$

You can derive the identities below from the ones above, or ... you'll have to memorize them! Well, it's best if you know how to get to them from the ones above using some basic algebra.

$$\tan(-x) = -\tan(x) \qquad \sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin\left(\frac{\pi}{2}-x\right) = \cos x \qquad \cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos\left(\frac{\pi}{2}-x\right) = \sin x \qquad \sin x\cos y = \frac{1}{2}[\sin(x+y)+\sin(x-y)]$$

$$\tan\left(\frac{\pi}{2}-x\right) = \cot x \qquad \sin x\sin y = -\frac{1}{2}[\cos(x+y)-\cos(x-y)]$$

$$\cos(2x) = 1-2\sin^2 x \qquad \cos x\cos y = \frac{1}{2}[\cos(x+y)+\cos(x-y)]$$

$$\cos(2x) = 2\cos^2 x - 1 \qquad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y} \qquad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

NOTES:

Exercise Set 2.

Use the method of Example 18, Example 20, Example 21 and the discussion following Definition 2 to **remove the absolute value** appearing in the values of the functions defined below. Note that, once the absolute value is removed, the function will be defined *in pieces*.

1. $f(x) = |x^2 - 1|$, for $-\infty < x < +\infty$.

2. $g(x) = |3x + 4|$, for $-\infty < x < +\infty$.

   Hint: Put the symbols $3x + 4$ in a box and use the idea in Example 18

3. $h(x) = x|x|$, for $-\infty < x < +\infty$.

4. $f(t) = 1 - |t|$, for $-\infty < t < +\infty$.

5. $g(w) = |\sin w|$, for $-\infty < w < +\infty$.

   Hint: $\sin w \geq 0$ when w is in Quadrants I and II, or, equivalently, when w is between 0 and π radians, 2π and 3π radians, etc.

6. $$f(x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

   for $|x| > 1$.

7. The *signum* function, whose name is simply *sgn* (and pronounced *the sign of* x) where

   $$sgn(x) = \frac{x}{|x|}$$

   for $x \neq 0$. The motivation for the name comes from the fact that the values of this function correspond to the *sign* of x (whether it is positive or negative).

8. $f(x) = x + |x|$, for $-\infty < x < +\infty$.

9. $f(x) = x - \sqrt{x^2}$, for $-\infty < x < +\infty$.

Suggested Homework Set 2. *Do problems* 2, 4, 6, 8, *above.*

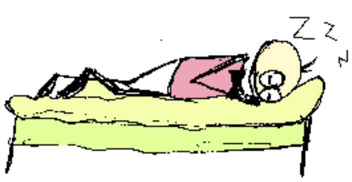
1. $\sin(A+B) = \sin A\cos(B) + \cos A\sin B$
2. $\cos(A+B) = \cos A\cos(B) - \sin A\sin B$
3. $\sin^2 x + \cos^2 x = 1$
4. $\sec^2 x - \tan^2 x = 1$
5. $\csc^2 x - \cot^2 x = 1$
6. $\cos 2x = \cos^2 x - \sin^2 x$
7. $\sin 2x = 2\sin x\cos x$
8. $\cos^2 x = \dfrac{1+\cos 2x}{2}$
9. $\sin^2 x = \dfrac{1-\cos 2x}{2}$

Table 1.1: Useful Trigonometric Identities

WHAT'S WRONG WITH THIS ??

$$
\begin{aligned}
-1 &= -1 \\
\sqrt{\frac{1}{-1}} &= \sqrt{\frac{-1}{1}} \\
\frac{\sqrt{1}}{\sqrt{-1}} &= \frac{\sqrt{-1}}{\sqrt{1}} \\
\sqrt{1}\,\sqrt{1} &= \sqrt{-1}\,\sqrt{-1} \\
(\sqrt{1})^2 &= (\sqrt{-1})^2 \\
1 &= -1
\end{aligned}
$$

Crazy, right?



NOTES:

A Basic Inequality

If
$$0 < \square \leq \triangle,$$
then
$$\frac{1}{\square} \geq \frac{1}{\triangle},$$
regardless of the meaning of the box or the triangle or what's in them!

OR

You reverse the inequality when you take reciprocals !

Table 1.2: Reciprocal Inequalities Among Positive Quantities

Inequalities among reciprocals

If
$$0 < \frac{1}{\square} \leq \frac{1}{\triangle},$$
then
$$\square \geq \triangle,$$
regardless of the meaning of the box or the triangle or what's in them!

OR

You still reverse the inequality when you take reciprocals !

Table 1.3: Another Reciprocal Inequality Among Positive Quantities

1.4 A Quick Review of Inequalities

In this section we will review basic inequalities because they are really important in Calculus. Knowing how to manipulate basic inequalities will come in handy when we look at how graphs of functions are sketched, in our examination of the monotonicity of functions, their convexity and many other areas. We leave the subject of reviewing the solution of basic and polynomial inequalities to Chapter 5. So, this is one section you should know well!

In this section, as in previous ones, we make heavy use of the generic symbols $\square$ and $\triangle$, that is, our box and triangle. Just remember that variables *don't have to be* called x, and any other symbol will do as well.

Recall that the **reciprocal** of a number is simply the number 1 divided by the number. Table 1.2 shows what happens when you **take the reciprocal** of each term in an inequality involving two **positive** terms. You see, you need to **reverse the sign!**. The same result is true had we started out with an inequality among reciprocals of positive quantities, see Table 1.3.

The results mentioned in these tables are really useful! For example,

Example 27. *Show that given any number $x \neq 0$,*

$$\frac{1}{x^2} > \frac{1}{x^2+1}.$$

Solution We know that $0 < x^2 < x^2 + 1$, and this is true regardless of the value of x, so long as $x \neq 0$, which we have assumed anyhow. So, if we put x^2 in the box in Table 1.2 and (make the triangle big enough so that we can) put $x^2 + 1$ in the triangle, then we'll find, as a conclusion, that

$$\frac{1}{x^2} > \frac{1}{x^2+1},$$

and this is true for any value of $x \neq 0$, whether x be positive or negative.

Example 28. *Solve the inequality $|2x - 1| < 3$ for x.*

Solution Recall that $|\,\Box\,| < \triangle$ is equivalent to $-\triangle < \Box < \triangle$ for any two symbols (which we denote by $\Box$ and $\triangle$). In this case, putting $2x - 1$ in the box and 3 in the triangle, we see that we are looking for x's such that $-3 < 2x - 1 < 3$. Adding 1 to all the terms gives $-2 < 2x < 4$. Finally, dividing by 2 right across the inequality we get $-1 < x < 2$ and this is our answer.

$$\begin{tikzcd} & \boxed{A\Box \le A\triangle} \\ \boxed{\Box \le \triangle} \arrow[ur, "A>0\ ?"] \arrow[dr, "A<0\ ?"'] & \\ & \boxed{A\Box \ge A\triangle} \end{tikzcd}$$

Figure 9.

Example 29. *Solve the inequality*

$$\left|\frac{x+1}{2x+3}\right| < 2$$

for x.

Solution Once again, by definition of the absolute value, this means we are looking for x's such that

$$-2 < \frac{x+1}{2x+3} < 2.$$

There now two main cases: **Case 1** where $2x+3 > 0$ OR **Case 2** where $2x+3 < 0$. Of course, when $2x + 3 = 0$, the fraction is undefined (actually infinite) and so this is not a solution of our inequality. We consider the cases in turn:

Case 1: $\boldsymbol{2x + 3 > 0}$ In this case we multiply the last display throughout by $2x+3$ and keep the inequalities as they are (by the rules in Figure 9). In other words, we must now have

$$-2(2x + 3) < x + 1 < 2(2x + 3).$$

Grouping all the x's in the "middle" and all the constants "on the ends" we find the two inequalities $-4x - 6 < x + 1$ and $x + 1 < 4x + 6$. Solving for x in both instances we get $5x > -7$ or $x > -\frac{7}{5}$ and $3x > -5$ or $x > -\frac{5}{3}$. Now what?

Well, let's recapitulate. We have shown that if $2x + 3 > 0$ then we must have $x > -\frac{7}{5} = -1.4$ and $x > -\frac{5}{3} \approx -1.667$. On the one hand if $x > -1.4$ then $x > -1.667$ for sure and so the two inequalities together imply that $x > -1.4$, or $x > -\frac{7}{5}$. But this is not all! You see, we still need to guarantee that $2x + 3 > 0$ (by the main assumption of this case)! That is, we need to make sure that we have BOTH $2x+3 > 0$ AND $x > -\frac{7}{5}$, *i.e.*, we need $x > -\frac{3}{2}$ and $x > -\frac{7}{5}$. These last two inequalities together imply that

$$x > -\frac{7}{5}.$$

Case 2: $\boldsymbol{2x + 3 < 0}$ In this case we still multiply the last display throughout by $2x + 3$ but now we must REVERSE the inequalities (by the rules in Figure 9, since

now $A = 2x + 3 < 0$). In other words, we must now have

$$-2(2x+3) > x+1 > 2(2x+3).$$

As before we can derive the two inequalities $-4x - 6 > x + 1$ and $x + 1 > 4x + 6$. Solving for x in both instances we get $5x < -7$ or $x < -\frac{7}{5}$ and $3x < -5$ or $x < -\frac{5}{3}$. But now, these two inequalities together imply that $x < -\frac{5}{3} \approx -1.667$. This result, combined with the basic assumption that $2x + 3 < 0$ or $x < -\frac{3}{2} = -1.5$, gives us that $x < -\frac{5}{3}$ (since $-1.667 < -1.5$). The solution in this case is therefore given by the inequality

$$x < -\frac{5}{3}.$$

Combining the two cases we get the final solution as (see the margin)

$$-1.4 = -\frac{7}{5} < x \quad \text{OR} \quad x < -\frac{5}{3} \approx -1.667.$$

In terms of intervals the answer is the set of points x such that

$$-\infty < x < -\frac{5}{3} \quad OR \quad -\frac{7}{5} < x < \infty.$$

1.4.1 The triangle inequalities

Let $x, y, a, \Box$ be any real numbers with $a \geq 0$. Recall that the statement

$$|\Box| \leq a \quad \text{is equivalent to} \quad -a \leq \Box \leq a \tag{1.4}$$

where the symbols $\Box, a$ may represent actual numbers, variables, function values, etc. Replacing a here by $|x|$ and $\Box$ by x we get, by (1.4),

$$-|x| \quad \leq x \leq \quad |x|. \tag{1.5}$$

We get a similar statement for y, that is,

$$-|y| \quad \leq y \leq \quad |y|. \tag{1.6}$$

There are 2 other really important inequalities called the **Triangle Inequalities**: If $\Box, \triangle$ are any 2 symbols representing real numbers, functions, etc. then

$$|\Box + \triangle| \leq |\Box| + |\triangle|,$$

and

$$|\Box - \triangle| \geq |\ |\Box| - |\triangle|\ |.$$

Since we can add inequalities together we can combine (1.5) and (1.6) to find

$$-|x| - |y| \quad \leq x + y \leq \quad |x| + |y|, \tag{1.7}$$

or equivalently

$$-(|x| + |y|) \quad \leq x + y \leq \quad |x| + |y|. \tag{1.8}$$

Now replace $x + y$ by $\Box$ and $|x| + |y|$ by a in (1.8) and apply (1.4). Then (1.8) is equivalent to the statement

$$\boxed{|x+y| \leq |x| + |y|, \tag{1.9}}$$

for any two real numbers x, y. This is called the **Triangle Inequality**.

The second triangle inequality is just as important as it provides a lower bound on the absolute value of a sum of two numbers. To get this we replace x by $x - y$ in (1.9) and re-arrange terms to find

$$|x - y| \geq |x| - |y|.$$

Similarly, replacing y in (1.9) by $y - x$ we obtain

$$|y - x| \geq |y| - |x| = -(|x| - |y|).$$

But $|x - y| = |y - x|$. So, combining the last two displays gives us

$$|x - y| \geq \pm(|x| - |y|),$$

and this statement is equivalent to the statement

$$\boxed{\Big||x| - |y|\Big| \leq |x - y|. \qquad (1.10)}$$

We may call this the **second triangle inequality**.

Example 30. *Show that if x is any number , $x \geq 1$, then*

$$\frac{1}{\sqrt{x}} \geq \frac{1}{|x|}.$$

Solution If $x = 1$ the result is clear. Now, everyone believes that, if $x > 1$, then $x < x^2$. OK, well, we can take the square root of both sides and use Figure 6 to get $\sqrt{x} < \sqrt{x^2} = |x|$. From this we get,

$$\text{If } x > 1, \qquad \frac{1}{\sqrt{x}} > \frac{1}{|x|}.$$

On the other hand, one has to be careful with the opposite inequality $x > x^2$ if $x < 1$ ... This is *true*, even though it doesn't seem right!

Example 31. *Show that if x is any number, $0 < x \leq 1$, then*

$$\frac{1}{\sqrt{x}} \leq \frac{1}{x}.$$

Solution Once again, if $x = 1$ the result is clear. Using Figure 6 again, we now find that if $x < 1$, then $\sqrt{x} > \sqrt{x^2} = |x|$, and so,

$$\text{If } 0 < x < 1, \qquad \frac{1}{\sqrt{x}} < \frac{1}{|x|} = \frac{1}{x}.$$

These inequalities can allow us to estimate how big or how small functions can be!

Example 32. *We know that $\square < \square + 1$ for any $\square$ representing a positive number. The box can even be a function! In other words, we can put a function of x inside the box, apply the reciprocal inequality of Table 1.2, (where we put the symbols $\square + 1$ inside the triangle) and get a new inequality, as follows. Since $\square < \square + 1$, then*

$$\frac{1}{\square} > \frac{1}{\square + 1}.$$

Now, put the function f defined by $f(x) = x^2 + 3x^4 + |x| + 1$ inside the box. Note that $f(x) > 0$ (this is really important!). It follows that, for example,

$$\frac{1}{x^2 + 3x^4 + |x| + 1} > \frac{1}{x^2 + 3x^4 + |x| + 1 + 1} = \frac{1}{x^2 + 3x^4 + |x| + 2}.$$

Multiplying inequalities by an unknown quantity

- If $A > 0$, is any symbol (variable, function, number, fraction, ...) and
$$\square \leq \triangle,$$
then
$$A\square \leq A\triangle,$$
- If $A < 0$, is any symbol (variable, function, number, fraction, ...) and
$$\square \leq \triangle,$$
then
$$A\square \geq A\triangle,$$
Don't forget to **reverse the inequality sign when $A < 0$!**

Table 1.4: Multiplying Inequalities Together

$\boxed{\textbf{Example 33.}}$ *How "big" is the function f defined by $f(x) = x^2 + \cos x$ if x is in the interval $[0, 1]$?*

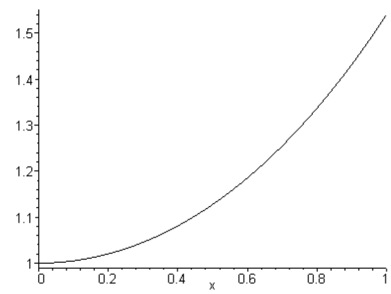


Figure 10.

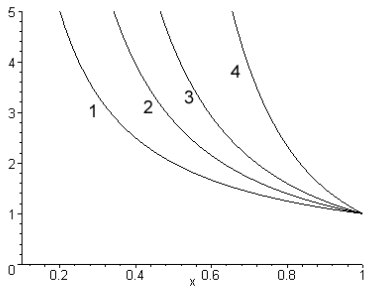
Solution The best way to figure out how big f is, is to try and estimate each term which makes it up. Let's leave x^2 alone for the time being and concentrate on the $\cos x$ term. We know from trigonometry that $|\cos x| \leq 1$ for any value of x. OK, since $\pm \cos x \leq |\cos x|$ by definition of the absolute value, and $|\cos x| \leq 1$ it follows that
$$\pm \cos x \leq 1$$
for any value of x. Choosing the plus sign, because that's what we want, we add x^2 to both sides and this gives
$$f(x) \ = \ x^2 + \cos x \leq x^2 + 1$$
and this is true for any value of x. But we're only given that x is between 0 and 1. So, we take the right-most term, the $x^2 + 1$, and *replace it by something "larger"*. The simplest way to do this is to notice that, since $x \leq 1$ (then $x^2 \leq 1$ too) and $x^2 + 1 \leq 1^2 + 1 = 2$. Okay, now we combine the inequalities above to find that, if $0 \leq x \leq 1$,
$$f(x) = x^2 + \cos x \leq x^2 + 1 \leq 2.$$
This shows that $f(x) \leq 2$ for such $x's$ and yet we never had to calculate the range of f to get this ... We just used inequalities! You can see this too by plotting its graph as in Figure 10.

NOTE: We have just shown that the so-called **maximum value of the function f over the interval** $[0, 1]$ denoted mathematically by the symbols
$$\max_{x \ in \ [0,1]} \ f(x)$$
is not greater than 2, that is,
$$\max_{x \ in \ [0,1]} \ f(x) \leq \ 2.$$

For a 'flowchart interpretation' of Table 1.4 see Figure 9 in the margin.

Example 34. *Show that if x is any real number, then $-x^3 \geq -x^2(x+1)$.*

Solution We know that, for any value of x, $x < x+1$ so, by Table 1.4, with $A = -2$ we find that $-2x > -2(x+1)$. You see that we reversed the inequality since we multiplied the original inequality by a negative number! We could also have multiplied the original inequality by $A = -x^2 \leq 0$, in which case we find, $-x^3 \geq -x^2(x+1)$ for any value of x, as being true too.



1. The graph of $1/x$
2. The graph of $1/x^{1.5}$
3. The graph of $1/x^{2.1}$
4. The graph of $1/x^{3.8}$

Figure 11.

Example 35. *Show that if $p \geq 1$, and $x \geq 1$, then*

$$\frac{1}{x^p} \leq \frac{1}{x}.$$

Solution Let $p \geq 1$ be any number, (e.g, $p = 1.657, p = 2, \ldots$). Then you'll believe that if $p-1 \geq 0$ and if $x \geq 1$ then $x^{p-1} \geq 1$ (for example, if $x = 2$ and $p = 1.5$, then $2^{1.5-1} = 2^{0.5} = 2^{1/2} = \sqrt{2} = 1.414\ldots > 1$). Since $x^{p-1} \geq 1$ we can multiply both sides of this inequality by $x > 1$, which is positive, and find, by Figure 9 with $A = x$, that $x^p \geq x$. From this and Table 1.2 we obtain the result

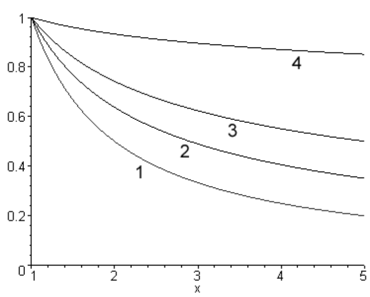
$$\frac{1}{x^p} \leq \frac{1}{x}, \qquad \textit{if } p \geq 1, \textit{ and } x \geq 1$$

Example 36. *Show that if $p > 1$, and $0 < x \leq 1$, then*

$$\frac{1}{x^p} \geq \frac{1}{x}.$$

Solution In this example we change the preceding example slightly by requiring that $0 < x \leq 1$. In this case we get the opposite inequality, namely, if $p > 1$ then $x^{p-1} \leq 1$ (e.g., if $x = 1/2, p = 1.5$, then $(1/2)^{1.5-1} = (1/2)^{0.5} = (1/2)^{1/2} = 1/\sqrt{2} = 0.707\ldots < 1$). Since $x^{p-1} \leq 1$ we can multiply both sides of this inequality by $x > 0$, and find, by Figure 9 with $A = x$, again, that $x^p \leq x$. From this and Table 1.2 we obtain the result (see Fig. 11)

$$\frac{1}{x^p} \geq \frac{1}{x}, \qquad \textit{if } p \geq 1, \textit{ and } 0 < x \leq 1$$



1. The graph of $1/x$
2. The graph of $1/x^{0.65}$
3. The graph of $1/x^{0.43}$
4. The graph of $1/x^{0.1}$

Figure 12.

Example 37. *Show that if $0 < p \leq 1$, and $x \geq 1$, then*

$$\frac{1}{x^p} \geq \frac{1}{x}.$$

Solution In this final example of this type we look at what happens if we change the p values of the preceding two examples and keep the x-values larger than 1. Okay, let's say that $0 < p \leq 1$ and $x \geq 1$. Then you'll believe that, since $p - 1 \leq 0$, $0 < x^{p-1} \leq 1$, (try $x = 2, p = 1/2$, say). Multiplying both sides by x and taking reciprocals we get the important inequality (see Fig. 12),

$$\frac{1}{x^p} \geq \frac{1}{x}, \qquad \text{if } 0 < p \leq 1, \text{ and } x \geq 1 \tag{1.11}$$

NOTES:

SNAPSHOTS

Example 38. *We know that* $\sin x \geq 0$ *in Quadrants I and II, by trigonometry. Combining this with Equation (1.11), we find that: If* x *is an angle expressed in radians and* $1 \leq x \leq \pi$ *then*

$$\frac{\sin x}{x^p} \geq \frac{\sin x}{x}, \qquad 0 < p \leq 1$$

Think about why we had to have some restriction like $x \leq \pi$ *here.*

Example 39. *On the other hand,* $\cos x \leq 0$ *if* $\pi/2 \leq x \leq \pi$ *(notice that* $x > 1$ *automatically in this case, since* $\pi/2 = 1.57... > 1$*). So this, combined with Equation (1.11), gives*

$$\frac{\cos x}{x^p} \leq \frac{\cos x}{x} \qquad 0 < p \leq 1$$

where we had to "reverse" the inequality (1.11) as $\cos x \leq 0$.

Example 40. *There is this really cool (and old) inequality called the AG- inequality, (meaning the Arithmetic-Geometric Inequality). It is an inequality between the "arithmetic mean" of two positive numbers,* $\square$ *and* $\triangle$*, and their "geometric mean". By definition, the* **arithmetic mean** *of* $\square$ *and* $\triangle$*, is* $(\square + \triangle)/2$*, or more simply, their "average". The* **geometric mean** *of* $\square$ *and* $\triangle$*, is, by definition,* $\sqrt{\square\triangle}$. *The inequality states that if* $\square \geq 0$, $\triangle \geq 0$, *then*

$$\frac{\square + \triangle}{2} \geq \sqrt{\square\triangle}$$

Do you see why this is true? Just start out with the inequality $(\sqrt{\square} - \sqrt{\triangle})^2 \geq 0$, *expand the terms, rearrange them, and then divide by* 2.

Example 41. *For example, if we set* x^2 *in the box and* x^4 *in the triangle and apply the AG-inequality (Example 40) to these two positive numbers we get the "new" inequality*

$$\frac{x^2 + x^4}{2} \geq x^3$$

valid for any value of x*, something that is not easy to see if we don't use the AG-inequality.*

We recall the **general form of the Binomial Theorem**. It states that if n is any positive integer, and $\square$ is any symbol (a function, the variable x, or a positive number, or negative, or even zero) then

$$(1+\square)^n = 1 + n\square + \frac{n(n-1)}{2!}\square^2 + \frac{n(n-1)(n-2)}{3!}\square^3 + \cdots + \frac{n(n-1)\cdots(2)(1)}{n!}\square^n \tag{1.12}$$

where the symbols that look like 3!, or n!, called **factorials**, mean that we multiply all the integers from 1 to n together. For example, $2! = (1)(2) = 2$, $3! = (1)(2)(3) = 6$, $4! = (1)(2)(3)(4) = 24$ and, generally, "n factorial" is defined by

$$n! = (1)(2)(3)\cdots(n-1)(n) \tag{1.13}$$

When $n = 0$ we all agree that $0! = 1$ . . . Don't worry about why this is true right now, but it has something to do with a function called the **Gamma Function**, which will be defined later when we study things called **improper integrals**. Using this we can arrive at identities like:

$$(1-x)^n \quad = \quad 1 - nx + \frac{n(n-1)}{2!}x^2 \pm \cdots + (-1)^n \frac{n(n-1)\cdots(2)(1)}{n!}x^n,$$

Remember, if

$$\square^2 \leq \triangle^2,$$

then

$$|\square| \leq |\triangle|,$$

regardless of the values of the variables or symbols involved. If $\square > 0$, this is also true for positive powers, p, other than 2. So, if, for example, $\square > 0$, and

$$\square^p \leq \triangle^p,$$

then

$$|\square| \leq |\triangle|.$$

This result is not true if $\square < 0$ since, for example, $(-2)^3 < 1^3$ but $|-2| > |1|$.

Figure 13.

obtained by setting $\square = -x$ in (1.12) or even

$$2^n = 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots + \frac{n(n-1)\cdots(2)(1)}{n!}$$

(just let $\square = 1$ in the Binomial Theorem), and finally,

$$(2x-3)^n = (-1)^n 3^n \left(1 - \frac{2nx}{3} + \frac{4n(n-1)x^2}{9\cdot 2!} + \cdots + (-1)^n \frac{2^n x^n}{3^n}\right),$$

found by noting that

$$(2x-3)^n = \left(-3(1-\frac{2x}{3})\right)^n = (-3)^n \left(1 - \frac{2x}{3}\right)^n = (-1)^n 3^n \left(1 - \frac{2x}{3}\right)^n$$

and then using the boxed formula (1.12) above with $\square = -\frac{2x}{3}$. In the above formulae note that

$$\frac{n(n-1)\cdots(2)(1)}{n!} = 1$$

by definition of the factorial symbol.

If you already know something about improper integrals then the Gamma Function can be written as,

$$\Gamma(p) = \int_0^\infty x^{p-1}\, e^{-x}\, dx$$

where $p \geq 1$. One can actually prove that $\Gamma(n+1) = n!$ if n is a positive integer. The study of this function dates back to Euler.

Exercise Set 3.

Determine which of the following 7 statements is true, if any. If the statement is false give an example that shows this. Give reasons either way.

1. If $-A < B$ then $-\frac{1}{A} > \frac{1}{B}$
2. If $-\frac{1}{A} < B$ then $-A > -B$
3. If $A < B$ then $A^2 < B^2$
4. If $A > B$ then $1/A < 1/B$
5. If $A < B$ then $-A < -B$
6. If $A^2 < B^2$ and $A > 0$, then $A < B$
7. If $A^2 > B^2$ and $A > 0$, then $|B| < A$
8. Start with the obvious $\sin x < \sin x + 1$ and find an interval of $x's$ in which we can conclude that
$$\csc x > \frac{1}{\sin x + 1}$$
9. How big is the function f defined by $f(x) = x^2 + 2\sin x$ if x is in the interval $[0, 2]$?
10. How big is the function g defined by $g(x) = 1/x$ if x is in the interval $[-1, 4]$?
11. Start with the inequality $x > x - 1$ and conclude that for $x > 1$, we have $x^2 > (x-1)^2$.
12. If x is an angle expressed in radians and $1 \leq x \leq \pi$ show that
$$\frac{\sin x}{x^{p-1}} \geq \sin x, \qquad p \leq 1.$$
13. Use the AG-inequality to show that if $x \geq 0$ then $x + x^2 \geq \sqrt{x^3}$. Be careful here, you'll need to use the fact that $2 > 1$! Are you allowed to "square both sides" of this inequality to find that, if $x \geq 0$, $(x + x^2)^2 \geq x^3$? Justify your answer.

14. Can you replace the $x's$ in the inequality $x^2 \geq 2x - 1$ by an arbitrary symbol, like $\square$? Under what conditions on the symbol?

15. Use the ideas surrounding Equation (1.11) to show that, if $p \leq 1$ and $|x| \geq 1$, then
$$\frac{1}{|x|^p} \geq \frac{1}{|x|}$$
Hint: Note that if $|x| \geq 1$ then $|x|^{1-p} \geq 1$.

16. In the theory of relativity developed by A. Einstein, H. Lorentz and others at the turn of this century, there appears the quantity γ, read as "gamma", defined as a function of the velocity, v, of an object by
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$
where c is the speed of light. Determine the conditions on v which give us that the quantity γ is a real number. In other words, find the natural domain of γ. Hint: This involves an inequality and an absolute value.

17. Show that for any integer $n \geq 1$ there holds the inequalities
$$2 \quad \leq \quad \left(1 + \frac{1}{n}\right)^n \; < \; 3.$$
Hint: 

This is a really hard problem! Use the Binomial Theorem, (1.12), with $\square = \frac{1}{n}$. But first of all, get a *feel* for this result by using your calculator and setting $n = 2, 3, 4, \ldots, 10$ and seeing that this works!

Suggested Homework Set 3. *At the very least, do problems* $3, 6, 11, 12, 14$

Web Links

Additional information on Functions may be found at:
http://www.coolmath.com/precalculus-review-calculus-intro/index.html

For more on inequalities see:
http://www.khanacademy.org/math/algebra/solving-linear-inequalities

More on the AG- inequality at:
http://www.cut-the-knot.com/pythagoras/corollary.html

NOTES:

Mathematics is not always done by mathematicians. For example, **Giordano Bruno**, 1543-1600, Renaissance Philosopher, once a Dominican monk, was burned at the stake in the year 1600 for heresy. He wrote around 20 books in many of which he upheld the view that Copernicus held, i.e., that of a sun centered solar system (called *heliocentric*). A statue has been erected in his honor in the *Campo dei Fiori*, in Rome. Awesome. The rest is history...What really impresses me about Bruno is his steadfastness in the face of criticism and ultimate torture and execution. It is said that he died without uttering a groan. Few would drive on this narrow road...not even **Galileo Galilei**, 1564 - 1642, physicist, who came shortly after him. On the advice of a Franciscan, Galileo retracted his support for the heliocentric theory when called before the Inquisition. As a result, he stayed under house arrest until his death, in 1642. The theories of Copernicus and Galileo would eventually be absorbed into Newton and Leibniz's Calculus as a consequence of the basic laws of motion. All this falls under the heading of *differential equations*.

1.5 Chapter Exercises

Use the methods of this Chapter to evaluate the following functions at the indicated point(s) or symbol.

1. $f(x) = 3x^2 - 2x + 1$. What is $f(-1)$? $f(0)$? $f(+1)$? $f(-1/2)$?
2. $g(t) = t^3 \cos t$. What is the value of $g(x^2 + 1)$?
3. $h(z) = z + 2 \sin z - \cos(z + 2)$. Evaluate $h(z + 3)$.
4. $f(x) = \cos x$. Find the value of
$$\frac{f(x+h) - f(x)}{h}$$
whenever $h \neq 0$. Simplify this expression as much as you can!
• Use a trigonometric identity for $\cos(A + B)$ with $A = x$, $B = h$.

Solve the following inequalities for the stated variable.

5. $\dfrac{3}{x} > 6, \quad x$
6. $3x + 4 \geq 0, \quad x$
7. $\dfrac{3}{2x-1} \leq 0, \quad x$
8. $x^2 > 5, \quad x$
9. $t^2 < \sqrt{5}, \quad t$
10. $\sin^2 x \leq 1, \quad x$
11. $z^p \geq 2, \quad z$, if $p > 0$
12. $x^2 - 9 \leq 0, \quad x$

Remove the absolute value (see Section 1.3 and Equation 1.3).

13. $f(x) = |x + 3|$
14. $g(t) = |t - 0.5|$
15. $g(t) = |1 - t|$
16. $f(x) = |2x - 1|$
17. $f(x) = |1 - 6x|$
18. $f(x) = |x^2 - 4|$
19. $f(x) = |3 - x^3|$
20. $f(x) = |x^2 - 2x + 1|$
21. $f(x) = |2x - x^2|$
22. $f(x) = |x^2 + 2|$
23. If x is an angle expressed in radians and $1 \leq x \leq \pi/2$ show that
$$\frac{\cos x}{x^{p-1}} \geq \cos x, \qquad p \leq 1$$
24. Use your calculator to tabulate the values of the quantity
$$\left(1 + \frac{1}{n}\right)^n$$
for $n = 1, 2, 3, \ldots, 10$ (See Exercise 17 of Exercise Set 3). Do the numbers you get seem to be getting close to something?

25. Use the AG-inequality to show that if $0 \le x \le \pi/2$, then
$$\frac{\sin x + \cos x}{\sqrt{2}} \ge \sqrt{\sin 2x}.$$

Suggested Homework Set 4. *Work out problems* $2, 4, 12, 17, 19, 21, 24$

1.6 Using Computer Algebra Systems (CAS),

Use your favorite Computer Algebra System (CAS), like Maple, MatLab, etc., or even a graphing calculator to answer the following questions:

Evaluate the functions at the following points:

1. $f(x) = \sqrt{x}$, for $x = -2, -1, 0, 1.23, 1.414, 2.7$. What happens when $x < 0$? Conclude that the natural domain of f is $[0, +\infty)$.
2. $g(x) = \sin(x\sqrt{2}) + \cos(x\sqrt{3})$, for $x = -4.37, -1.7, 0, 3.1415, 12.154, 16.2$. Are there any values of x for which $g(x)$ is not defined as a real number? Explain.
3. $f(t) = \sqrt[3]{t}$, for $t = -2.1, 0, 1.2, -4.1, 9$. Most CAS define power functions only when the base is positive, which is not the case if $t < 0$. In this case the natural domain of f is $(-\infty, +\infty)$ even though the CAS wants us to believe that it is $[0, \infty)$. So, be careful when reading off results using a CAS.
4. $g(x) = \dfrac{x+1}{x-1}$. Evaluate $g(-1), g(0), g(0.125), g(1), g(1.001), g(20), g(1000)$. Determine the behavior of g near $x = 1$. To do this use values of x just less than 1 and then values of x just larger than 1.
5. Define a function f by
$$f(t) = \begin{cases} \dfrac{t+\sqrt{t}}{\sqrt{t}}, & \text{if} \quad t > 0, \\ 1, & \text{if} \quad t = 0. \end{cases}$$

   Evaluate $f(1)$, $f(0)$, $f(2.3)$, $f(100.21)$. Show that $f(t) = \sqrt{t} + 1$ for every value of $t \ge 0$.
6. Let $f(x) = 1 + 2\cos^2(\sqrt{x+2}) + 2\sin^2(\sqrt{x+2})$.

   a) Evaluate $f(-2)$, $f(0.12)$, $f(-1.6)$, $f(3.2)$, $f(7)$.
   b) Explain your results.
   c) What is the natural domain of f?
   d) Can you conclude something simple about this function? Is it a constant function? Why?
7. To solve the inequality $|2x - 1| < 3$ use your CAS to
   a) Plot the graphs of $y = |2x - 1|$ and $y = 3$ and superimpose them on one another
   b) Find their points of intersection, and
   c) Solve the inequality (see the figure below)

The answer is: $-1 < x < 2$.

Evaluate the following inequalities graphically using a CAS:

a) $|3x - 2| < 5$
b) $|2x - 2| < 4.2$
c) $|(1.2)x - 3| > 2.61$
d) $|1.3 - (2.5)x| = 0.5$
e) $|1.5 - (5.14)x| > 2.1$

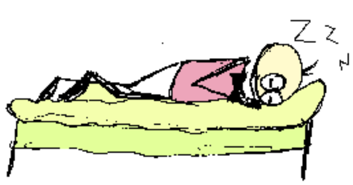
8. Find an interval of x's such that
a) $|\cos x| < \dfrac{1}{2}$
b) $\sin x + 2\cos x < 1$
c) $\sin(x\sqrt{2}) - \cos x > -\dfrac{1}{2}$.

Hint: Plot the functions on each side of the inequality separately, superimpose their graphs, estimate their points of intersection visually, and solve the inequality.

9. Plot the values of
$$f(x) = x\ \sin\left(\frac{1}{x}\right)$$
for small x's such as $x = 0.1,\ 0.001,\ -0.00001, 0.000001, -0.00000001$ etc. Guess what happens to the values of $f(x)$ as x gets closer and closer to zero (regardless of the direction, *i.e.,* regardless of whether $x > 0$ or $x < 0$.)

10. Let $f(x) = 4x - 4x^2$, for $0 \le x \le 1$. Use the Box method to evaluate the following terms, called the *iterates* of f for $x = x_0 = 0.5$:
$$f(0.5),\ f(f(0.5)),\ f(f(f(0.5))),\ f(f(f(f(0.5)))),\ldots$$
where each term is the image of the preceding term under f. Are these values approaching any specific value? Can you find values of $x = x_0$ (*e.g.,* $x_0 = 0.231, 0.64, \ldots$) for which these iterations actually seem to be approaching some specific number? This is an example of a *chaotic sequence* and is part of an exciting area of mathematics called "Chaos".

11. Plot the graphs of $y = x^2,\ (1.2)x^2,\ 4x^2,\ (10.6)x^2$ and compare these graphs with those of
$$y = x^{\frac{1}{2}},\ (1.2)x^{\frac{1}{2}},\ 4x^{\frac{1}{2}},\ (10.6)x^{\frac{1}{2}}.$$
Use this graphical information to guess the general shape of graphs of the form $y = x^p$ for $p > 1$ and for $0 < p < 1$. Guess what happens if $p < 0$?



12. Plot the graphs of the family of functions $f(x) = \sin(ax)$ for $a = 1,\ 10,\ 20,\ 40,\ 50$.

a) Estimate the value of those points in the interval $0 \le x \le \pi$ where $f(x) = 0$ (these x's are called "zeros" of f).
b) How many are there in each case?
c) Now find the position and the number of exact zeros of f inside this interval $0 \le x \le \pi$.

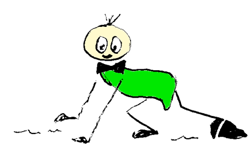
NOTES:

Chapter 2

Limits and Continuity

The Big Picture

The notion of a 'limit' permeates the universe around us. In the simplest cases, the *speed of light*, denoted by 'c', in a vacuum is the upper *limit* for the velocities of any object. Photons always travel with speed c but electrons can never reach this speed exactly no matter how much energy they are given. That's life! In another vein, let's look at the speed barrier for the $100m$ dash in Track & Field. World records rebound and are broken in this, the most illustrious of all races. But there *must be a limit* to the time in which one can run the $100m$ dash, right? For example, it is clear that none will ever run this in a record time of, say, 3.00 seconds! On the other hand, it *has* been run in a record time of 9.58 seconds. So, there must be a *limiting time* that no one will ever be able to reach but the records will get closer and closer to! It is the author's guess that this limiting time is around 9.50 seconds (although we thought it was 9.7 seconds, once). In a sense, this time interval of 9.60 seconds between the start of the race and its end, may be considered a *limit* of human locomotion. We can't seem to run at a constant speed of $100/9.50 = 10.5$ meters per second. Of course, the actual 'speed limit' of any human may sometimes be slightly higher than 10.5 m/sec, but not over the whole race. If you look at the Records Table below, you can see why we *could* consider this number, 9.50, a kind of limit.



0. Usain Bolt	Jamaica	9.58	09/08/16	Berlin, Worlds
1. Asafa Powell	Jamaica	9.74	07/09/09	Rieti, Italy
2. Tim Montgomery	USA	9.78	02/09/14	Paris, France
3. Maurice Greene	USA	9.79	99/06/16	Athens
4. Donovan Bailey	CAN	9.84	96/07/26	Atlanta
5. Leroy Burrell	USA	9.85	94/07/06	Lausanne
6. Carl Lewis	USA	9.86	91/08/25	Tokyo, Worlds
7. Frank Fredericks	NAM	9.86	96/07/03	Lausanne
8. Linford Christie	GBR	9.87	93/08/15	Stuttgart
9. Ato Boldon	TRI	9.89	97/05/10	Modesto
10. Florence Griffith-Joyner	USA	10.49	88/07/16	Indianapolis

On the other hand, in the world of temperature we have another 'limit', namely, something called *absolute zero* equal to -273^oC, or, by definition, $0K$ where K stands for Kelvin. This temperature is a lower *limit* for the temperature of any object under normal conditions. Normally, we may remove as much heat as we want from an object but we'll never be able to remove all of it, so to speak, so the object

will never attain this limiting temperature of $0\ K$.

These are physical examples of limits and long ago some guy called **Karl Weierstrass** (1815-1897), decided he would try to make sense out of all this limit stuff mathematically. So he worked really hard and created this method which we now call the *epsilon-delta* method which most mathematicians today use to prove that such and such a number is, in fact, the limit of some given function. We don't always have to prove it when we're dealing with applications, but if you want to know how to use this method you can look at the *Advanced Topics* later on. Basically, Karl's idea was that you could call some number L a 'limit' of a given function if the values of the function managed to get close, really close, always closer and closer to this number L but never really reach L. He just made this last statement meaningful using symbols.

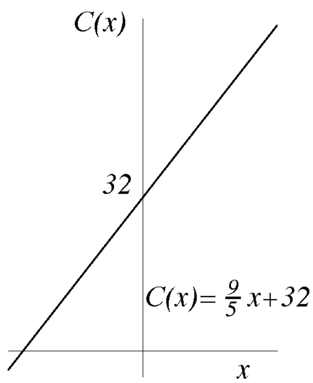


Figure 14.

In many practical situations functions may be given in different formats: that is, their graphs may be unbroken curves or even broken curves. For example, the function C which converts the temperature from degrees Centigrade, x, to degrees Fahrenheit, $C(x)$, is given by the straight line $C(x) = \frac{9}{5}x + 32$ depicted in Fig. 14.

This function's graph is an unbroken curve and we call such graphs the **graph of a continuous function** (as the name describes).

Example 42. If a taxi charges you \$3 as a flat fee for stepping in and 10 cents for every minute travelled, then the graph of the cost $c(t)$, as a function of time t (in minutes), is shown in Fig. 15.

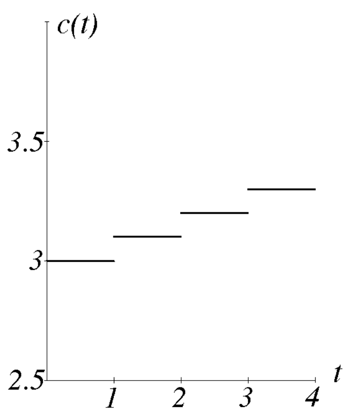


Figure 15.

When written out symbolically this function, c, in Fig. 15 is given by

$$c(t) = \begin{cases} 3, & 0 \le t < 1 \\ 3.1, & 1 \le t < 2 \\ 3.2, & 2 \le t < 3 \\ 3.3, & 3 \le t < 4 \\ \dots & \dots \end{cases}$$

or, more generally, as:

$$c(t) = 3 + \frac{n}{10}, \text{ if } n \le t \le n+1$$

where $n = 0, 1, 2, \ldots$.

In this case, the graph of c is a broken curve and this is an example of a **discontinuous function** (because of the 'breaks' it cannot be continuous). It is also called a **step-function** for obvious reasons.

These two examples serve to motivate the notion of continuity. Sometimes functions describing real phenomena are not continuous but we "turn" them into continuous functions as they are easier to visualize graphically.

Example 43. For instance, in Table 2.1 above we show the plot of the frequency of Hard X-rays versus time during a Solar Flare of 6th March, 1989:

The actual X-ray count per centimeter per second is an integer and so the plot should consist of points of the form $(t, c(t))$ where t is in seconds and $c(t)$ is the X-ray count, which is an *integer*. These points are grouped tightly together over small time intervals in the graph and "consecutive" points are joined by a line segment (which is quite short, though). The point is that even though these signals are *discrete* we tend to *interpolate* between these data points by using these small line segments. It's a fair thing to do but is it the *right* thing to do? Maybe nature doesn't

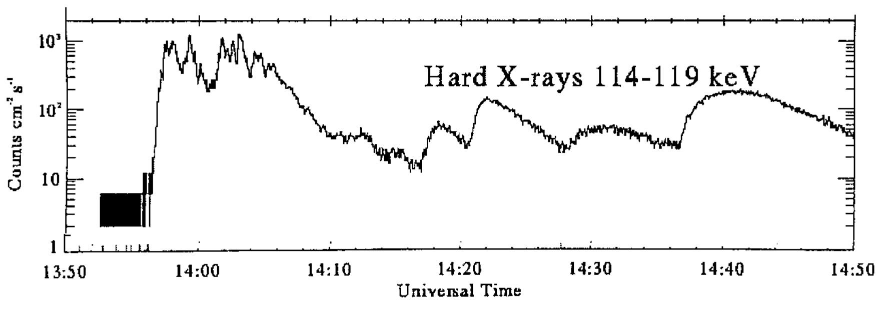


Table 2.1: The Mathematics of Solar Flares

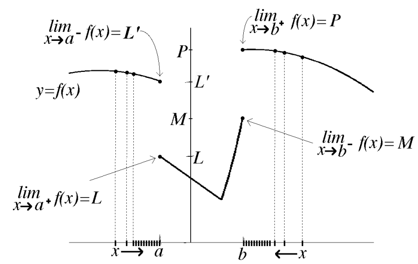


Figure 16.

like straight lines! The resulting graph of $c(t)$ is now the graph of a **continuous** function (there are no "breaks").

As you can gather from Fig. 15, the size of the **break in the graph of $c(t)$** is given by **subtracting neighbouring values of the function c around $t = a$**.

To make this idea more precise we define the **limit from the left** and the **limit from the right** of function f at the point $x = a$, see Tables 2.2 & 2.3 (and an optional chapter for the rigorous definitions).

2.1 One-Sided Limits of Functions

Limits from the Right

We say that the function f **has a limit from the right at $x = a$** (or the right-hand limit of f exists at $x = a$) whose value is L and denote this symbolically by

$$f(a+0) = \lim_{x\to a^+} f(x) = L$$

if BOTH of the following statements are satisfied:

1. Let $x > a$ and x be very close to $x = a$.
2. As x approaches a ("from the right" because "$x > a$"), the values of $f(x)$ approach the value L.

(For a more rigorous definition see the **Advanced Topics**, later on.)

Table 2.2: One-Sided Limits From the Right

For example, the function H defined by

$$H(x) = \begin{cases} 1, & for\ x \geq 0 \\ 0, & for\ x < 0 \end{cases}$$

called the **Heaviside Function** (named after *Oliver Heaviside*, (1850 - 1925) an

electrical engineer) has the property that

$$\lim_{x \to 0^+} H(x) = 1$$

Why? This is because we can set $a = 0$ and $f(x) = H(x)$ in the definition (or in Table 2.2) and apply it as follows:

a) Let $x > 0$ and x be very close to 0;

b) As x approaches 0 we need to ask the question: "What are the values, $H(x)$, doing?"

Well, we know that $H(x) = 1$ for **any** $x > 0$, so, as long as $x \neq 0$, the values $H(x) = 1$, (see Fig. 17), so this will be true "in the limit" as x approaches 0.

Limits from the Left

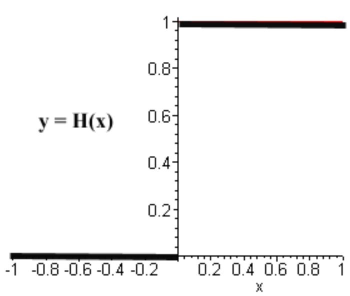
We say that the function f **has a limit from the left at** $x = a$ (or the left-hand limit of f exists at $x = a$) and is equal to L and denote this symbolically by

$$f(a-0) = \lim_{x \to a^-} f(x) = L$$

if BOTH of the following statements are satisfied:

1. Let $x < a$ and x be very close to $x = a$.
2. As x approaches a ("from the left" because "$x < a$"), the values of $f(x)$ approach the value L.

Table 2.3: One-Sided Limits From the Left



The graph of the Heaviside Function, $H(x)$.

Figure 17.

Returning to our Heaviside function, $H(x)$, (see Fig. 17), defined earlier we see that

$$\lim_{x \to 0^-} H(x) = 0$$

Why? In this case we set $a = 0$, $f(x) = H(x)$ in the definition (or Table 2.3), as before:

a) Let $x < 0$ and x very close to 0;

b) As x approaches 0 the values $H(x) = 0$, right? (This is because $x < 0$, and by definition, $H(x) = 0$ for such x). The same must be true of the "limit" and so we have

$$\lim_{x \to 0^-} H(x) = 0$$

OK, but how do you find these limits?

In practice, the idea is to choose some specific values of x near a (to the 'right' or to the 'left' of a) and, using your calculator, find the corresponding values of the function near a.

Example 44. Returning to the Taxi problem, Example 42 above, find the values of $c(1+0)$, $c(2-0)$ and $c(4-0)$, (See Fig. 15).

Solution By definition, $c(1+0) = \lim_{t\to 1^+} c(t)$. But this means that we want the values of $c(t)$ as $t \to 1$ from the right, *i.e.,* the values of $c(t)$ for $t > 1$ (just slightly bigger than 1) and $t \to 1$. Referring to Fig. 15 and Example 42 we see that $c(t) = 3.1$ for such t's and so $c(1+0) = 3.1$. In the same way we see that $c(2-0) = \lim_{t\to 2^-} c(t) = 3.1$ while $c(4-0) = \lim_{t\to 4^-} c(t) = 3.3$.

Example 45.

The function f is defined by:

$$f(x) = \begin{cases} x+1, & x < -1 \\ -2x, & -1 \le x \le +1 \\ x^2, & x > +1 \end{cases}$$

Evaluate the following limits whenever they exist and justify your answers.

$$\text{i) } \lim_{x\to -1^-} f(x); \text{ ii) } \lim_{x\to 0^+} f(x); \text{ iii) } \lim_{x\to 1^+} f(x)$$

Numerical evidence for Example 45, (iii). You can think of 'x' as being the name of an athlete and '$f(x)$' as being their record at running the 100 m. dash.

x	$f(x)$
1.5000	2.2500
1.2500	1.5625
1.1000	1.2100
1.0500	1.1025
1.0033	1.0067
1.0020	1.0040
1.0012	1.0025
1.0010	1.0020
1.0001	1.0002
...	...

Figure 18.

Solution **i)** We want a left-hand limit, right? This means that $x < -1$ and x should be very close to -1 (according to the definition in Table 2.3).

Now as x approaches -1 (from the 'left', i.e., with x always less than -1) we see that $x+1$ approaches 0, that is, $f(x)$ approaches 0. Thus,

$$\lim_{x\to -1^-} f(x) = 0.$$

ii) We want a right-hand limit here. This means that $x > 0$ and x must be very close to 0 (according to the definition in Table 2.2). Now for values of $x > 0$ and close to 0 the value of $f(x)$ is $-2x$ ... OK, this means that as x approaches 0 then $-2x$ approaches 0, or, equivalently $f(x)$ approaches 0. So

$$\lim_{x\to 0^+} f(x) = 0$$

iii) In this case we need $x > 1$ and x very close to 1. But for such values, $f(x) = x^2$ and so if we let x approach 1 we see that $f(x)$ approaches $(1)^2 = 1$. So,

$$\lim_{x\to 1^+} f(x) = 1.$$

See Figure 18, in the margin, where you can 'see' the values of our function, f, in the second column of the table while the x's which are approaching one (from the right) are in the first column. Note how the numbers in the second column get closer and closer to 1. This table is **not a proof** but it does make the limit we found believable.

Example 46. Evaluate the following limits and explain your answers.

$$f(x) = \begin{cases} x+4 & x \le 3 \\ 6 & x > 3 \end{cases}$$

$$\text{i) } \lim_{x\to 3^+} f(x) \text{ and ii) } \lim_{x\to 3^-} f(x)$$

Solution **i)** We set $x > 3$ and x very close to 3. Then the values are all $f(x) = 6$, by definition, and these don't change with respect to x. So

$$\lim_{x\to 3^+} f(x) = 6$$

ii) We set $x < 3$ and x very close to 3. Then the values $f(x) = x + 4$, by definition, and as x approaches 3, we see that $x + 4$ approaches $3 + 4 = 7$. So

$$\lim_{x \to 3^-} f(x) = 7$$

Example 47. Evaluate the following limits, if they exist.

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x \leq 0 \end{cases}$$

i) $\displaystyle\lim_{x \to 0^-} f(x)$; ii) $\displaystyle\lim_{x \to 0^+} f(x)$

Solution **i)** Let $x < 0$ and x very close to 0. Since $x < 0$, $f(x) = -x^2$ and $f(x)$ is very close to $-0^2 = 0$ since x is. Thus

$$\lim_{x \to 0^-} f(x) = 0$$

ii) Let $x > 0$ and x very close to 0. Since $x > 0$, $f(x) = x^2$ and $f(x)$ is very close to 0 too! Thus

$$\lim_{x \to 0^+} f(x) = 0$$

NOTE: In this example the graph of f has no breaks whatsoever since $f(0) = 0$.. In this case we say that the function f is **continuous** at $x = 0$. Had there been a 'break' or some points 'missing' from the graph we would describe f as **discontinuous** whenever those 'breaks' or 'missing points' occurred.

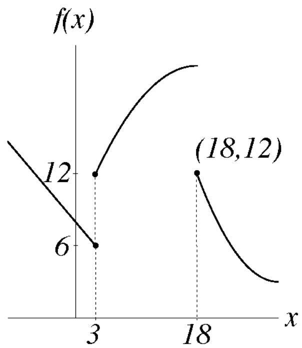


Figure 19.

Example 48. Use the graph in Figure 19 to determine the value of the required limits.

i) $\displaystyle\lim_{x \to 3^-} f(x)$; ii) $\displaystyle\lim_{x \to 3^+} f(x)$; iii) $\displaystyle\lim_{x \to 18^+} f(x)$

Solution **i)** Let $x < 3$ and let x be very close to 3. The point $(x, f(x))$ which is **on** the curve $y = f(x)$ now approaches a definite point as x approaches 3. Which point? The graph indicates that this point is $(3, 6)$. Thus

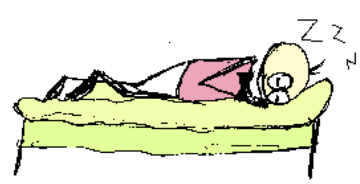
$$\lim_{x \to 3^-} f(x) = 6$$

ii) Let $x > 3$ and let x be very close to 3. In this case, as x approaches 3, the points $(x, f(x))$ travel down towards the point $(3, 12)$. Thus

$$\lim_{x \to 3^+} f(x) = 12$$

iii) Here we let $x > 18$ and let x be very close to 18. Now as x approaches 18 the points $(x, f(x))$ on the curve are approaching the point $(18, 12)$. Thus

$$\lim_{x \to 18^+} f(x) = 12$$



Exercise Set 4.

Evaluate the following limits and justify your conclusions.

1. $\lim_{x\to 2^+} (x+2)$
2. $\lim_{x\to 0^+} (x^2+1)$
3. $\lim_{x\to 1^-} (1-x^2)$
4. $\lim_{t\to 2^+} \left(\frac{1}{t-2}\right)$
5. $\lim_{x\to 0^+} (x|x|)$
6. $\lim_{x\to 0^-} \frac{x}{|x|}$
7. $\lim_{x\to 0^+} x\sin x$
8. $\lim_{x\to \pi^+} \left(\frac{\cos x}{x}\right)$
9. $\lim_{x\to 2^+} \left(\frac{x-2}{x+2}\right)$
10. $\lim_{x\to 1^-} \frac{x}{|x-1|}$
11. $\lim_{x\to 1^-} \left(\frac{x-1}{x+2}\right)$
12. $\lim_{x\to 3^+} \left(\frac{x-3}{x^2-9}\right)$ (Hint: Factor the denominator)

13. Let the function f be defined as follows:

$$f(x) = \begin{cases} 1-|x| & x<1 \\ x & x\geq 1 \end{cases}$$

Evaluate i) $\lim_{x\to 1^-} f(x)$; ii) $\lim_{x\to 1^+} f(x)$

Conclude that the graph of $f(x)$ must have a 'break' at $x=1$.

14. Let g be defined by

$$g(x) = \begin{cases} x^2+1 & x<0 \\ 1-x^2 & 0\leq x\leq 1 \\ x & x>1 \end{cases}$$

Evaluate

i). $\lim_{x\to 0^-} g(x)$ ii). $\lim_{x\to 0^+} g(x)$

iii). $\lim_{x\to 1^-} g(x)$ iv). $\lim_{x\to 1^+} g(x)$

v) Conclude that the graph of g has no breaks at $x=0$ but it does have a break at $x=1$.

15. Use the graph in Figure 20 to determine the value of the required limits. (The function f is composed of parts of 2 functions).

Evaluate

i). $\lim_{x\to -1^+} f(x)$ ii). $\lim_{x\to -1^-} f(x)$

iii). $\lim_{x\to 1^-} f(x)$ iv). $\lim_{x\to 1^+} f(x)$

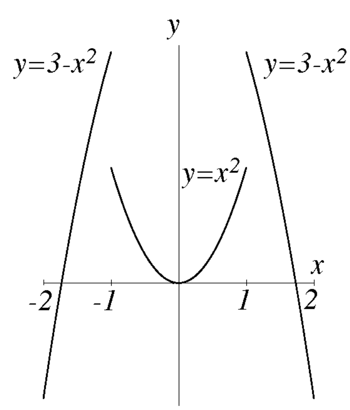


Figure 20.

2.2 Two-Sided Limits and Continuity

At this point we know how to determine the values of the limit from the left (or right) of a given function f at a point $x = a$. We have also seen that whenever

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

then there is a 'break' in the graph of f at $x = a$. The **absence of breaks or holes in the graph** of a function is what the notion of **continuity** is all about.

One of the key mathematical figures during the first millennium was a monk called **Alcuin of York** (735 - 804) who was Charlemagne's scribe and general advisor. He wrote a very influential book in Latin which contained many mathematical problems passed on from antiquity. In this book of his you'll find the following (paraphrased) problem:

A dog chases a rabbit who has a head-start of 150 feet. All you know is that every time the dog leaps 9 feet, the rabbit leaps 7 feet. How many leaps will it take for the dog to pass the rabbit?

Definition of the limit of a function at $x = a$.

We say that **a function f has the (two-sided) limit L as x approaches a** if

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

When this happens, we write (for brevity)

$$\lim_{x \to a} f(x) = L$$

and read this as: the limit of $f(x)$ as x approaches a is L (L may be infinite here).

NOTE: So, in order for a limit to exist both the right- and left-hand limits must exist and be equal. Using this notion we can now define the 'continuity of a function f at a point $x = a$.'

We say that f is **continuous** at $x = a$ if all the following conditions are satisfied:

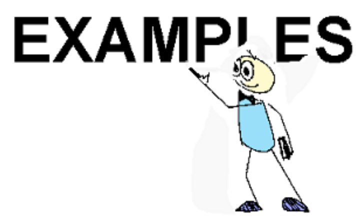
1. f is defined at $x = a$ (i.e., $f(a)$ is finite)
2. $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ ($= L$, their common value) and
3. $L = f(a)$.

NOTE: These three conditions must be satisfied in order for a function f to be continuous at a given point $x = a$. If any one or more of these conditions is not satisfied we say that f is **discontinuous** at $x = a$. In other words, we see from the Definition above (or in Table 2.7) that the one-sided limits from the left and right must be equal in order for f to be continuous at $x = a$ but that this equality, in itself, is **not enough to guarantee continuity** as there are 2 other conditions that need to be satisfied as well.

Example 49. Show that the given function is continuous at the given points, $x = 1$ and $x = 2$, where f is defined by

$$f(x) = \begin{cases} x + 1 & 0 \leq x \leq 1 \\ 2x & 1 < x \leq 2 \\ x^2 & x > 2 \end{cases}$$

Solution To show that f is continuous at $x = 1$ we need to verify 3 conditions (according to Table 2.7):



1. **Is f defined at $x = 1$?** Yes, its value is $f(1) = 1 + 1 = 2$ by definition.
2. **Are the one-sided limits equal?** Let's check this: (See Fig. 21)

$$\lim_{x \to 1^-} f(x) \quad = \quad \underbrace{\lim_{x \to 1^-}}_{\text{because } f(x)=x+1 \text{ for } x \le 1} (x+1) = 1 + 1 = 2$$

   Moreover,

$$\lim_{x \to 1^+} f(x) \quad = \quad \underbrace{\lim_{x \to 1^+}}_{\text{because } f(x)=2x \text{ for } x>1, \text{ and close to } 1} (2x) = 2 \cdot 1 = 2$$

   The one-sided limits are equal to each other and their common value is $L = 2$.
3. **Is $L = f(1)$?** By definition $f(1) = 1 + 1 = 2$, so OK, this is true, because $L = 2$.

Thus, by definition f is continuous at $x = 1$.

We proceed in the same fashion for $x = 2$. Remember, we still have to verify 3 conditions ...

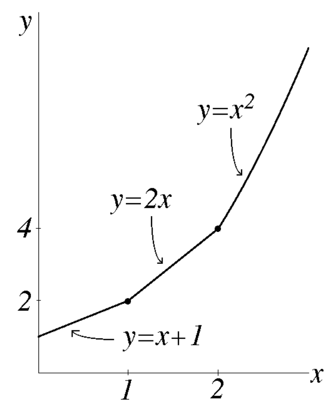


Figure 21.

1. **Is f defined at $x = 2$?** Yes, because its value is $f(2) = 2 \cdot 2 = 4$.
2. **Are the one-sided limits equal**? Let's see:

$$\lim_{x \to 2^+} f(x) \quad = \quad \underbrace{\lim_{x \to 2^+}}_{\text{because } f(x)=x^2 \text{ for } x>2} x^2 = 2^2 = 4$$

   and

$$\lim_{x \to 2^-} f(x) \quad = \quad \lim_{x \to 2^-} (2x) = 2 \cdot 2 = 4$$
$$\text{because } f(x)=2x \text{ for } x \le 2 \text{ and close to } 2$$

   So they are both equal and their common value, $L = 4$.
3. **Is $L = f(2)$?** We know that $L = 4$ and $f(2) = 2 \cdot 2 = 4$ by definition so, OK.

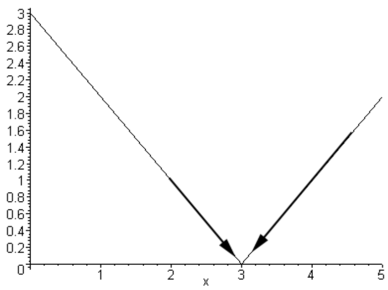
Thus, by definition (Table 2.7), f is continuous at $x = 2$.

Remarks:

1. The **existence of the limit of a function f at $x = a$ is equivalent to requiring that both one-sided limits be equal (to each other).**
2. The existence of the limit of a function f at $x = a$ doesn't imply that f is continuous at $x = a$.
   Why? Because the value of this limit may be different from $f(a)$, or, worse still, $f(a)$ may be infinite.
3. It follows from (1) that

$$\boxed{\text{If } \lim_{x \to a^+} f(x) \ne \lim_{x \to a^-} f(x), \text{ then } \lim_{x \to a} f(x) \text{ does not exist,}}$$

so, in particular, f cannot be continuous at $x = a$.



The function $f(x) = |x - 3|$

Figure 22.

Example 50. Show that the function f defined by $f(x) = |x-3|$ is continuous at $x = 3$ (see Figure 22).

Solution By definition of the absolute value we know that

$$f(x) = |x-3| = \begin{cases} x-3 & x \geq 3 \\ 3-x & x < 3 \end{cases}$$

(Remember: $|symbol| = symbol$ if $symbol \geq 0$ and $|symbol| = -symbol$ if $symbol < 0$ where '*symbol*' is any expression involving some variable...) OK, now

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x-3) = 3-3 = 0$$

and

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (3-x) = 3-3 = 0$$

so $\lim_{x\to 3} f(x)$ exists and is equal to 0 (by definition).

Is $0 = f(3)$? Yes (because $f(3) = |3-3| = |0| = 0$). Of course $f(3)$ is defined. We conclude that f is continuous at $x = 3$.

Remark: In practice it is easier to remember the statement:

$$\boxed{\begin{array}{c} \boldsymbol{f \text{ is continuous at } x = a \text{ if}} \\ \boldsymbol{\lim_{x\to a} f(x) = f(a)} \end{array}}$$

whenever all the 'symbols' here have meaning (*i.e.* the limit exists, $f(a)$ exists etc.).

The **Arabic numerals** or those 10 basic symbols we use today in the world of mathematics seem to have been accepted in Europe and subsequently in the West, sometime during the period 1482-1494, as can be evidenced from old merchant records from the era (also known as the *High Renaissance* in art). Prior to this, merchants and others used Roman numerals (X=10, IX=9, III = 3, etc.) in their dealings.

Example 51. Determine whether or not the following functions have a limit at the indicated point.

a) $f(x) = x^2 + 1$ at $x = 0$

b) $f(x) = 1 + |x-1|$ at $x = 1$

c) $f(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$ at $t = 0$

d) $f(x) = \dfrac{1}{x}$ at $x = 0$

e) $g(t) = \dfrac{t}{t+1}$ at $t = 0$

Solution **a)**

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} (x^2+1) = 0^2 + 1 = 1 \\ \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} (x^2+1) = 0^2 + 1 = 1 \end{aligned}$$

Thus $\lim_{x\to 0} f(x)$ exists and is equal to 1. *i.e.* $\lim_{x\to 0} f(x) = 1$.

b)
$$\begin{aligned}
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (1+|x-1|) \\
&= \lim_{x\to 1^+} (1+(x-1)) \quad \text{(because } |x-1| = x-1 \text{ as } x>1) \\
&= \lim_{x\to 1^+} x \\
&= 1 \\
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (1+|x-1|) \\
&= \lim_{x\to 1^-} (1+(1-x) \quad \text{(because } |x-1| = 1-x \text{ as } x<1) \\
&= \lim_{x\to 1^-} (2-x) \\
&= 2-1 \\
&= 1
\end{aligned}$$

Thus $\lim_{x\to 1} f(x)$ exists and is equal to 1, *i.e.* $\lim_{x\to 1} f(x) = 1$.

c)
$$\begin{aligned}
\lim_{t\to 0^+} f(t) &= \lim_{t\to 0^+} (1) \quad \text{(as } f(t)=1 \text{ for } t>0) \\
&= 1 \\
\lim_{t\to 0^-} f(t) &= \lim_{t\to 0^-} (0) \quad \text{(as } f(t)=0 \text{ for } t<0) \\
&= 0
\end{aligned}$$

Since $\lim_{t\to 0^+} f(t) \neq \lim_{t\to 0^-} f(t)$, it follows that $\lim_{t\to 0} f(t)$ does not exist. (In particular, f cannot be continuous at $t=0$.)

d)
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = +\infty$$

because "division by zero" does not produce a real number, in general. On the other hand

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty \text{ (since } x<0)$$

Since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$ the limit does not exist at $x=0$.

The **Sandwich Theorem** states that, if

$$g(x) \le f(x) \le h(x)$$

for all (sufficiently) large x and for some (extended) real number A, and

$$\lim_{x\to a} g(x) = A, \qquad \lim_{x\to a} h(x) = A$$

then f also has a limit at $x=a$ and

$$\lim_{x\to a} f(x) = A$$

In other words, f is "sandwiched" between two values that are ultimately the same and so f must also have the same limit.

e)
$$\begin{aligned}
\lim_{t\to 0^+} g(t) &= \lim_{t\to 0^+} \frac{t}{t+1} = \frac{0}{0+1} = 0 \\
\lim_{t\to 0^-} g(t) &= \lim_{t\to 0^-} \frac{t}{t+1} = \frac{0}{0+1} = 0
\end{aligned}$$

and so $\lim_{t\to 0} g(t)$ exists and is equal to 0, *i.e.* $\lim_{t\to 0} g(t) = 0$.

The rigorous method of handling these examples is presented in an optional Chapter, **Advanced Topics**. Use of your calculator will be helpful in determining some limits but cannot substitute a theoretical proof. The reader is encouraged to consult the *Advanced Topics* for more details.

Remark:

It follows from Table 2.4 that continuous functions themselves have similar properties, being based upon the notion of limits. For example it is true that:

1. The sum or difference of two continuous functions (at $x=a$) is again continuous (at $x=a$).

2. The product or quotient of two continuous functions (at $x = a$) is also continuous (provided the quotient has a non-zero denominator at $x = a$).
3. A multiple of two continuous functions (at $x = a$) is again a continuous function (at $x = a$).

Properties of Limits of Functions

Let f, g be two given functions, $x = a$ be some (finite) point. The following statements hold (but will not be proved here):

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and are **finite**.

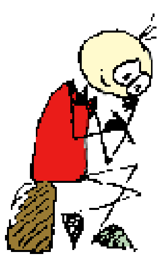
Then

a) The limit of a sum is the sum of the limits.
$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

b) The limit of a difference is the difference of the limits.
$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

c) The limit of a multiple is the multiple of the limit.
If c is any number, then $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$

d) The limit of a quotient is the quotient of the limits.
If $\lim_{x \to a} g(x) \neq 0$ then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$



e) The limit of a product is the product of the limits.
$\lim_{x \to a} f(x)g(x) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$

f) If $f(x) \leq g(x)$ then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

Table 2.4: Properties of Limits of Functions

Now, the Properties in Table 2.4 and the following Remark allow us to make some very important observations about some classes of functions, such as polynomials.

How? Well, let's take the simplest polynomial $f(x) = x$. It is easy to see that for some given number $x = a$ and setting $g(x) = x$, Table 2.4 Property (e), implies that the function $h(x) = f(x)g(x) = x \cdot x = x^2$ is also continuous at $x = a$. In the same way we can show that $k(x) = f(x)h(x) = x \cdot x^2 = x^3$ is also continuous at $x = a$, and so on.

In this way we can prove (using, in addition, Property (a)), that **any polynomial whatsoever is continuous at $x = a$, where a is any real number**. We summarize this and other such consequences in Table 2.8.

NOTES:

SUMMARY: One-Sided Limits from the Right

We say that the function ***f* has a limit from the right at $x = a$** (or the right-hand limit of f exists at $x = a$) whose value is L and denote this symbolically by

$$\lim_{x \to a^+} f(x) = L$$

if BOTH the following statements are satisfied:

1. Let $x > a$ and x be very close to $x = a$.
2. As x approaches a ("from the right" because "$x > a$"), the values of $f(x)$ approach the value L.

(For a more rigorous definition see the **Advanced Topics**)

Table 2.5: SUMMARY: One-Sided Limits From the Right

SUMMARY: One-Sided Limits from the Left

We say that the function ***f* has a limit from the left at $x = a$** (or the left-hand limit of f exists at $x = a$) and is equal to L and denote this symbolically by

$$\lim_{x \to a^-} f(x) = L$$

if BOTH the following statements are satisfied:

1. Let $x < a$ and x be very close to $x = a$.
2. As x approaches a ("from the left" because "$x < a$"), the values of $f(x)$ approach the value L.

Table 2.6: SUMMARY: One-Sided Limits From the Left

Exercise Set 5.

Determine whether the following limits exist. Give reasons.

Nicola Oresme, (1323-1382), Bishop of Lisieux, in Normandy, wrote a tract in 1360 (this is before the printing press) where, for the first time, we find the introduction of perpendicular xy-axes drawn on a plane. His work is likely to have influenced **René Descartes** (1596-1650), the founder of modern *Analytic Geometry*.

1. $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} x+2 & x \le 0 \\ x & x > 0 \end{cases}$
2. $\lim_{x \to 1} (x+3)$
3. $\lim_{x \to -2} \left(\dfrac{x+2}{x} \right)$
4. $\lim_{x \to 0} x \sin x$
5. $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} \sin(x-1) & 0 \le x \le 1 \\ |x-1| & x > 1 \end{cases}$
6. $\lim_{x \to 0} \left(\dfrac{x+1}{x} \right)$
7. $\lim_{x \to 0} \left(\dfrac{2}{x} \right)$

SUMMARY: Continuity of f at $x = a$.
We say that f is **continuous** at $x = a$ if all the following conditions are satisfied: 1. f is defined at $x = a$ (i.e., $f(a)$ is finite) 2. $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ ($= L$, their common value) and 3. $L = f(a)$. These three conditions must be satisfied in order for a function f to be continuous at a given point $x = a$. If any one or more of these conditions is not satisfied we say that f is **discontinuous** at $x = a$.

Table 2.7: SUMMARY: Continuity of a Function f at a Point $x = a$

8. $\lim_{x\to 1} \left(\dfrac{x}{x+1}\right)$

9. $\lim_{x\to 2} (2 + |x - 2|)$

10. $\lim_{x\to 0} f(x) \quad$ where $f(x) = \begin{cases} 3 & x \le 0 \\ 2 & x > 0 \end{cases}$

11. Are the following functions continuous at 0? Give reasons.
    a) $f(x) = |x|$
    b) $g(t) = t^2 + 3t + 2$
    c) $h(x) = 3 + 2|x|$
    d) $f(x) = \dfrac{2}{x+1}$
    e) $f(x) = \dfrac{x^2+1}{x^2-2}$

12. **Hard** Let f be defined by

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \ne 0 \\ 0 & x = 0 \end{cases}$$

Show that f is continuous at $x = 0$.

(*Hint*: Do this in the following steps:

a) Show that for $x \ne 0$, $|x\sin\left(\frac{1}{x}\right)| \le |x|$.

b) Use (a) and the Sandwich Theorem to show that

$$0 \le \lim_{x\to 0} \left| x\sin\left(\frac{1}{x}\right)\right| \le 0$$

and so

$$\lim_{x\to 0} \left| x\sin\left(\frac{1}{x}\right)\right| = 0$$

c) Conclude that $\lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = 0$.

d) Verify the other conditions of continuity.)

Some Continuous Functions

Let $x = a$ be a given point.

a) The polynomial p of degree n, with fixed coefficients, given by

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

is continuous at any real number $x = a$.

b) The rational function, r, where $r(x) = \frac{p(x)}{q(x)}$ where p, q are both polynomials is continuous at $x = a$ provided $q(a) \neq 0$ or equivalently, provided $x = a$ is not a root of $q(x)$. Thus

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0}$$

is continuous at $x = a$ provided the denominator is not equal to zero at $x = a$.

c) If f is a continuous function, so is its **absolute value** function, $|f|$, and if

$$\lim_{x \to a} |f(x)| = 0, \quad \text{then}$$

$$\lim_{x \to a} f(x) = 0$$

(The proof of (c) uses the ideas in the Advanced Topics chapter.)

Table 2.8: Some Continuous Functions

What about discontinuous functions?

In order to show that a function is discontinuous somewhere we need to show that at least one of the three conditions in the definition of continuity (Table 2.7) is not satisfied.

Remember, to show that f is continuous requires the verification of all three conditions in Table 2.7 whereas to show some function is discontinuous only requires that **one** of the three conditions for continuity is not satisfied.

Example 52. Determine which of the following functions are discontinuous somewhere. Give reasons.

a) $f(x) = \begin{cases} x & x \leq 0 \\ 3x + 1 & x > 0 \end{cases}$

b) $f(x) = \dfrac{x}{|x|}, \; f(0) = 1$

c) $f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$

d) $f(x) = \dfrac{1}{|x|}, \; x \neq 0$

Solution **a)**

$$\text{Note that } \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x) = 0$$
$$\text{while } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3x+1) = 1$$

Thus the $\lim_{x\to 0} f(x)$ does not exist and so f cannot be continuous at $x = 0$, or, equivalently, f is discontinuous at $x = 0$.

How can a function f be discontinuous at $x = a$? If any one (or more) of the following occurs . . .

1. $f(a)$ is not defined (*e.g.*, it is infinite, or we are dividing by 0, or extracting the root of a negative number, . . .) (See Example 52 (d))
2. If either one of the left- and right-limits of f at $x = a$ is infinite (See Example 52 (d))
3. $f(a)$ is defined but the left- and right-limits at $x = a$ are *unequal* (See Example 52 (a), (b))
4. $f(a)$ is defined, the left- and right-limits are *equal* to L but $L = \pm\infty$
5. $f(a)$ is defined, the left- and right-limits are *equal* to L but $L \neq f(a)$ (See Example 52 (c))

Then f is discontinuous at $x = a$.

What about the other points, $x \neq 0$?

Well, if $x \neq 0$, and $x < 0$, then $f(x) = x$ is a polynomial, right? Thus f is continuous at each point x where $x < 0$. On the other hand, if $x \neq 0$ and $x > 0$ then $f(x) = 3x + 1$ is also a polynomial. Once again f is continuous at each point x where $x > 0$.

Conclusion: f is continuous at every point x except at $x = 0$.

b) Since $f(0) = 1$ is defined, let's check for the existence of the limit at $x = 0$. (You've noticed, of course, that at $x = 0$ the function is of the form $\frac{0}{0}$ **which is not defined as a real number** and this is why an additional condition was added there to make the function defined for **all** x and not just those $x \neq 0$.)

Now,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{|x|} = \underbrace{\lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} (1)}_{(\text{because } x\neq 0)}$$
$$= 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{(-x)} = \lim_{x\to 0^-} (-1)$$
$$= -1$$

(since $|x| = -x$ if $x < 0$ by definition). Since the one-sided limits are different it follows that

$$\lim_{x\to 0} f(x) \text{ does not exist.}$$

Thus f is discontinuous at $x = 0$.

What about the other points? Well, for $x \neq 0$, f is nice enough. For instance, if $x > 0$ then

$$f(x) = \frac{x}{|x|} = \frac{x}{x} = +1$$

for each such $x > 0$. Since f is a constant it follows that f is continuous for $x > 0$. On the other hand, if $x < 0$, then $|x| = -x$ so that

$$f(x) = \frac{x}{|x|} = \frac{x}{(-x)} = -1$$

and once again f is continuous for such $x < 0$.

Conclusion: f is continuous for each $x \neq 0$ and at $x = 0$, f is discontinuous.

The graph of this function is shown in Figure 23.

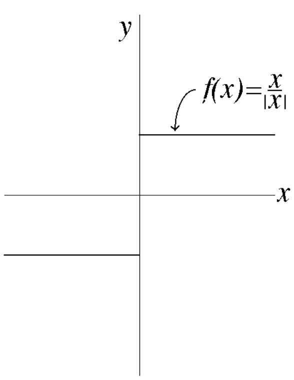


Figure 23.

c) Let's look at f for $x \neq 0$ first, (it doesn't really matter how we start). For $x \neq 0$, $f(x) = x^2$ is a polynomial and so f is continuous for each such $x \neq 0$, (Table 2.8).

What about $x = 0$? We are given that $f(0) = 1$ so f is defined there. What about the limit of f as x approaches $x = 0$. Does this limit exist?

Let's see

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0^2 = 0$$

and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0^2 = 0$$

OK, so $\lim_{x\to 0} f(x)$ exists and is equal to 0. But note that

$$\lim_{x\to 0} f(x) = 0 \neq f(0) = 1$$

So, in this case, f is discontinuous at $x = 0$, (because even though conditions (1) and (2) of Table 2.7 are satisfied the final condition (3) is not!)

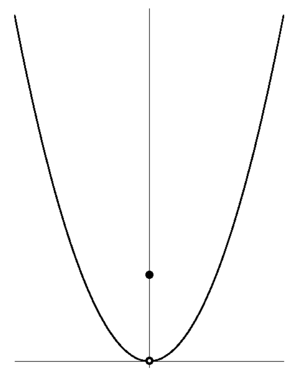
d) In this case we see that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = +\infty$$

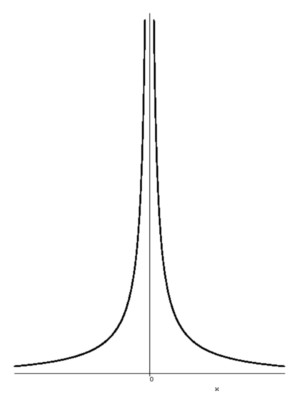
and $f(0) = +\infty$ as well! Ah, but now $f(0)$ is not defined as a real number (thus violating condition (1)). Thus f is discontinuous at $x = 0 \ldots$ and the other points? Well, for $x < 0$, $f(x) = -\frac{1}{x}$ is a quotient of two polynomials and any x (since $x \neq 0$) is not a root of the denominator. Thus f is a continuous function for such $x < 0$. A similar argument applies if $x > 0$.

Conclusion: f is continuous everywhere except at $x = 0$ where it is discontinuous.

We show the graph of the functions defined in (c), (d) in Figure 24.



The graph of Example 52(c)



The graph of Example 52(d)

Figure 24.

Exercise Set 6.

Determine the points of discontinuity of each of the following functions.

1. $f(x) = \dfrac{|x|}{x} + 1$ for $x \neq 0$ and $f(0) = 2$
2. $g(x) = \begin{cases} x & x < 0 \\ 1 + x^2 & x \geq 0 \end{cases}$
3. $f(x) = \dfrac{x^2 + 3x + 3}{x^2 - 1}$
   (*Hint*: Find the zeros of the denominator.)
4. $f(x) = \begin{cases} x^3 + 1 & x \neq 0 \\ 2 & x = 0 \end{cases}$
5. $f(x) = \dfrac{1}{x} + \dfrac{1}{x^2}$ for $x \neq 0$, $f(0) = +1$
6. $f(x) = \begin{cases} 1.62 & x < 0 \\ 2x & x \geq 0 \end{cases}$

Before proceeding with a study of some trigonometric limits let's recall some fundamental notions about trigonometry. Recall that the measure of angle called the **radian** is equal to $\frac{360^o}{2\pi} \approx 57^o$. It is also that angle whose arc is numerically equal to the radius of the given circle. (So 2π radians correspond to 360^o, π radians correspond to 180^o, 1 radian corresponds to $\approx 57^o$, etc.) Now, to find the area of a

Continuity of various trigonometric functions

(**Recall**: Angles x are in **radians**)

1. The functions f, g defined by $f(x) = \sin x$, $g(x) = \cos x$ are continuous everywhere (i.e., for each real number x).
2. The functions $h(x) = \tan x$ and $k(x) = \sec x$ are continuous at every point which is not an odd multiple of $\frac{\pi}{2}$. At such points h, k are discontinuous. (*i.e.* at $-\frac{\pi}{2}$, $\frac{3\pi}{2}$, $-\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\ldots$)
3. The functions 'csc' and 'cot' are continuous whenever x is not a multiple of π, and discontinuous whenever x is a multiple of π. (*i.e.* at $x = \pi$, $-\pi$, 2π, -5π, etc.)

Table 2.9: Continuity of Various Trigonometric Functions

sector of a circle of radius r subtending an angle θ at the center we note that the area is proportional to this central angle so that

$$\begin{aligned} \frac{2\pi}{\text{Area of circle}} &= \frac{\theta}{\text{Area of sector}} \\ \text{Area of sector} &= \frac{\theta}{2\pi}(\pi r^2) \\ &= \frac{r^2\theta}{2} \end{aligned}$$

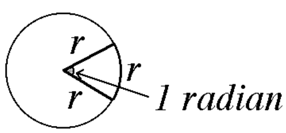


Figure 25.

We conclude that the area of a sector subtending an angle θ at the center is given by $\frac{r^2\theta}{2}$ where θ is in radians and summarize this in Table 2.10.

The area of a sector subtending an angle θ (in radians) at the center of a circle of radius r is given by

$$\text{Area of a sector} = \frac{r^2\theta}{2}$$

Table 2.10: Area of a Sector of a Circle

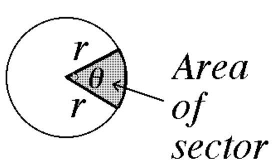


Figure 26.

Next we **find some relationships between triangles in order to deduce a very important limit in the study of calculus**.

We begin with a circle $\mathcal{C}$ of radius 1 and a sector subtending an angle, $x < \frac{\pi}{2}$ in radians at its center, labelled O. Label the extremities of the sector along the arc by A and B as in the adjoining figure, Fig. 27.

At A produce an altitude which meets OB extended to C. Join AB by a line segment. The figure now looks like Figure 28.

We call the 'triangle' whose side is the arc AB and having sides AO, OB a "**curvilinear triangle**" for brevity. (It is also a "sector"!)

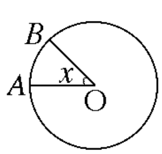


Figure 27.

Let's compare areas. Note that

$$\begin{aligned}
\text{Area of } \triangle ACO &> \text{Area of curvilinear triangle}\\
&> \text{Area of } \triangle ABO\\
\text{Now the area of } \triangle ACO &= \frac{1}{2}(1)|AC| = \frac{1}{2}\tan x\\
\text{Area of curvilinear triangle} &= \text{Area of the sector with central angle } x\\
&= \frac{1}{2}(1^2)\cdot x \text{ (because of Table 2.10 above)}\\
&= \frac{x}{2}
\end{aligned}$$

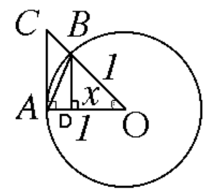


Figure 28.

Finally, from Figure 28,

$$\begin{aligned}
\text{Area of triangle } ABO &= \frac{1}{2}(\text{altitude from base } AO)(\text{base length})\\
&= \frac{1}{2}(\text{length of BD})\cdot(1) = \frac{1}{2}(\sin x)\cdot(1)\\
&= \frac{\sin x}{2}
\end{aligned}$$

(by definition of the sine of the angle x.) Combining these inequalities we get

$$\frac{1}{2}\tan x > \frac{x}{2} > \frac{\sin x}{2} \text{ (for } 0 < x < \frac{\pi}{2}, \text{ remember?)}$$

or

$$\sin x < x < \tan x$$

from which we can derive

$$\boxed{\cos x < \frac{\sin x}{x} < 1 \quad \text{for } 0 < x \le \frac{\pi}{2}}$$

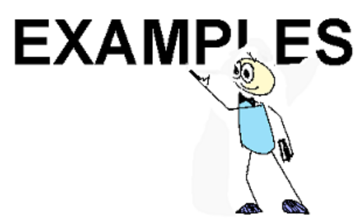
since all quantities are positive. This is a fundamental inequality in trigonometry.

We now apply Table 2.4(f) to this inequality to show that

$$\underbrace{\lim_{x\to 0^+} \cos x}_{1} \le \lim_{x\to 0^+} \frac{\sin x}{x} \le \underbrace{\lim_{x\to 0^+} 1}_{1}$$

and we conclude that

$$\boxed{\lim_{x\to 0^+} \frac{\sin x}{x} = 1}$$



If, on the other hand, $-\frac{\pi}{2} < x < 0$ (or x is a negative angle) then, writing $x = -x_0$, we have $\frac{\pi}{2} > x_0 > 0$. Next

$$\frac{\sin x}{x} = \frac{\sin(-x_0)}{-x_0} = \frac{-\sin(x_0)}{-x_0} = \frac{\sin x_0}{x_0}$$

where we have used the relation $\sin(-x_0) = -\sin x_0$ valid for any angle x_0 (in radians, as usual). Hence

$$\begin{aligned}
\lim_{x\to 0^-}\left(\frac{\sin x}{x}\right) &= \lim_{x\to 0^-}\frac{\sin x_0}{x_0} \\
&= \lim_{-x_0\to 0^-}\frac{\sin x_0}{x_0} \\
&= \lim_{x_0\to 0^+}\frac{\sin x_0}{x_0} \quad \text{(because if } -x_0 < 0 \text{ then } x_0 > 0 \\
&\qquad\qquad\qquad \text{and } x_0 \text{ approaches } 0^+) \\
&= 1.
\end{aligned}$$

Since both one-sided limits are equal it follows that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Another important limit like the one in Table 2.11 is obtained by using the basic

If the symbol $\square$ represents any continuous function then, so long as we can let $\square \to 0$, we have

$$\lim_{\square\to 0}\frac{\sin\square}{\square} = 1$$

Table 2.11: Limit of $(\sin\square)/\square$ as $\square \to 0$

identity

$$1-\cos\theta = \frac{1-\cos^2\theta}{1+\cos\theta} = \frac{\sin^2\theta}{1+\cos\theta}$$

Dividing both sides by θ and rearranging terms we find

$$\frac{1-\cos\theta}{\theta} = \frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}$$

Now, we know that

$$\lim_{\square\to 0}\frac{1-\cos\square}{\square} = 0.$$

Table 2.12: Limit of $(1-\cos\square)/\square$ as $\square \to 0$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

and

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

exists (because it is the limit of the quotient of 2 continuous functions, the denominator not being 0 as $\theta \to 0$). Furthermore it is easy to see that

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \frac{\sin 0}{1+\cos 0} = \frac{0}{1+1} = 0$$

It now follows from Table 2.4(e) that

$$\begin{aligned}\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\right) = 1\cdot 0\\ &= 0\end{aligned}$$

and we conclude that

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$$

If you want, you can replace 'θ' by 'x' in the above formula or any other '*symbol*' as in Table 2.11. Hence, we obtain Table 2.12,

NOTES:

Example 53. Evaluate the following limits.

a) $\lim_{x\to 0} \frac{\sin(3x)}{x}$

b) $\lim_{x\to 0} \frac{1-\cos(2x)}{x}$

c) $\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$

d) $\lim_{x\to 0^+} \frac{\sin(\sqrt{x})}{\sqrt{x}}$

One of the first complete introductions to *Trigonometry* was written by one **Johannes Müller of Königsberg**, (1436 - 1476), also known as *Regiomontanus*. The work, written in Latin, is entitled *De Triangulis omnimodus* first appeared in 1464.

Solution **a)** We use Table 2.11. If we let $\Box = 3x$, we also need the symbol $\Box$ in the denominator, right? In other words, $x = \frac{\Box}{3}$ and so

$$\frac{\sin 3x}{x} = \frac{\sin(\Box)}{(\frac{\Box}{3})} = 3 \cdot \frac{\sin(\Box)}{\Box}$$

Now, as $x \to 0$ it is clear that, since $\Box = 3x$, $\Box \to 0$ as well. Thus

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(3x)}{x} &= \lim_{\Box\to 0} 3 \cdot \frac{\sin(\Box)}{\Box} \\ &= 3 \lim_{\Box\to 0} \frac{\sin(\Box)}{\Box} \quad \text{(by Table 2.4(c))} \\ &= 3 \cdot 1 \quad \text{(by Table 2.11)} \\ &= 3 \end{aligned}$$

b) We use Table 2.12 because of the form of the problem for $\Box = 2x$ then $x = \frac{\Box}{2}$. So

$$\frac{1-\cos(2x)}{x} = \frac{1-\cos\Box}{\frac{\Box}{2}} = 2 \cdot \frac{1-\cos\Box}{\Box}$$

As $x \to 0$, we see that $\Box \to 0$ too! So

$$\begin{aligned} \lim_{x\to 0} \frac{1-\cos(2x)}{x} &= \lim_{\Box\to 0} (2 \cdot \frac{1-\cos(\Box)}{\Box}) \\ &= 2 \lim_{\Box\to 0} (\frac{1-\cos(\Box)}{\Box}) = 2 \cdot 0 = 0 \end{aligned}$$

c) This type of problem is not familiar at this point and all we have is Table 2.11 as reference ... The idea is to rewrite the quotient as something that is *more familiar*. For instance, using plain algebra, we see that

$$\frac{\sin 2x}{\sin 3x} = (\frac{\sin 2x}{2x})(\frac{2x}{3x})(\frac{3x}{\sin 3x}),$$

so that some of the $2x$'s and $3x$-cross-terms **cancel out** leaving us with the original expression.

OK, now as $x \to 0$ it is clear that $2x \to 0$ and $3x \to 0$ too! So,

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = (\lim_{x\to 0} \frac{\sin 2x}{2x})(\lim_{x\to 0} \frac{2x}{3x})(\lim_{3x\to 0} \frac{3x}{\sin 3x})$$

(because "the limit of a product is the product of the limits" cf., Table 2.4.) Therefore

$$\begin{aligned} \lim_{x\to 0} \frac{\sin 2x}{\sin 3x} &= (\lim_{x\to 0} \frac{\sin 2x}{2x}) \lim_{x\to 0} \frac{2x}{3x} (\lim_{3x\to 0} \frac{3x}{\sin 3x}) \\ &= 1 \cdot \frac{2}{3} \cdot 1 = \frac{2}{3} \end{aligned}$$

(by Tables 2.11& 2.4). Note that the middle-term, $\frac{2x}{3x} = \frac{2}{3}$ since $x \neq 0$.

Using the 'Box' method we can rewite this argument more briefly as follows: We have **two** symbols, namely '$2x$' and '$3x$', so if we are going to use Table 2.11 we need to introduce these symbols into the expression as follows: (**Remember**, $\Box$ and $\triangle$ are just '*symbols*'...).

Let $\Box = 2x$ and $\triangle = 3x$. Then

$$\frac{\sin 2x}{\sin 3x} = \frac{\sin \Box}{\sin \triangle} = (\frac{\sin \Box}{\Box})(\frac{\Box}{\triangle})(\frac{\triangle}{\sin \triangle})$$

So that $\Box$'s and $\triangle$'s **cancel out** leaving the original expression.

OK, now as $x \to 0$ it is clear that $\Box \to 0$ and $\triangle \to 0$ too! So

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = (\lim_{\Box\to 0} \frac{\sin \Box}{\Box})(\lim_{x\to 0} \frac{\Box}{\triangle})(\lim_{\triangle\to 0} \frac{\triangle}{\sin \triangle})$$

(because "the limit of a product is the product of the limits" , cf., Table 2.4.) Therefore

$$\begin{aligned} \lim_{x\to 0} \frac{\sin 2x}{\sin 3x} &= (\lim_{\Box\to 0} \frac{\sin \Box}{\Box}) \lim_{x\to 0} \frac{2x}{3x} (\lim_{\triangle\to 0} \frac{\triangle}{\sin \triangle}) \\ &= 1 \cdot \frac{2}{3} \cdot 1 = \frac{2}{3} \end{aligned}$$

(by Tables 2.11& 2.4).

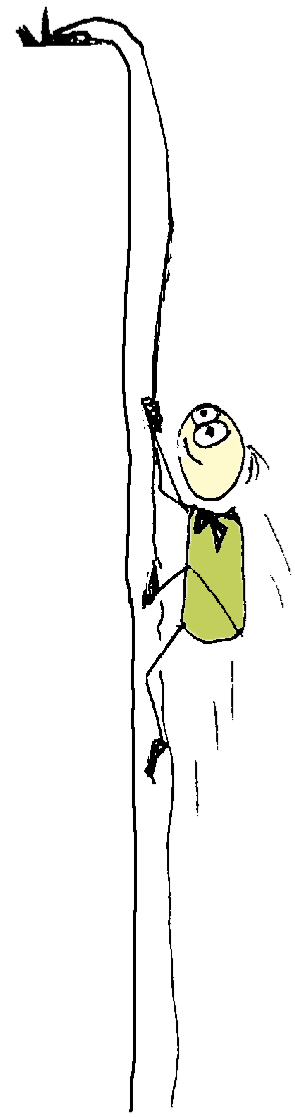
d) In this problem we let $\Box = \sqrt{x}$. As $x \to 0^+$ we know that $\sqrt{x} \to 0^+$ as well. Thus

$$\lim_{x\to 0^+} \frac{\sin \sqrt{x}}{\sqrt{x}} = \lim_{\Box\to 0^+} \frac{\sin \Box}{\Box} = 1$$

(by Table 2.11).

Philosophy:

Learning mathematics has a lot to do with learning the rules of the interaction between symbols, some recognizable (such as 1, 2, $\sin x$, ...) and others not (such as $\Box$, $\triangle$, etc.) Ultimately these are all 'symbols' and we need to recall **how** they interact with one another.
Sometimes it is helpful to replace the commonly used symbols 'y', 'z' , etc. for variables, by other, not so commonly used ones, like $\Box$, $\triangle$ or '*squiggle*' etc. It doesn't matter **how** we denote something, what's important is **how it interacts** with other symbols.



NOTES:

Limit questions can be approached in the following way.

You want to find

$$\lim_{x\to a} f(x).$$

Option 1 **Take the value to which x tends, *i.e.* $x = a$, and evaluate the expression (function) at that value, *i.e.* $f(a)$.**

Three possibilities arise:

a) You obtain a number like $\frac{B}{A}$, with $A \neq 0$ and the question is answered (if the function is continuous at $x = a$), the answer being $\frac{B}{A}$.

b) You get $\frac{B}{0}$, with $B \neq 0$ which implies that the limit exists and is plus infinity $(+\infty)$ if $B > 0$ and minus infinity $(-\infty)$ if $B < 0$.

c) You obtain something like $\frac{0}{0}$ which means that the limit being sought may be "in disguise" and we need to move onto Option 2 below.

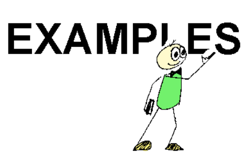
Option 2 If the limit is of the form $\frac{0}{0}$ proceed as follows:
We need to **play around** with the expression, that is you may have to factor some terms, use trigonometric identities, substitutions, simplify, rationalize the denominator, multiply and divide by the same symbol, etc. until you can return to Option 1 and repeat the procedure there.

Option 3 If 1 and 2 fail, then check the left and right limits.

a) If they are equal, the limit exists and go to Option 1.

b) If they are unequal, the limit does not exist. Stop here, that's your answer.

Table 2.13: Three Options to Solving Limit Questions



Exercise Set 7.

Determine the following limits if they exist. Explain.

1. $\displaystyle\lim_{x\to\pi} \frac{\sin x}{x-\pi}$
(*Hint:* Write $\Box = x - \pi$. Note that $x = \Box + \pi$ and as $x \to \pi$, $\Box \to 0$.)

2. $\displaystyle\lim_{x\to\frac{\pi}{2}} (x - \frac{\pi}{2})\tan x$
(*Hint:* $(x - \dfrac{\pi}{2})\tan x = \dfrac{(x-\frac{\pi}{2})}{\cos x}\sin x$. Now set $\Box = x - \frac{\pi}{2}$,so that $x = \Box + \frac{\pi}{2}$ and note that, as $x \to \frac{\pi}{2}$, $\Box \to 0$.)

3. $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{2x}$

4. $\lim_{x \to 0} \dfrac{1 - \cos(3x)}{4x}$
5. $\lim_{x \to 0} \dfrac{\sin(4x)}{\sin(2x)}$
6. $\lim_{x \to 1^+} \dfrac{\sin\sqrt{x-1}}{\sqrt{x-1}}$

Web Links

For a very basic introduction to Limits see:
www.en.wikibooks.org/wiki/Calculus/Limits

Section 2.1: For one-sided limits and quizzes see:

www.math.montana.edu/frankw/ccp/calculus/estlimit/onesided/learn.htm

More about limits can be found at:

http://calculusapplets.com/ (neat applets)

http://www.math.psu.edu/dlittle/java/calculus/

The proofs of the results in Table 2.4 can be found at:

http://archives.math.utk.edu/visual.calculus/1/limits.18/index.html

Exercise Set 8.

Find the following limits whenever they exist. Explain.

1. $\lim_{x \to 2} \dfrac{x-2}{x}$
2. $\lim_{x \to 0^+} \sqrt{x}\cos x$
3. $\lim_{x \to 3} \left(\dfrac{x-3}{x^2-9}\right)$
4. $\lim_{x \to \frac{\pi}{2}} \left(x - \dfrac{\pi}{2}\right)\sec x$
5. $\lim_{x \to \frac{\pi}{2}} \left(\dfrac{2x-\pi}{\cos x}\right)$
6. $\lim_{x \to 2^-} \sin(\pi x)$
7. $\lim_{x \to 0} \dfrac{\sin(3x)}{x}$
8. $\lim_{x \to \pi^+} \left(\dfrac{\cos x}{x-\pi}\right)$
9. $\lim_{x \to \pi^-} \left(\dfrac{\cos x}{x-\pi}\right)$
10. $\lim_{x \to 0^+} x\,|\,x\,|$

Hints:
3) Factor the denominator (Table 2.13, Option 2).
4) Write $\square = x - \frac{\pi}{2}$, $x = \square + \frac{\pi}{2}$ and simplify (Table 2.13, Option 2).
5) Let $\square = x - \frac{\pi}{2}$, $x = \square + \frac{\pi}{2}$ and use a formula for the cosine of the sum of two angles.
9) See Table 2.13, Option 1(b).

Find the points of discontinuity, if any, of the following functions f:

11. $f(x) = \dfrac{\cos x}{x - \pi}$

12. $f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ -1 & x = 0 \end{cases}$

13. $f(x) = x^3 + x^2 - 1$

14. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

15. $f(x) = \dfrac{x - 2}{|\, x^2 - 4 \,|}$

Evaluate the following limits, whenever they exist. Explain.

16. $\displaystyle\lim_{x \to 0} \frac{\cos x - \cos 2x}{x^2}$

    (*Hint:* Use the trigonometric identity

    $$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

    along with Table 2.11.)

17. $\displaystyle\lim_{x \to 0} \frac{\tan x - \sin x}{x^2}$

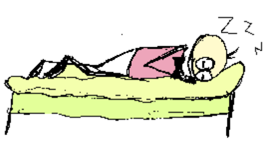
    (*Hint:* Factor the term '$\tan x$' out of the numerator and use Tables 2.11 & 2.12.)

18. $\displaystyle\lim_{x \to 1} \frac{x^2 + 1}{(x-1)^2}$

19. Find values of a and b such that

    $$\lim_{x \to \pi} \frac{ax + b}{2 \sin x} = \frac{\pi}{4}$$

    (*Hint:* It is necessary that $a\pi + b = 0$, why? Next, use the idea of Exercise Set 7 #1.)



20. $\displaystyle\lim_{x \to 0^+} \frac{1}{\sqrt{x+1} - \sqrt{x}}$

NOTES:

2.3 Important Theorems About Continuous Functions

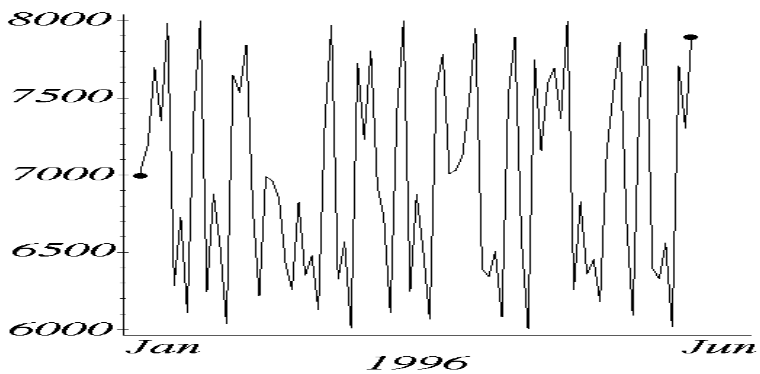
There are two main results (one being a consequence of the other) in the basic study of continuous functions. These are based on the property that the graph of a continuous function on a given interval has no 'breaks' in it.

Basically one can think of such a graph as a string which joins 2 points, say $(a, f(a))$ to $(b, f(b))$ (see Figure 29).

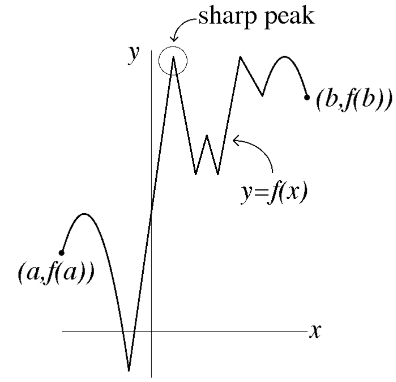
In Figure 29(a), the graph may have "sharp peaks" and may also look "smooth" and still be the graph of a continuous function (as is Figure 29(b)).

The **Intermediate Value Theorem** basically says that if you are climbing a mountain and you stop at 1000 meters and you want to reach 5000 meters, then at some future time you will pass, say the 3751 meter mark! This is obvious, isn't it? But this basic observation allows you to understand this deep result about continuous functions.

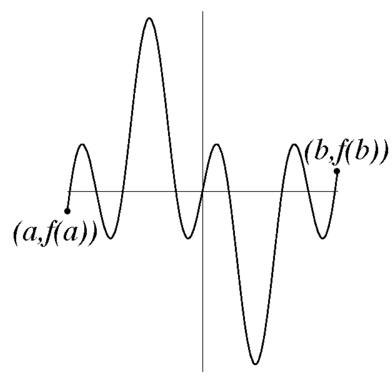
For instance, the following graph may represent the fluctuations of your local Stock Exchange over a period of 1 year.



Assume that the index was 7000 points on Jan. 1, 1996 and that on June 30 it was 7900 points. Then sometime during the year the index passed the 7513 point mark at least once . . .



(a)



(b)

Figure 29.

OK, so what does this theorem say mathematically?

Intermediate Value Theorem (IVT)

Let f be continuous at each point of a closed interval $[a, b]$. Assume
1. $f(a) \neq f(b)$;
2. Let z be a point between $f(a)$ and $f(b)$.
Then there is at least one value of c between a and b such that

$$f(c) = z$$

The idea behind this Theorem is that **any** horizontal line that intersects the graph of a continuous function must intersect it at a point of its domain! This sounds and looks obvious (see Figure 30), but it's NOT true if the graph is NOT that of a continuous function (see Example 31). One of the most important **consequences** of this Intermediate Value Theorem (IVT) is sometimes called **Bolzano's Theorem** (after *Bernhard Bolzano (1781-1848)* mathematician, priest and philosopher).

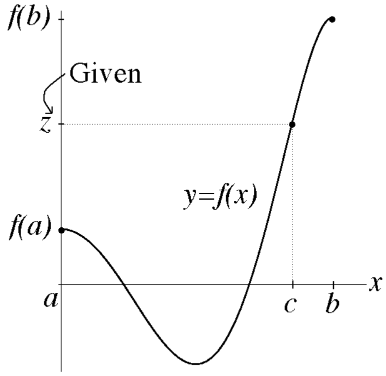


Figure 30.

Theorem 2.3.1. *(Bolzano's Theorem)*

Let f be **continuous on a closed interval** *$[a, b]$ (i.e., at each point x in $[a, b]$). If $f(a)f(b) < 0$, then there is at least one point c between a and b such that $f(c) = 0$.*

Bolzano's Theorem is especially useful in determining the **location of roots** of polynomials or general (continuous) functions. Better still, it is also helpful in determining **where the graphs of functions intersect** each other.

For example, at which point(s) do the graphs of the functions given by $y = \sin x$ and $y = x^2$ intersect? In order to find this out you need to equate their values, so that $\sin x = x^2$ which then means that $x^2 - \sin x = 0$ so the points of intersection are roots of the function whose values are given by $y = x^2 - \sin x$.

Example 54. Show that there is one root of the polynomial $p(x) = x^3 + 2$ in the interval $-2 \le x \le -1$.

Solution We note that $p(-2) = -6$ and $p(-1) = 1$. So let $a = -2$, $b = -1$ in Bolzano's Theorem. Since $p(-2) < 0$ and $p(-1) > 0$ it follows that $p(x_0) = 0$ for some x_0 in $[-2, -1]$ which is what we needed to show.

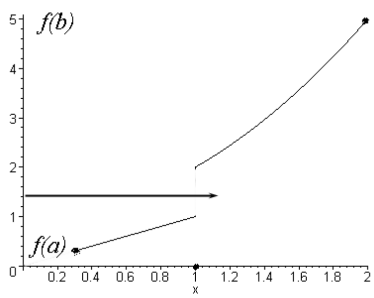


Figure 31.

Remark:

If you're not given the interval where the root of the function may be you need to find it! Basically **you look for points a and b where $f(a) < 0$ and $f(b) > 0$** and then you can refine your estimate of the root by "narrowing down" your interval.

Example 55. The distance between 2 cities A and B is 270 km. You're driving along the superhighway between A and B with speed limit 100 km/h hoping to get to your destination as soon as possible. You quickly realize that after one and one-half hours of driving you've travelled 200 km so you decide to stop at a rest area to relax. All of a sudden a police car pulls up to yours and the officer hands you a speeding ticket! Why?

Solution Well, the officer didn't actually **see** you speeding but saw you leaving A. Had you been travelling at the speed limit of 100 km/h it should have taken you 2 hours to get to the rest area. The officer quickly realized that somewhere along the highway you must have travelled at speeds of around 133 km/hr $\left(= (\dfrac{200 \text{ km}}{90\text{min}} \times 60\text{min} \right)$. As a check notice that if you were travelling at a constant speed of say 130 km/h then you would have travelled a distance of only $130 \times 1.5 = 195$ km short of your mark.

A typical graph of your journey appears in Figure 32. Note that your "speed" must be related to the amount of "steepness" of the graph. The faster you go, the

"steeper" the graph. This motivates the notion of a **derivative** which you'll see in the next chapter.

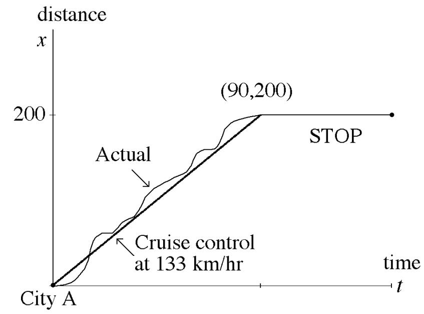


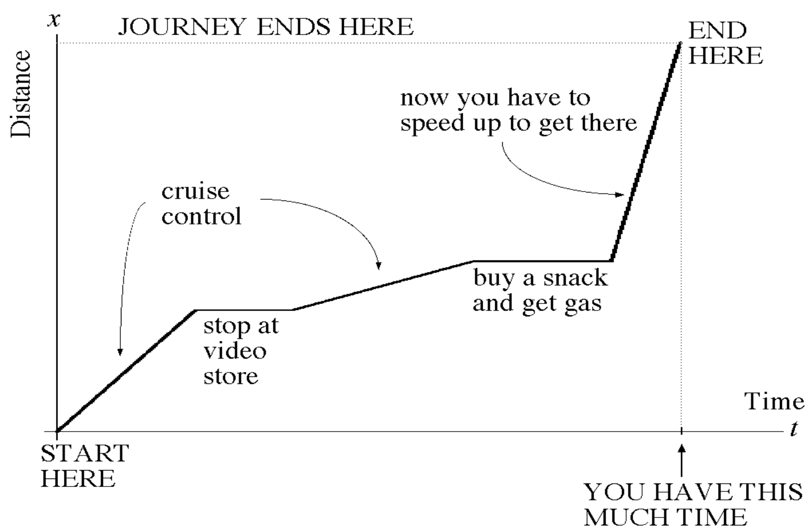
Figure 32.

Philosophy

Actually one uses a form of the Intermediate Value Theorem almost daily. For instance, do you find yourself asking: "Well, based on this and that, such and such must happen somewhere between 'this' and 'that'?"

When you're driving along in your car you make decisions based on your speed, right? Will you get to school or work on time? Will you get to the store on time? You're always assuming (correctly) that your **speed is a continuous function of time** (of course you're not really **thinking** about this) and you make these quick mental calculations that will verify whether or not you'll get "there" on time. Basically you know what time you started your trip and you have an idea about when it should end and then figure out where you have to be in between...

Another result which you know about continuous functions is this: **If f is continuous on a closed interval $[a, b]$, then it has a maximum value and a minimum value, and these values are attained by some points in $[a, b]$.**



Since total distance travelled is a continuous function of time it follows that there exists at least one time t at which you were at the video store (this is true) and some other time t at which you were "speeding" on your way to your destination (also true!)... all applications of the IVT.

Finally, we should mention that since the definition of a continuous function depends on the notion of a limit it is immediate that many of the properties of limits should reflect themselves in similar properties of continuous functions. For example, from Table 2.4 we see that sums, differences, and products of continuous functions are continuous functions. The same is true of quotients of continuous functions provided the denominator is not zero at the point in question!

Web Links

For an application of the IVT to Economics see :

http://hadm.sph.sc.edu/Courses/Econ/irr/irr.html

For proofs of the main theorems here see:

www.cut-the-knot.com/Generalization/ivt.html
www.cut-the-knot.com/fta/brodie.html

Special Exercise Set

1. According to a famous book of world records a Venom GT racing car accelerated along a flat road from 0-60 mph in 3.05 seconds early in January, 2013. Assuming that 1 mile = 1.61 km, show that the car must have traveled across the 25 meter mark at some time.
2. The height of the mountain Ketu (a.k.a K2) is 8611 meters above sea level. Assuming that its sides form a continuous curve prove that somewhere along your journey to the top you must have passed the 5000 meter mark.
3. You're sitting at a table with your friends enjoying a refreshment when all of a sudden you notice that the table wobbles when you lean your elbow on it and your friends start freaking out (their drinks start spilling)! Assuming that the table has four legs of equal length show them how you can fix this *uneven floor problem* easily by *rotating* and perhaps *sliding* the table (without putting length extenders below on one or more legs).
4. Show that if the temperature in a room is 36^o C at some point and 14^o C near your open freezer, then there must be a point in the room where the temperature must be a perfect 20^o C!
5. Given that Toronto and New York city are 491 miles apart and that the maximum speed limit along the highways is 65 mph show that getting to either one from the other in 6 hours means that you broke the law somewhere along your trip. In fact, prove that somewhere on the highway you were traveling at a speed of at least 81 mph!
6. Show that Bolzano's theorem (Theorem 2.3.1) is a consequence of the Intermediate Value Theorem (IVT).

Suggested Homework Set 5. *Problems* $1, 3, 5, 6$

NOTES:

2.4 Evaluating Limits at Infinity

In this section we introduce some basic ideas as to when the variable tends to **plus infinity** $(+\infty)$ or **minus infinity** $(-\infty)$. Note that limits at infinity are always one-sided limits, (why?). This section is intended to be a prelude to a later section on **L'Hospital's Rule** which will allow you to evaluate many of these limits by a neat trick involving the function's **derivatives.**

For the purposes of evaluating limits at infinity, the symbol '∞' has the following properties:

PROPERTIES

1. It is an 'extended' real number (same for '$-\infty$').
2. For any real number c (including 0), and $r > 0$,
$$\lim_{x\to\infty} \frac{c}{x^r} = 0$$
(Think of this as saying that $\frac{c}{\infty^r} = 0$ and $\infty^r = \infty$ for $r > 0$.)
3. The symbol $\frac{\infty}{\infty}$ is undefined and **can only be defined in the limiting sense** using the procedure in Table 2.13, Stage 2, some insight and maybe a little help from your calculator. We'll be using this procedure a little later when we attempt to evaluate limits at $\pm\infty$ using extended real numbers.

Table 2.14: Properties of $\pm\infty$

Basically, the limit symbol "$x \to \infty$" means that the real variable x can be made "larger" than any real number!

A similar definition applies to the symbol "$x \to -\infty$" except that now the real variable x may be made "smaller" than any real number. The next result is very useful in evaluating limits involving oscillating functions where it may not be easy to find the limit.

The **Sandwich Theorem** (mentioned earlier) is also valid for **limits at infinity**, that is, if
$$g(x) \le f(x) \le h(x)$$
for all (sufficiently) large x and for some (extended) real number A,
$$\lim_{x\to\infty} g(x) = A, \qquad \lim_{x\to\infty} h(x) = A$$
then f has a limit at infinity and
$$\lim_{x\to\infty} f(x) = A$$

Table 2.15: The Sandwich Theorem

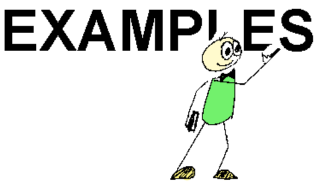
Example 56. Evaluate the following limits at infinity.

a) $\lim_{x\to\infty} \frac{\sin(2x)}{x}$

b) $\lim\limits_{x\to\infty} \dfrac{3x^2 - 2x + 1}{x^2 + 2}$

c) $\lim\limits_{x\to-\infty} \dfrac{1}{x^3 + 1}$

d) $\lim\limits_{x\to\infty} (\sqrt{x^2 + x + 1} - x)$



Solution **a)** Let $f(x) = \dfrac{\sin(2x)}{x}$. Then $|f(x)| = \dfrac{|\sin(2x)|}{x}$ and $|f(x)| \le \dfrac{1}{x}$ since $|\sin(2x)| \le 1$ for every real number x. Thus

$$0 \le \lim_{x\to\infty} |f(x)| \le \lim_{x\to\infty} \frac{1}{x} = 0$$

(where we have set $g(x) = 0$ and $h(x) = \dfrac{1}{x}$ in the statement of the Sandwich Theorem.) Thus,

$$\lim_{x\to\infty} |f(x)| = 0$$

which means that

$$\lim_{x\to\infty} f(x) = 0$$

(See Table 2.8 (c)).

b) Factor the term 'x^2' out of both numerator and denominator. Thus

$$\begin{aligned}\frac{3x^2 - 2x + 1}{x^2 + 2} &= \frac{x^2(3 - \frac{2}{x} + \frac{1}{x^2})}{x^2(1 + \frac{2}{x^2})} \\ &= \frac{(3 - \frac{2}{x} + \frac{1}{x^2})}{(1 + \frac{2}{x^2})}\end{aligned}$$

Now

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{x^2 + 2} &= \frac{\lim_{x\to\infty}\left(3 - \frac{2}{x} + \frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(1 + \frac{2}{x^2}\right)} \\ &\quad \text{(because the limit of a quotient is the quotient of the limits)} \\ &= \frac{3 - 0 + 0}{1 + 0} \\ &= 3\end{aligned}$$

where we have used the Property 2 of limits at infinity, Table 2.14.

c) Let $f(x) = \dfrac{1}{x^3 + 1}$. We claim $\lim\limits_{x\to-\infty} f(x) = 0 \ldots$ Why?

Well, as $x \to -\infty$, $x^3 \to -\infty$ too, right? Adding 1 won't make any difference, so $x^3 + 1 \to -\infty$ too (remember, this is true because $x \to -\infty$). OK, now $x^3 + 1 \to -\infty$ which means $(x^3 + 1)^{-1} \to 0$ as $x \to -\infty$.

d) As it stands, letting $x \to \infty$ in the expression $x^2 + x + 1$ also gives ∞. So $\sqrt{x^2 + x + 1} \to \infty$ as $x \to \infty$. So we have to calculate a "difference of two infinities" *i.e.*,

$$\begin{array}{ccc} & f(x) = \sqrt{x^2 + x + 1} - x & \\ \swarrow & & \searrow \\ \infty \text{ as } x \to \infty & & \infty \text{ as } x \to \infty \end{array}$$

There is no way of doing this so we have to simplify the expression (see Table 2.13, Stage 2) by rationalizing the expression ... So,

$$\begin{aligned}
\sqrt{x^2+x+1}-x &= \frac{(\sqrt{x^2+x+1}-x)(\sqrt{x^2+x+1}+x)}{\sqrt{x^2+x+1}+x} \\
&= \frac{(x^2+x+1)-x^2}{\sqrt{x^2+x+1}+x} \\
&= \frac{x+1}{\sqrt{x^2+x+1}+x}
\end{aligned}$$

The form still isn't good enough to evaluate the limit directly. (We would be getting a form similar to $\frac{\infty}{\infty}$ if we took limits in the numerator and denominator separately.)

OK, so we keep simplifying by factoring out 'x's from both numerator and denominator ... Now,

$$\begin{aligned}
\sqrt{x^2+x+1}-x &= \frac{x+1}{\sqrt{x^2+x+1}+x} \\
&= \frac{x(1+\frac{1}{x})}{x\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1\right)} \\
&= \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1}
\end{aligned}$$

OK, now we can let $x \to \infty$ and we see that

$$\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^2+x+1}-x) &= \lim_{x\to\infty}\frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1} \\
&= \frac{1+0}{\sqrt{1+0+0}+1} \\
&= \frac{1}{2}
\end{aligned}$$

As a **quick check** let's use a calculator and some large values of x: e.g. $x = 10$, 100, 1000, 10000, ... This gives the values: $f(10) = 0.53565$, $f(100) = 0.50373$, $f(1000) = 0.50037$, $f(10000) = 0.500037$, ... which gives a sequence whose limit appears to be $0.500... = \frac{1}{2}$, which is our theoretical result.

Exercise Set 9.

Evaluate the following limits (a) numerically and (b) theoretically.

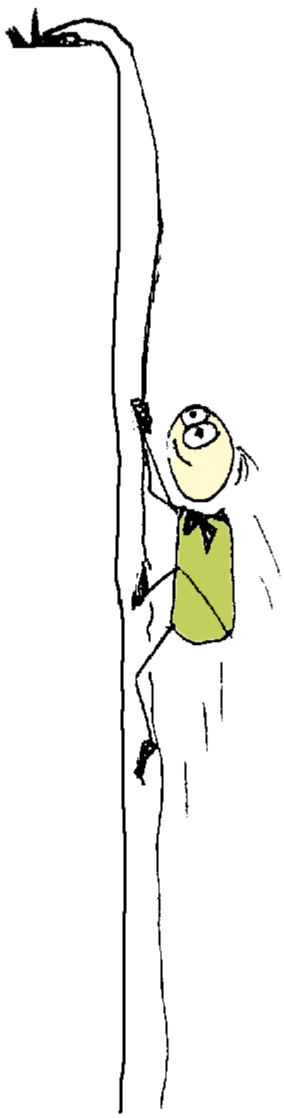
1. $\lim_{x\to\infty}\frac{\sin(3x)}{2x}$ (Remember: x is in **radians** here.)
2. $\lim_{x\to-\infty}\frac{x}{x^3+2}$
3. $\lim_{x\to\infty}\frac{x^3+3x-1}{x^3+1}$
4. $\lim_{x\to\infty}\sqrt{x}(\sqrt{x+1}-\sqrt{x})$
5. $\lim_{x\to-\infty}\frac{\cos x}{x^2}$

6. **Hard** Show that

$$\lim_{x\to\infty} \sin x$$

does **not** exist by giving a **graphical** argument.
(*Hint:* Use the ideas developed in the Advanced Topics to **prove this theoretically**.)

2.5 How to Guess a Limit



We waited for this part until you learned about limits in general. Here we'll show you a quick and quite reliable way of *guessing* or *calculating* some limits at infinity (or "minus" infinity). Strictly speaking, you still need to 'prove' that your guess *is* right, even though it *looks* right. See Table 2.16. Later on, in Section 3.10, we will see a method called *L'Hospital's Rule* that can be used effectively, under some mild conditions, to evaluate limits involving *indeterminate forms*.

OK, now just a few words of caution before you start manipulating infinities. If an operation between infinities and reals (or another infinity) is not among those listed in Table 2.16, it is called an **indeterminate form**.

The most common indeterminate forms are:

$$\boxed{0\cdot(\pm\infty),\quad \pm\frac{\infty}{\infty},\quad \infty-\infty,\quad (\pm\infty)^0,\quad 1^{\pm\infty},\quad \frac{0}{0},\quad 0^0}$$

When you meet these forms in a limit you can't do much except simplify, rationalize, factor, etc. and then see if the form becomes "determinate".

FAQ about Indeterminate Forms

Let's have a closer look at these *indeterminate* forms: They are called indeterminate because we cannot assign a single real number (once and for all) to any one of those expressions. For example,

Question 1: Why can't we define $\frac{\infty}{\infty} = 1$? After all, this looks okay ...

Answer 1: If that were true then,

$$\lim_{x\to\infty} \frac{2x}{x} = 1,$$

but this is impossible because, for any real number x no matter how large,

$$\frac{2x}{x} = 2,$$

and so, in fact,

$$\lim_{x\to\infty} \frac{2x}{x} = 2,$$

and so we can't define $\frac{\infty}{\infty} = 1$. Of course, we can easily modify this example to show that if r is any real number, then

$$\lim_{x\to\infty} \frac{rx}{x} = r,$$

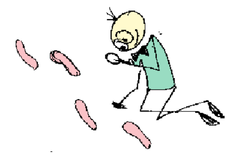
which seems to imply that $\frac{\infty}{\infty} = r$. But r is also arbitrary, and so these numbers r can't all be equal because we can choose the r's to be different! This shows that we cannot define the quotient $\frac{\infty}{\infty}$. Similar reasoning shows that we cannot define the quotient $-\frac{\infty}{\infty}$.

Question 2: All right, but surely $1^\infty = 1$, since $1 \times 1 \times 1 \times ... = 1$?

Answer 2: No. The reason for this is that there is an infinite number of 1's here and this statement about multiplying 1's together is only true if there is a *finite* number of 1's. Here, we'll give some **numerical evidence** indicating that $1^\infty \neq 1$, necessarily.

Let $n \geq 1$ be a positive integer and look at some of the values of the expression

$$\left(1 + \frac{1}{n}\right)^n, \quad n = 1, 2, 3, ..., 10,000$$



These values below:

n	$\left(1+\frac{1}{n}\right)^n$	*value*
$n = 1$	$\left(1+\frac{1}{1}\right)^1$	2
$n = 2$	$\left(1+\frac{1}{2}\right)^2$	2.25
$n = 3$	$\left(1+\frac{1}{3}\right)^3$	2.37
$n = 4$	$\left(1+\frac{1}{4}\right)^4$	2.44
$n = 5$	$\left(1+\frac{1}{5}\right)^5$	2.48
$n = 10$	$\left(1+\frac{1}{10}\right)^{10}$	2.59
$n = 50$	$\left(1+\frac{1}{50}\right)^{50}$	2.69
$n = 100$	$\left(1+\frac{1}{100}\right)^{100}$	2.7048
$n = 1,000$	$\left(1+\frac{1}{1000}\right)^{1000}$	2.7169
$n = 10,000$	$\left(1+\frac{1}{10000}\right)^{10000}$	2.71814
$\cdots$	$\cdots$	$\cdots$

Well, you can see that the values do not appear to be approaching 1! In fact, they seem to be getting closer to some number whose value is around 2.718. More on this special number later, in Chapter 4. Furthermore, we saw in Exercise 17, of Exercise Set 3, that these values must always lie between 2 and 3 and so, once again cannot converge to 1. This shows that, generally speaking, $1^\infty \neq 1$. In this case one can show that, in fact,

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818284590...$$

is a special number called *Euler's Number*, (see Chapter 4).

Question 3: What about $\infty - \infty = 0$?

Answer 3: No. This isn't true either since, to be precise, ∞ is NOT a real number, and so we cannot apply real number properties to it. The simplest example that shows that this difference between two infinities is not zero is the following. Let n be an integer (not infinity), for simplicity. Then

$$\begin{aligned} \infty - \infty &= \lim_{n\to\infty} [n-(n-1)] \\ &= \lim_{n\to\infty} [n-n+1] \\ &= \lim_{n\to\infty} 1 \\ &= 1. \end{aligned}$$

The same argument can be used to find examples where $\infty - \infty = r$, where r is any given real number. It follows that we cannot assign a real number to the expression $\infty - \infty$ and so this is an indeterminate form.

Question 4: Isn't it true that $\frac{0}{0} = 0$?

Answer 4: No, this isn't true either. See the example in Table 2.11 and the discussion preceding it. The results there show that

$$\begin{aligned} \frac{0}{0} &= \frac{\sin 0}{0} \\ &= \lim_{x\to 0} \frac{\sin x}{x} \\ &= 1 \end{aligned}$$

in this case. So we cannot assign a real number to the quotient "zero over zero".

Question 5: Okay, but it must be true that $\infty^0 = 1$!?

Answer 5: Not generally. An example here is harder to construct but it can be done using the methods in Chapter 4.

The Numerical Estimation of a Limit

At this point we'll be guessing limits of indeterminate forms by performing numerical calculations. See Example 59, below for their *theoretical*, rather than *numerical* calculation.

Example 57. Guess the value of each of the following limits at infinity:

a) $\lim_{x\to\infty} \frac{\sin(2x)}{x}$

b) $\lim_{x\to-\infty} \frac{x^2}{x^2+1}$

c) $\lim_{x\to\infty} \left(\sqrt{x+1} - \sqrt{x}\right)$

Solution **a)** Since $x \to \infty$, we only need to try out **really large values** of x. So, just set up a table such as the one below and look for a pattern ...

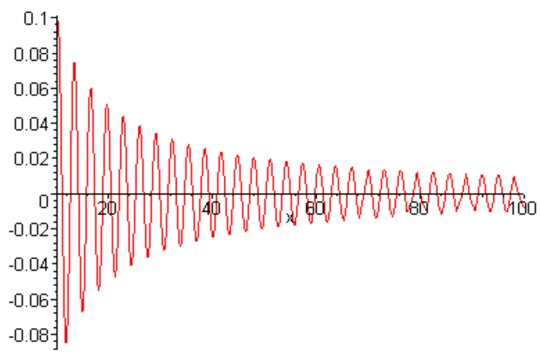
Some values of x	The values of $f(x) = \frac{\sin 2x}{x}$
10	.0913
100	−.00873
1,000	0.000930
10,000	0.0000582
100,000	−0.000000715
1,000,000	−0.000000655
...	...

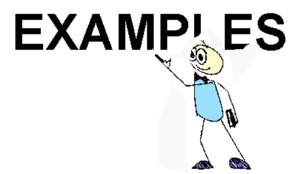
We note that even though the values of $f(x)$ here alternate in sign, they are always getting smaller. In fact, they seem to be approaching $f(x) = 0$, as $x \to \infty$. This is our guess and, on this basis, we can claim that

$$\lim_{x \to \infty} \frac{\sin 2x}{x} = 0.$$

See Example 59 a), for another way of seeing this.

Below you'll see a graphical depiction (made by using your favorite software package or the Plotter included with this book), of the function $f(x)$ over the interval $[10, 100]$.





Note that the *oscillations* appear to be dying out, that is, they are getting smaller and smaller, just like the oscillations of your car as you pass over a bump! We guess that the value of this limit is 0.

b) Now, since $x \to -\infty$, we only need to try out really *small* (and negative) values of x. So, we set up a table like the one above and look for a pattern in the values.

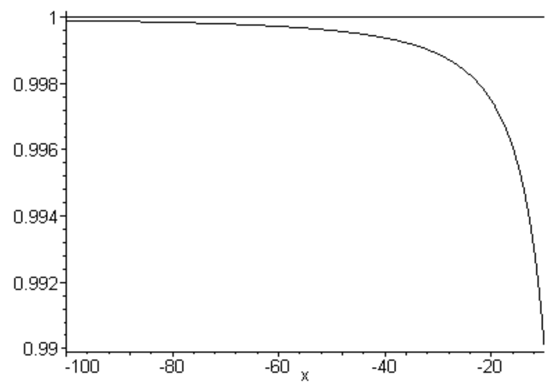
Some values of x	The values of $f(x) = \dfrac{x^2}{x^2+1}$
-10	0.9900990099
-100	0.9999000100
$-1,000$	0.9999990000
$-10,000$	0.9999999900
$-100,000$	0.9999999999
$-1,000,000$	1.0000000000
...	...

In this case the values of $f(x)$ all have the same sign, they are always *positive*. Furthermore, they seem to be approaching $f(x) = 1$, as $x \to \infty$. This is our guess and, on this numerical basis, we can claim that

$$\lim_{x \to -\infty} \frac{x^2}{x^2+1} = 1.$$

See Example 59 b), for another way of seeing this.

A graphical depiction of this function $f(x)$ over the interval $[-100, -10]$ appears below.



In this example, the values of the function appear to increase steadily towards the line whose equation is $y = 1$. So, we guess that the value of this limit is 1.

c) Once again $x \to +\infty$, we only need to try out really *large* (and positive) values of x. Our table looks like:

Some values of x	The values of $f(x) = \sqrt{x+1} - \sqrt{x}$
10	0.15434713
100	0.04987562
$1,000$	0.01580744
$10,000$	0.00499999
$100,000$	0.00158120
$1,000,000$	0.00050000
...	...

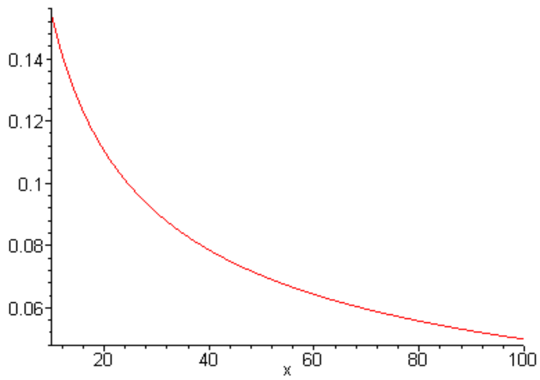
In this case the values of $f(x)$ all have the same sign, they are always *positive*. Furthermore, they seem to be approaching $f(x) = 0$, as $x \to \infty$. We can claim that

$$\lim_{x\to\infty} \left(\sqrt{x+1} - \sqrt{x}\right) = 0.$$

See Example 59 d), for another way of seeing this. A graphical depiction of this function $f(x)$ over the interval $[10, 100]$ appears below.

Note that larger values of x are not necessary since we have a *feeling* that they'll just be closer to our limit. We can believe that the values of $f(x)$ are always getting closer to 0 as x gets larger. So 0 should be the value of this limit. In the graph below we see that the function is getting smaller and smaller as x increases but it

always stays positive. Nevertheless, its values never reach the number 0 exactly, but only in the *limiting sense* we described in this section.



Watch out!

This numerical way of "guessing" limits **doesn't always work!** It works well when the function *has* a limit, but it doesn't work if the limit doesn't exist (see the previous sections).

For example, the function $f(x) = \sin x$ has NO limit as $x \to \infty$. But how do you know this? The table could give us a hint;

Some values of x	The values of $f(x) = \sin x$
10	-0.5440211109
100	-0.5063656411
$1,000$	$+0.8268795405$
$10,000$	-0.3056143889
$100,000$	$+0.0357487980$
$1,000,000$	-0.3499935022
$\ldots$	$\ldots$

As you can see, these values do not seem to have a pattern to them. They don't seem to "converge" to any particular value. We should be suspicious at this point and claim that the limit doesn't exist. But remember: **Nothing can replace a rigorous (theoretical) argument for the existence or non-existence of a limit!** Our guess may not coincide with the reality of the situation as the next example will show!

Now, we'll manufacture a function **with the property that, based on our numerical calculations,** *it seems to have a limit* **(actually $= 0$) as $x \to \infty$, but, in reality, its limit is SOME OTHER NUMBER!**

Example 58. Evaluate the following limit using your calculator,

$$\lim_{x\to\infty} \left(\frac{1}{x} + 10^{-12}\right).$$

Solution Setting up the table gives us:

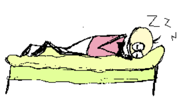
Some values of x	The values of $f(x) = \left(\frac{1}{x} + 10^{-12}\right)$
10	0.100000000
100	0.010000000
1,000	0.001000000
10,000	0.000100000
100,000	0.000001000
999,999,999	0.000000001
...	...

Well, if we didn't know any better we would think that this limit should be 0. But this is **only because we are limited by the number of digits displayed upon our calculator!** The answer, based upon our knowledge of limits, should be the number 10^{-12} (but this number would display as 0 on most hand-held calculators). That's the real problem with using calculators for finding limits. You must be careful!!

Web Links

More (solved) examples on limits at infinity at:

http://tutorial.math.lamar.edu/Classes/CalcI/LimitsAtInfinityI.aspx
http://www.sosmath.com/calculus/limcon/limcon04/limcon04.html



NOTES:

Finding Limits using Extended Real Numbers

At this point we'll be guessing limits of indeterminate forms by performing a new arithmetic among *infinite quantities*! In other words, we'll define addition and multiplication of infinities and then use these ideas to actually *find* limits at (plus or minus) infinity. This material is not standard in Calculus Texts and so can be omitted if so desired. However, it does offer an alternate method for actually guessing limits correctly every time!

If you think that adding and multiplying 'infinity' is nuts, you should look at the work of **Georg Cantor** (1845-1918), who actually developed an arithmetic of *transfinite cardinal numbers*, (or numbers that are infinite). He showed that 'different infinities' exist and actually set up rules of arithmetic for them. His work appeared in 1833.

As an example, the totality of all the integers (one type of infinite number), is different from the totality of all the numbers in the interval $[0, 1]$ (another 'larger' infinity). In a very specific sense, there are more "real numbers" than "integers".

Operations on the Extended Real Number Line

The **extended real number line** is the collection of all (usual) real numbers plus **two new symbols**, namely, $\pm\infty$ (**called extended real numbers**) which have the following properties:

Let x be any real number. Then

a) $x + (+\infty) = (+\infty) + x = +\infty$

b) $x + (-\infty) = (-\infty) + x = -\infty$

c) $x \cdot (+\infty) = (+\infty) \cdot x = +\infty$ if $x > 0$

d) $x \cdot (-\infty) = (-\infty) \cdot x = -\infty$ if $x > 0$

e) $x \cdot (+\infty) = (+\infty) \cdot x = -\infty$ if $x < 0$

f) $x \cdot (-\infty) = (-\infty) \cdot x = +\infty$ if $x < 0$

The operation $0 \cdot (\pm\infty)$ is undefined and requires further investigation.

Operations between $+\infty$ and $-\infty$

g) $(+\infty) + (+\infty) = +\infty$

h) $(-\infty) + (-\infty) = -\infty$

i) $(+\infty) \cdot (+\infty) = +\infty = (-\infty) \cdot (-\infty)$

j) $(+\infty) \cdot (-\infty) = -\infty = (-\infty) \cdot (+\infty)$

Quotients and powers involving $\pm\infty$

k) $\dfrac{x}{\pm\infty} = 0$ for **any** real x

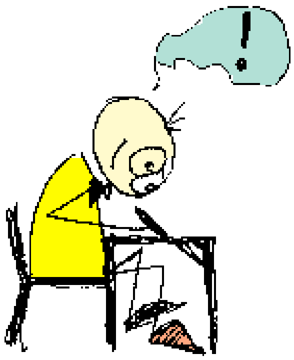
l) $\infty^r = \begin{cases} \infty & r > 0 \\ 0 & r < 0 \end{cases}$

m) $a^\infty = \begin{cases} \infty & a > 1 \\ 0 & 0 \le a < 1 \end{cases}$

Table 2.16: Properties of Extended Real Numbers

The **extended real number line** is, by definition, the ordinary (positive and negative) real numbers with the addition of two idealized points denoted by $\pm\infty$ (and called the points at infinity). The way in which infinite quantities interact with each other and with real numbers is summarized briefly in Table 2.16 above.

It is important to note that any basic operation that is not explicitly mentioned in Table 2.16 is to be considered an *indeterminate form*, unless it can be derived from one or more of the basic axioms mentioned there.



Evaluating Limits of Indeterminate Forms

OK, now what? Well, you want

$$\lim_{x \to \pm\infty} f(x)$$

Basically, you look at $f(\pm\infty)$ respectively.

This is an expression involving "infinities" which you simplify (if you can) using the rules of arithmetic of the extended real number system listed in Table 2.16. If you get an indeterminate form you need to factor, rationalize, simplify, separate terms etc. until you get something more manageable.

$\boxed{\textbf{Example 59.}}$ Evaluate the following limits involving indeterminate forms:

a) $\displaystyle\lim_{x\to\infty} \frac{\sin 2x}{x}$

b) $\displaystyle\lim_{x\to\infty} \frac{x^2}{x^2+1}$

c) $\displaystyle\lim_{x\to-\infty} \frac{x^3+1}{x^3-1}$

d) $\displaystyle\lim_{x\to\infty} \sqrt{x+1}-\sqrt{x}$

e) $\displaystyle\lim_{x\to 0^+} \frac{x}{\sin x}$

Solution **a)** Let $f(x) = \dfrac{\sin(2x)}{x}$, then $f(\infty) = \dfrac{\sin(2\infty)}{\infty}$. Now use Table 2.16 on the previous page. Even though $\sin(2\infty)$ doesn't really have a meaning, we can safely take it that the "$\sin(2\infty)$ is something less than or equal to 1", because the sine of any *finite* angle has this property. So $f(\infty) = \dfrac{\text{something}}{\infty} = 0$, by property (k), in Table 2.16.

We conclude our guess which is:

$$\lim_{x\to\infty} \frac{\sin 2x}{x} = 0$$

> **Remember:** This is just an educated guess; you really have to *prove* this to be sure. This method of guessing is far better than the numerical approach of the previous subsection since it gives the right answer in case of Example 58, where the numerical approach failed!

b) Let $f(x) = \frac{x^2}{x^2+1}$. Then $f(\infty) = \frac{\infty}{\infty}$ by properties (l) and (a) in Table 2.16. So we have to simplify, etc. There is no other recourse ... Note that

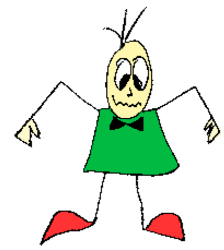
$$\begin{aligned}
f(x) &= \frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1} \\
\text{So } \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} 1 - \frac{1}{x^2+1} \\
&= 1 - \frac{1}{\infty} \quad \text{(by property (l) and (a))} \\
&= 1 - 0 \quad \text{(by property (k))} \\
&= 1
\end{aligned}$$

c) Let $f(x) = \dfrac{x^3+1}{x^3-1}$. Then

$$f(-\infty) = \frac{(-\infty)^3+1}{(-\infty)^3-1} = \frac{-\infty^3+1}{-\infty^3-1} = \frac{-\infty+1}{-\infty-1} = \frac{-\infty}{-\infty} = \frac{\infty}{\infty}$$

is an indeterminate form! So we have to simplify ... Dividing the numerator by the denominator using long division, we get

$$\begin{aligned}
\frac{x^3+1}{x^3-1} &= 1 + \frac{2}{x^3-1} \\
\text{Hence } \lim_{x\to-\infty} \frac{x^3+1}{x^3-1} &= \lim_{x\to\infty} (1 + \frac{2}{x^3-1}) \\
&= 1 + \frac{2}{\infty} \quad \text{(by property (e) and (a))} \\
&= 1 + 0 \quad \text{(by property (k), in Table 2.16)} \\
&= 1
\end{aligned}$$



d) Let $f(x) = \sqrt{x+1} - \sqrt{x}$. Then

$$\begin{aligned}
f(\infty) &= \sqrt{\infty+1} - \sqrt{\infty} \\
&= \sqrt{\infty} - \sqrt{\infty} \quad \text{(by property (a))} \\
&= \infty - \infty \quad \text{(by property (k), in Table 2.16)}
\end{aligned}$$

It follows that $f(\infty)$ is an indeterminate form. Let's simplify ... By rationalizing the numerator we know that

$$\begin{aligned}
\sqrt{x+1} - \sqrt{x} &= \frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x+1}+\sqrt{x}}.
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} \\
&= \frac{1}{\sqrt{\infty+1}+\sqrt{\infty}} \\
&= \frac{1}{\sqrt{\infty}+\sqrt{\infty}} \quad \text{(by property (a))} \\
&= \frac{1}{\infty+\infty} \quad \text{(by property (h))} \\
&= \frac{1}{\infty} \quad \text{(by property (g))} \\
&= 0 \quad \text{(by property (k))}
\end{aligned}$$

d) In this case the function can be seen to be of *more than one indeterminate form*: For example,

$$0 \cdot \frac{1}{\sin 0} = 0 \cdot \frac{1}{0} = 0 \cdot \infty,$$

which is indeterminate (by definition), or

$$\frac{0}{\sin 0} = \frac{0}{0},$$

which is also indeterminate. But we have already seen in Table 2.11 that when this indeterminate form is interpreted as a limit, it is equal to 1.

2.6 Limits equal to $\pm\infty$

The Big Picture

Now we look at those limits that CAN be equal to infinity. There is a big difference between limits that exist and are equal to plus or minus infinity and limits that don't exist at all (depending on the context of what number system is used, whether *real* or *extended*). In other words, when we use the *extended* real number system, we allow for those limits that EXIST and whose values are equal to either $+\infty$ or $-\infty$.

So, we'll be looking at two kinds of limits: **Limits that exist and are equal to** $\pm\infty$ and functions whose **limits do not exist** at all (regardless of the number system being used)! For example here are functions that have no limits at all (i.e., they don't exist) ...

$$\lim_{x\to\infty} \sin x, \quad \lim_{x\to\infty} \frac{2^x}{\cos x}, \quad \lim_{x\to\infty} (\cos x - \sin x).$$

When you approximate the graph of each of these functions using your graphing calculator you realize something quickly, that is, the curves are *oscillating* no matter how far you go and they do not appear to be approaching any specific (even extended) real number. This is the hallmark of a **limit that doesn't exist**! If you want to know more about such limits see the theoretical section in this book, Section **??**.

OPTIONAL BOX:

If you know something about **sequences** and their limits, then all you need to do in order **to show that the limit**

$$\lim_{x\to a} f(x)$$

does NOT exist is to find TWO different infinite sequences BOTH of whose limits are different! That is, you need to find an infinite sequence x_n and another infinite sequence z_n (each in the domain of f) such that, as $n \to \infty$ we have $x_n \to a$ AND $z_n \to a$ AND $f(x_n) \to L$ AND $f(z_n) \to M$ where $L \neq M$ and L, M are in the extended real number system.

Anyhow, on the other hand, the following limits actually exist but are equal to either $+\infty$ or $-\infty$.

$$\lim_{x\to 1^+} \frac{1}{x-1}, \quad \lim_{x\to+\infty} x^2, \quad \lim_{x\to-\infty} x.$$

Each one of the preceding limits actually *exists* but the values of the limits are infinite. In this book it's okay for a limit to be infinite, so long as it exists. We add a few more examples.

Example 60. Evaluate the following limits, if possible. In other words, determine whether the limit exists and is infinite or whether the limit doesn't exist (at all). Give reasons:

a) $\displaystyle\lim_{x\to\infty} \frac{\sin 2x}{\sin 4x}$

b) $\displaystyle\lim_{x\to\infty} 3^x \cos x$

c) $\displaystyle\lim_{x\to-\infty} \frac{x^4}{x^3+1}$

d) $\displaystyle\lim_{x\to 0^+} \frac{\cos x}{x}$

e) $\lim_{x\to 0} \frac{\cos x}{x}$

Solution **a)** We use the trigonometric identity $\sin 4x = 2\sin 2x \cos 2x$ (why can we do this? Because $\sin 2\Box = 2\sin\Box\cos\Box$ and just put $2x$ in the box!). So, for x such that the denominator is not equal to zero we can write,

$$\begin{aligned}\frac{\sin 2x}{\sin 4x} &= \frac{\sin 2x}{2\sin 2x\cos 2x}\\ &= \frac{1}{2\cos 2x}.\end{aligned}$$

provided $\sin 2x \neq 0$ too (so we can cancel the numerator and denominator). Now $\cos 2x$ does not approach a limit as $x \to \infty$ (it just keeps oscillating between -1 and 1). In addition, the numbers 1 and 2 do not change as $x \to \infty$. It follows that the limit does not exist!

b) Here $\cos x$ has no limit at all as $x \to \infty$ (as it also keeps oscillating between -1 and 1), but $3^x \to \infty$ as $x \to \infty$. If you look at the graph of $3^x \cos x$ you'll see that the oscillations get larger and larger both positively and negatively, so they're "not approaching a number" (or "not going anywhere"). It follows that the limit does not exist.

c) As $x \to -\infty$ we have that x is large and negative and so it can't be zero. Thus,

$$\begin{aligned}\frac{x^4}{x^3+1} &= \frac{x^4}{x^3(1+\frac{1}{x^3})}\\ &= \frac{x}{(1+1/x^3)}.\end{aligned}$$

Now as $x \to -\infty$ we know that $1/x^3 \to 0$ (as the form is $1/(-\infty) = 0$). It follows that

$$\lim_{x\to-\infty} \frac{x}{(1+1/x^3)} = -\infty,$$

so the limit actually exists and is equal to $-\infty$.

d) First note that this is a *one-sided limit*, from the right. Next, we also note that the cosine function is continuous at $x = 0$ so it doesn't matter HOW x is approaching the number 0 (whether from the *right* or from the *left*), the quantity $\cos x \to 1$ (because $\cos 0 = 1$). What about the denominator? Well, here as $x \to 0^+$ (or x is approaching zero from the right, which means that $x > 0$ as it approaches zero) then $\cos x/x \to 1/0 = +\infty$. There is no other possibility for a limit since x cannot be negative by hypothesis. So the one-sided limit is equal to $+\infty$.

e) Now note that this is a *two-sided limit*, okay? Next, as before, note that the cosine function is continuous at $x = 0$ so it doesn't matter HOW x is approaching the number 0 (whether from the *right* or from the *left*), the quantity $\cos x \to 1$ (because $\cos 0 = 1$). But now in the denominator it's clear that things get pretty bad in the sense that as $x \to 0^+$ (which means that $x > 0$ as it approaches zero) then $\cos x/x \to 1/0 = +\infty$. On the other hand, if $x \to 0^-$ (i.e., from the left, or here, $x < 0$) then $\cos x/x \to 1/0 = -\infty$, see? Since both these limits are different, the two-sided limit does not exist!

NOTE: You CAN'T DEFINE 1/0 to be any specific extended real number. The previous example shows that depending on how you approach zero, the quotient of something (not zero) divided by "zero" can go to either plus or minus infinity. YOU HAVE TO REMEMBER THIS!

Special Exercise Set

Use the methods of this section and the previous one to determine whether the following limits exist and then guess their values as extended real numbers.

1. $\lim_{x\to 0^+} \frac{x-1}{x}$
2. $\lim_{x\to 0^+} \frac{2+x}{x}$
3. $\lim_{x\to 0} \frac{3-x}{x}$
4. $\lim_{x\to 0^+} \frac{2x+1}{x}$
5. $\lim_{x\to 0^-} \frac{x^2+1}{x}$
6. $\lim_{x\to 0} \frac{x+1}{|x|}$
7. $\lim_{x\to 0} \frac{2x^2+x}{|x|}$
8. $\lim_{x\to 1} \frac{x}{x-1}$
9. $\lim_{x\to 2^+} \frac{1+\sin x}{x-2}$
10. $\lim_{x\to -3^+} \frac{1}{x+3}$
11. $\lim_{x\to 1/2^-} \frac{x}{2x-1}$
12. $\lim_{x\to 2} \frac{\cos(x-2)}{x-2}$
13. $\lim_{x\to 2} \frac{\cos(x-2)}{|x-2|}$
14. $\lim_{x\to \infty} \frac{x^4}{2x^3-1}$
15. $\lim_{x\to -\infty} \frac{2\sin x}{\sin 2x}$
16. $\lim_{x\to \infty} 2^x \sin x$
17. $\lim_{x\to \infty} \left(\frac{3}{1+x^2}+10^{-11}\right)$
18. $\lim_{x\to -\infty} \left(-\frac{1}{x^3}-10^{-8}\right)$
19. $\lim_{x\to -\infty} \left(\frac{1}{x}-x\right)$
20. $\lim_{x\to \infty} \left(\frac{2.718}{x}+x\right)$
21. $\lim_{x\to 0^+} \left(x-\frac{\cos x}{x}\right)$
22. $\lim_{x\to \infty} \left(\frac{1}{2x-\sqrt{x^2}}\right)$
23. $\lim_{x\to -\infty} \left(\sin^2(x)+\cos^2(x)\right)$
24. $\lim_{x\to \pi/2} (3\tan(x)-x)$
25. $\lim_{x\to \pi^-} (3x+\cot x)$

Suggested Homework Set 6. *Problems* $1, 3, 5, 7, 8, 11, 15, 17, 22, 23$

NOTES:

2.7 Chapter Exercises

Use the methods of this Chapter to decide the continuity of the following functions at the indicated point(s).

1. $f(x) = 3x^2 - 2x + 1$, at $x = 1$
2. $g(t) = t^3 \cos(t)$, at $t = 0$
3. $h(z) = z + 2\sin(z) - \cos(z + 2)$ at $z = 0$
4. $f(x) = 2\cos(x)$ at $x = \pi$
5. $f(x) = |x + 1|$ at $x = -1$

Evaluate the limits of the functions from Exercises 1-5 above and justify your conclusions.

6. $\lim_{x\to 1}(3x^2 - 2x + 1)$
7. $\lim_{t\to 0} t^3 \cos(t)$
8. $\lim_{z\to 0} (z + 2\sin(z) - \cos(z + 2))$
9. $\lim_{x\to \pi} 2\cos(x)$
10. $\lim_{x\to -1} |x + 1|$

Evaluate the following limits

11. $\lim_{t\to 2^+} \left(\frac{t-2}{t+2}\right)$
12. $\lim_{x\to 4^+} \left(\frac{x-4}{x^2-16}\right)$
13. $\lim_{t\to 2^+} \left(\frac{1}{t-2}\right)$
14. $\lim_{x\to 1^+} \left(\frac{x-1}{|x-1|}\right)$
15. $\lim_{x\to 0^+} \left(1 + \frac{1}{x^3}\right)$

16. Let g be defined as

$$g(x) = \begin{cases} x^2 + 1 & x < 0 \\ 1 - |x| & 0 \le x \le 1 \\ x & x > 1 \end{cases}$$

Evaluate

i). $\lim_{x\to 0^-} g(x)$ ii). $\lim_{x\to 0^+} g(x)$

iii). $\lim_{x\to 1^-} g(x)$ iv). $\lim_{x\to 1^+} g(x)$

v) Conclude that the graph of g has no breaks at $x = 0$ but it does have a break at $x = 1$.

Determine whether the following limits exist. Give reasons

17. $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} 2-x & x \le 0 \\ x+1 & x > 0 \end{cases}$
18. $\lim_{x\to 1} |x-3|$
19. $\lim_{x\to -2} \left(\dfrac{x+2}{x+1}\right)$
20. $\lim_{x\to 0} x^2 \sin x$
21. $\lim_{x\to 1} f(x) \quad$ where $f(x) = \begin{cases} \dfrac{\sin(x-1)}{x-1}, & 0 \le x < 1 \\ 1, & x = 1 \\ |x-1| & x > 1 \end{cases}$

Determine the points of discontinuity of each of the following functions.

22. $f(x) = \dfrac{|x|}{x} - 1$ for $x \ne 0$ and $f(0) = 1$
23. $g(x) = \begin{cases} \dfrac{x}{|x|} & x < 0 \\ \\ 1 + x^2 & x \ge 0 \end{cases}$
24. $f(x) = \dfrac{x^2 - 3x + 2}{x^3 - 1}$, $for\ x \ne 1; f(1) = -1/3.$
25. $f(x) = \begin{cases} x^4 - 1 & x \ne 0 \\ -0.99 & x = 0 \end{cases}$
26. $f(x) = 1.65 + \dfrac{1}{x^2} \quad$ for $x \ne 0,\ f(0) = +1$

Determine whether the following limits exist. If the limits exist, find their values in the extended real numbers.

27. $\lim_{x\to 0} \dfrac{\sin(ax)}{bx}$, where $a \ne 0, b \ne 0$
28. $\lim_{x\to 0} \dfrac{\cos(2x)}{|x|}$
29. $\lim_{x\to 0} \dfrac{x\ \sin(x)}{\sin(2x)}$
30. $\lim_{x\to 3^-} \dfrac{\sin\sqrt{3-x}}{\sqrt{3-x}}$
31. $\lim_{x\to 0} \dfrac{bx}{\sin(ax)}$, where $a \ne 0, b \ne 0$
32. $\lim_{x\to\infty} \dfrac{\cos 3x}{4x}$
33. $\lim_{x\to -\infty} x\ \sin x$
34. $\lim_{x\to\infty} \sqrt{x^2+1} - x$

35.Use Bolzano's Theorem and your pocket calculator to prove that the function f defined by $f(x) = x\ \sin x + \cos x$ has a root in the interval $[-5, 1]$.

36. Use Bolzano's Theorem and your pocket calculator to prove that the function f defined by $f(x) = x^3 - 3x + 2$ has a root in the interval $[-3, 0]$. Can you find it?

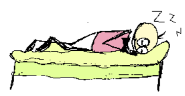
Are there any others? (Idea: Find smaller and smaller intervals and keep applying Bolzano's Theorem)

37. Find an interval of $x's$ containing the $x-$coordinates of the point of intersection of the curves $y = x^2$ and $y = \sin x$. Later on, when we study **Newton's Method** you'll see how to calculate these intersection points very accurately.

Hint: Use Bolzano's Theorem on the function $y = x^2 - \sin x$ over an appropriate interval (you need to find it).

Suggested Homework Set 7. *Problems* $1, 9, 12, 17, 22, 27, 34, 36$

NOTES:



Chapter 3

The Derivative of a Function

The Big Picture

This chapter contains material which is fundamental to the further study of Calculus. Its basis dates back to the great Greek scientist Archimedes (287-212 B.C.) who first considered the problem of the *tangent line*. Much later, attempts by the key historical figures Kepler (1571-1630), Galileo (1564-1642), and Newton (1642-1727) among others, to understand the motion of the planets in the solar system and thus the speed of a moving body, led them to the problem of *instantaneous velocity* which translated into the mathematical idea of a *derivative*. Through the geometric notion of a tangent line we will introduce the concept of the **ordinary derivative of a function**, itself another function with certain properties. Its interpretations in the physical world are so many that this book would not be sufficient to contain them all. Once we know what a derivative is and how it is used we can formulate many problems in terms of these, and the natural concept of an **ordinary differential equation** arises, a concept which is central to most applications of Calculus to the sciences and engineering. For example, the motion of every asteroid, planet, star, comet, or other celestial object is governed by a differential equation. Once we can solve these equations we can describe the motion. Of course, this is hard in general, and if we can't solve them exactly we can always approximate the solutions which give the orbits by means of some, so-called, **numerical approximations**. This is the way it's done these days ... We can send probes to Mars because we have a very good idea of where they should be going in the first place, because we know the mass of Mars (itself an amazing fact) with a high degree of accuracy.

Most of the time we realize that things are in motion and this means that certain physical quantities are changing. These changes are best understood through the derivative of some underlying function. For example, when a car is moving its distance from a given point is changing, right? The "rate at which the distance changes" is the derivative of the distance function. This brings us to the notion of "instantaneous velocity". Furthermore, when a balloon is inflated, its volume is changing and the "rate" at which this volume is changing is approximately given by the derivative of the original volume function (its units would be $meters^3/sec$). In a different vein, the stock markets of the world are full of investors who delve into **stock options** as a means of furthering their investments. Central to all this business is the **Black-Sholes equation**, a complicated differential equation, which won their discoverer(s) a Nobel Prize in Economics a few years ago.

> **Review**
>
> Look over your notes on functions in Chapter 1, especially the whole thing dealing with functions f being evaluated at abstract symbols other than x, like $f(x+2)$. To really understand derivatives you should review Chapter 2, in particular the part on the definition of continuity and right/left- handed limits.

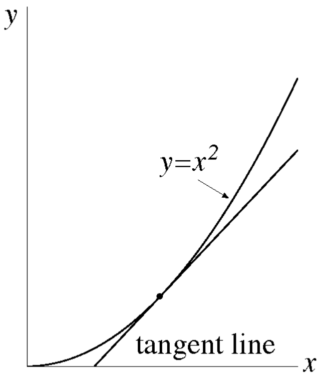


Figure 33.

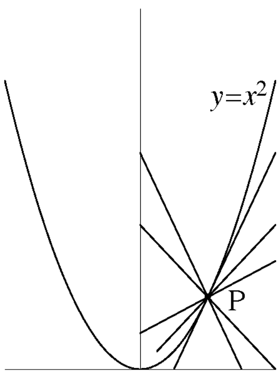


Figure 34.

3.1 Motivation

We begin this chapter by motivating the notation of the **derivative of a function,** itself another function with certain properties.

First, we'll define the notion of a **tangent to a curve.** In the phrase that describes it, a tangent at a given point P on the graph of the curve $y = f(x)$ is a straight line segment which intersects the curve $y = f(x)$ at P and is 'tangent' to it (think of the ordinary tangents to a circle, see Figure 33).

Example 61. Find the equation of the line tangent to the curve $y = x^2$ at the point $(1, 1)$.

Solution Because of the shape of this curve we can see from its graph that every straight line crossing this curve will do so in at most two points, and we'll actually show this below. Let's choose a point P, say, $(1, 1)$ on this curve for ease of exposition. We'll **find the equation of the tangent line to P** and we'll do this in the following steps:

1. Find the equation of all the straight lines through P.
2. Show that there exists, among this set of lines, a **unique** line which is tangent to P.

OK, the equation of every line through $P(1, 1)$ has the form

$$y = m(x-1)+1$$

where m is its slope, right? (Figure 34).

Since we want the straight line to intersect the curve $y = x^2$, we must set $y = x^2$ in the preceding equation to find

$$x^2 = m(x-1)+1$$

or the quadratic

$$x^2 - mx + (m-1) = 0$$

Finding its roots gives 2 solutions (the two x-coordinates of the point of intersection we spoke of earlier), namely,

$$x = m-1 \text{ and } x = 1$$

The second root $x = 1$ is clear to see as all these straight lines go through $P(1, 1)$. The first root $x = m-1$ gives a new root which is related to the **slope** of the straight line through $P(1, 1)$.

OK, we want only **one point of intersection,** right? (Remember, we're looking for a **tangent**). This means that the two roots must coincide! So we set $m-1 = 1$ (as the two roots are equal) and this gives $m = 2$.

Thus the line whose slope is 2 and whose equation is

$$y = 2(x-1)+1 = 2x-1$$

is the equation of the line tangent to P$(1,1)$ for the curve $y = x^2$. Remember that at the point $(1,1)$ this line has slope $m = 2$. This will be useful later.

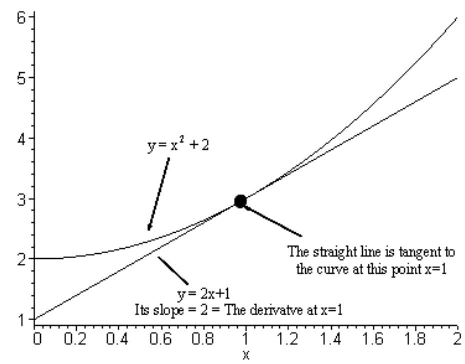
OK, but this is only an example of a tangent line to a curve .... How do you define this in general?

Well, let's take a function f, look at its graph and choose some point P(x_0, y_0) on its graph where $y_0 = f(x_0)$. Look at a nearby point Q$(x_0+h, f(x_0+h))$. **What is the equation of the line** joining P to Q? Its form is

$$y - y_0 = m(x - x_0)$$

But $y_0 = f(x_0)$ and m, the slope, is equal to the quotient of the difference between the y-coordinates and the x-coordinates (of Q and P), that is,

$$m = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}.$$



OK, so the equation of this line is

$$y = \{\frac{f(x_0+h) - f(x_0)}{h}\}(x - x_0) + f(x_0)$$

From this equation you can see that the slope of this line must change with "h". So, if we let h approach 0 as a limit, this line may approach a "limiting line" and it is this limiting line that we call the **tangent line to the curve $\boldsymbol{y = f(x)}$ at $\boldsymbol{P(x_0, y_0)}$** (see the figure in the margin on the right). The **slope** of this "tangent line" to the curve $y = f(x)$ at (x_0, y_0) defined by

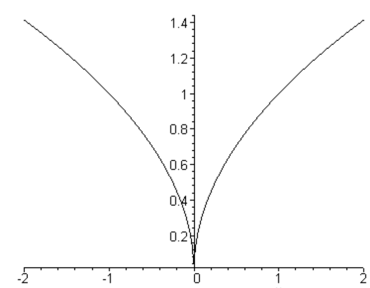
$$m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

(whenever this limit exists and is finite) is called the **derivative of f at x_0.** It is a number!!

Notation for Derivatives The following notations are all adopted universally for the derivatives of f at x_0:

$$f'(x_0),\ \frac{df}{dx}(x_0),\ D_x f(x_0),\ Df(x_0)$$

All of these have the same meaning.



This graph has a vertical tangent line, namely $x = 0$, at the origin.

Consequences!

1. If the limit as $h \to 0$ does not exist as a two-sided limit or it is infinite we say that the **derivative does not exist.** This is equivalent to saying that there is no uniquely defined tangent line at $(x_0, f(x_0))$, (Example 64).
2. The derivative, $f'(x_0)$ when it exists, is **the slope of the tangent line** at $(x_0, f(x_0))$ on the graph of f.
3. There's nothing special about these tangent lines to a curve in the sense that **the same line can be tangent to other points on the same curve.** (The simplest example occurs when $f(x) = ax + b$ is a straight line. Why?)

In this book we will use the symbols '$f'(x_0),\ Df(x_0)$' to mean the derivative of f at x_0 where

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \\ &= \text{The slope of the tangent line at } x = x_0 \\ &= \text{The instantaneous rate of change of } f \text{ at } x = x_0. \end{aligned}$$

whenever this (two-sided) limit exists and is finite.

Table 3.1: Definition of the Derivative as a Limit

4. If either one or both one-sided limits defined by $f'_{\pm}(x_0)$ is infinite, the tangent line at that point $P(x_0, f(x_0))$ is **vertical** and given by the equation $x = x_0$, (See the margin).

The key idea in finding the derivative using Table 3.1, here, is always to

SIMPLIFY first, THEN pass to the LIMIT

The concept of a left and right-derivative of f at $x = x_0$ is defined by the left and right limits of the expression on the right in Table 3.1. So, for example,

$$\begin{aligned} f'_{-}(x_0) &= \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}, \\ f'_{+}(x_0) &= \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}, \end{aligned}$$

define the left and right-derivative of f at $x = x_0$ respectively, whenever these limits exist and are finite.

Example 62. In Example 61 we showed that the slope of the tangent line to the curve $y = x^2$ at $(1, 1)$ is equal to 2. Show that the derivative of f where $f(x) = x^2$ at $x = 1$ is also equal to 2 (using the limit definition of the derivative, Table 3.1).

Solution By definition, the derivative of f at $x = 1$ is given by

$$f'(x) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

provided this limit exists and is finite. OK, then calculate

$$\begin{aligned} \frac{f(1+h)-f(1)}{h} &= \frac{(1+h)^2-1^2}{h} \\ &= \frac{1+2h+h^2-1}{h} \\ &= 2+h \end{aligned}$$

Since this is true for each value of $h \neq 0$ we can let $h \to 0$ and find

$$\begin{aligned} \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}(2+h) \\ &= 2 \end{aligned}$$

and so $f'(1) = 2$, as well. Remember, this also means that the slope of the tangent line at $x = 1$ is equal to 2, which is what we found earlier.

Example 63. Find the slope of the tangent line at $x = 2$ for the curve whose equation is $y = 1/x$.

Solution OK, we set $f(x) = 1/x$. As we have seen above, the slope's value, m_{tan}, is given by

$$m_{tan} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}.$$

Remember to simplify this ratio as much as possible (without the "lim" symbol). For $h \neq 0$ we have,

$$\begin{aligned}
\frac{f(2+h)-f(2)}{h} &= \frac{\frac{1}{(2+h)}-\frac{1}{2}}{h}\\
&= \frac{\frac{2}{2(2+h)}-\frac{(2+h)}{2(2+h)}}{h}\\
&= \frac{2-(2+h)}{2h(2+h)}\\
&= \frac{-h}{2h(2+h)}\\
&= \frac{-1}{2(2+h)}, \quad \textit{since } h \neq 0.
\end{aligned}$$

Since this is true for each $h \neq 0$, we can pass to the limit to find,

$$\begin{aligned}
m_{tan} &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\\
&= \lim_{h\to 0}\frac{-1}{2(2+h)}\\
&= -\frac{1}{4}.
\end{aligned}$$

Leonardo da Vinci, 1452-1519, who has appeared in a recent film on Cinderella, is the ideal of the Italian *Risorgimento*, the Renaissance: Painter, inventor, scientist, engineer, mathematician, pathologist etc., he is widely accepted as a universal genius, perhaps the greatest ever. What impresses me the most about this extremely versatile man is his ability to assimilate nature into a quantifiable whole, his towering mind, and his insatiable appetite for knowledge. He drew the regular polytopes (three-dimensional equivalents of the regular polygons) for his friend **Fra Luca Pacioli**, priest and mathematician, who included the hand-drawn sketches at the end of the original manuscript of his book on the golden number entitled *De divina proportione*, published in 1509, and now in Torino, Italy.

Example 64. We give examples of the following:

- **a)** A function f whose derivative does not exist (as a two-sided limit).
- **b)** A function f with a vertical tangent line to its graph $y = f(x)$ at $x = 0$, ('infinite' derivative at $x = 0$, i.e., both one-sided limits of the derivative exist but are infinite).
- **c)** A function f with a horizontal tangent line to its graph $y = f(x)$ at $x = 0$, (the derivative is equal to zero in this case).

Solution a) Let

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This function is the same as $f(x) = |x|$, the absolute value of x, right? The idea is that in order for the two-sided (or ordinary) limit of the "derivative" to exist at some point, it is necessary that both one-sided limits (from the right and the left) each exist and both be equal, remember? The point is that **this function's derivative has both one-sided limits existing at** $x = 0$ **but unequal.** Why? Let's use Table 3.1 and try to find its "limit from the right" at x = 0.

For this we suspect that we need $h > 0$, as we want the limit from the **right**, and we're using the same notions of right and left limits drawn from the theory of continuous functions.

$$\begin{aligned}
\frac{f(0+h)-f(0)}{h} &= \frac{f(h)-f(0)}{h}\\
&= \frac{h-0}{h} \quad (\text{because } \ f(h) = h \text{ if } h > 0)\\
&= 1, \quad (\text{since } h \neq 0).
\end{aligned}$$

This is true for each possible value of $h > 0$. So,

$$\lim_{h \to 0+} \frac{f(h) - f(0)}{h} = 1,$$

and so this limit from the right, also called the **right derivative of f at $x = 0$**, exists and is equal to 1. This also means that as $h \to 0$, the slope of the tangent line to the graph of $y = |x|$ approaches the value 1.

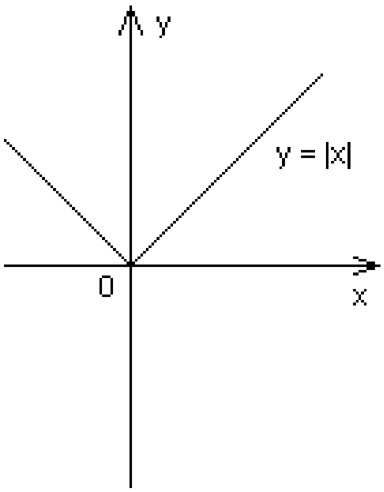


Figure 35.

OK, now let's find its limit from the **left** at $x = 0$. For this we want $h < 0$, right? Now

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{f(h) - f(0)}{h} \\ &= \frac{-h - 0}{h} \quad \text{(because } f(h) = -h \text{ if } h < 0) \\ &= -1 \quad (\text{since } h \neq 0) \end{aligned}$$

This is true for each possible value of $h < 0$. So,

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = -1.$$

This, so-called, **left-derivative of f at** $x = 0$ exists and its value is -1, a different value than 1 (which is the value of the right derivative of our f at $x = 0$). Thus

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist as a two-sided limit. The graph of this function is shown in Figure 35. Note the cusp/ sharp point/v-shape at the origin of this graph. This graphical phenomenon guarantees that the derivative does not exist there.

Note that there is **no uniquely defined tangent line** at $x = 0$ (as **both** $y = x$ and $y = -x$ should qualify, so there is no actual "tangent line").

Solution b) We give an example of a function whose derivative is infinite at $x = 0$, say, so that its tangent line is $x = 0$ (if its derivative is infinite at $x = x_0$, then its tangent line is the vertical line $x = x_0$).

SIMPLIFY first, then GO to the LIMIT

Define f by

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -\sqrt{-x}, & x < 0. \end{cases}$$

The graph of f is shown in Figure 36.

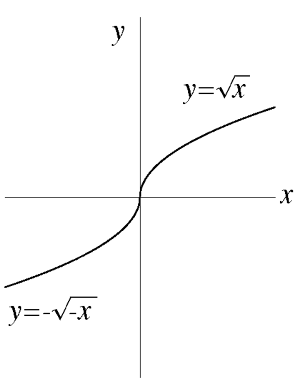


Figure 36.

Let's calculate its left- and right-derivative at $x = 0$. For $h < 0$, at $x_0 = 0$,

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{f(h) - f(0)}{h} \\ &= \frac{-\sqrt{-h} - 0}{h} \quad (\text{ because } f(h) = -\sqrt{-h} \text{ if } h < 0) \\ &= \frac{-\sqrt{-h}}{-(-h)} \\ &= \frac{1}{\sqrt{-h}}. \end{aligned}$$

So we obtain,

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0^-} \frac{1}{\sqrt{-h}} \\ &= +\infty, \end{aligned}$$

$f'(x_0)$	Tangent Line Direction	Remarks
$+$	$\swarrow \quad \nearrow$	"rises", bigger means steeper up
$-$	$\nwarrow \quad \searrow$	"falls", smaller means steeper down
0	$\longleftrightarrow$	horizontal tangent line
$\pm\infty$	$\updownarrow$	vertical tangent line

Table 3.2: Geometrical Properties of the Derivative

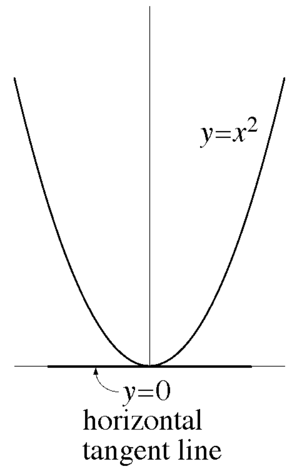


Figure 37.

and, similarly, for $h > 0$,

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+} \frac{1}{\sqrt{h}} \\ &= +\infty. \end{aligned}$$

Finally, we see that

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

both exist and are equal to $+\infty$.

Note: The line $x = 0$ acts as the 'tangent line' to the graph of f at $x = 0$.

Solution c) For an example of a function with a horizontal tangent line at some point (*i.e.* $f'(x) = 0$ at, say, $x = 0$) consider f defined by $f(x) = x^2$ at $x = 0$, see Figure 37. Its derivative $f'(0)$ is given by

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$$

and since the derivative of f at $x = 0$ is equal to the slope of the tangent line there, it follows that the tangent line is horizontal, and given by $y = 0$.

Example 65. On the surface of our moon, an object P falling from rest will fall a distance of approximately $5.3t^2$ feet in t seconds. Find its instantaneous velocity at $t = a$ sec, $t = 1$ sec, and at $t = 2.6$ seconds.

Solution We'll need to calculate its instantaneous velocity, let's call it, "v", at $t = a$ seconds. Since, in this case, $f(t) = 5.3t^2$, we have, by definition,

$$v = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Now, for $h \neq 0$,

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{5.3(a+h)^2 - 5.3a^2}{h} \\ &= \frac{5.3a^2 + 10.6ah + 5.3h^2 - 5.3a^2}{h} \\ &= 10.6a + 5.3h \end{aligned}$$

So,

$$v = \lim_{h\to 0} (10.6a + 5.3h) = 10.6a$$

$f'(x_0)$ does not exist (right and left derivatives not equal)

$f'(x_1) = 0$, (horizontal tangent line).

$f'(x_2)$ does not exist (left derivative at $x = x_2$ is infinite).

$f'(x_3) < 0$, (tangent line "falls")

$f'(x_4) = 0$, (horizontal tangent line)

$f'(x_5) > 0$, (tangent line "rises").

Table 3.3: Different Derivatives in Action: See Figure 38

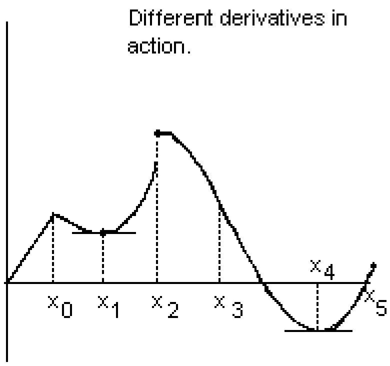
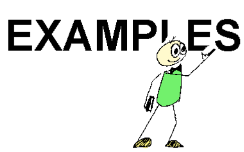


Figure 38.

feet per second. It follows that its instantaneous velocity at $t = 1$ second is given by $(10.6) \cdot (1) = 10.6$ feet per second, obtained by setting $a = 1$ in the formula for v. Similarly, $v = (10.6) \cdot (2.6) = 27.56$ feet per second. From this and the preceding discussion, you can conclude that an object falling from rest on the surface of the moon will fall at approximately *one-third* the rate it does on earth (neglecting air resistance, here).

Example 66. How long will it take the falling object of Example 65 to reach an instantaneous velocity of 50 feet per second?

Solution We know from Example 65 that $v = 10.6a$, at $t = a$ seconds. Since, we want $10.6a = 50$ we get $a = \frac{50}{10.6} = 4.72$ seconds.



Example 67. Different derivatives in action, see Figure 38, and Table 3.3.

Example 68. Evaluate the derivative of the function f defined by $f(x) = \sqrt{5x+1}$ at $x = 3$.

Solution By definition,

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}.$$

Now, we **try to simplify as much as possible before passing to the limit**. For $h \neq 0$,

$$\begin{aligned} \frac{f(3+h) - f(3)}{h} &= \frac{\sqrt{5(3+h)+1} - \sqrt{5(3)+1}}{h} \\ &= \frac{\sqrt{16+5h} - 4}{h}. \end{aligned}$$

Now, to simplify this last expression, we *rationalize* the numerator (by multiplying both the numerator *and* denominator by $\sqrt{16+5h} + 4$). Then we'll find,

Think BIG here: Remember that rationalization gives

$$\sqrt{\Box} - \sqrt{\triangle} = \frac{\Box - \triangle}{\sqrt{\Box} + \sqrt{\triangle}}$$

for any two positive symbols, $\Box, \triangle$.

$$\begin{aligned} \frac{\sqrt{16+5h} - 4}{h} &= \{\frac{\sqrt{16+5h} - 4}{h}\}\{\frac{\sqrt{16+5h} + 4}{\sqrt{16+5h} + 4}\} \\ &= \frac{16 + 5h - 16}{h(\sqrt{16+5h} + 4)} \\ &= \frac{5}{\sqrt{16+5h} + 4}, \quad \textit{since } h \neq 0. \end{aligned}$$

We can't simplify this any more, and now the expression "looks good" if we set $h = 0$ in it, so we can pass to the limit as $h \to 0$, to find,

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{5}{\sqrt{16+5h}+4} \\ &= \frac{5}{\sqrt{16}+4} \\ &= \frac{5}{8}. \end{aligned}$$

Summary

The derivative of a function f at a point $x = a$, (or $x = x_0$), denoted by $f'(a)$, or $\frac{df}{dx}(a)$, or $Df(a)$, is defined by the two equivalent definitions

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x-a}. \end{aligned}$$

whenever either limit exists (in which case so does the other). You get the second definition from the first by setting $h = x - a$, so that the statement "$h \to 0$" is the same as "$x \to a$".

SIMPLIFY first, then GO to the LIMIT

The **right-derivative** (resp. **left-derivative**) is defined by the right- (resp. left-hand) limits

$$\begin{aligned} f'_+(a) &= \lim_{h\to 0+} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{x\to a+} \frac{f(x) - f(a)}{x-a}, \end{aligned}$$

and

$$\begin{aligned} f'_-(a) &= \lim_{h\to 0-} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{x\to a-} \frac{f(x) - f(a)}{x-a}. \end{aligned}$$

NOTES

Exercise Set 10.

Evaluate the following limits

1. $\displaystyle\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$ where $f(x)=x^2$
2. $\displaystyle\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$ where $f(x)=|x|$
3. $\displaystyle\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ where $f(x)=\begin{cases} 0 & x=0 \\ \frac{1}{x} & x\neq 0\end{cases}$

4.a) $\displaystyle\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}$

b) $\displaystyle\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$ where $f(x)=\begin{cases} x+1 & x\geq 1 \\ x & 0\leq x<1\end{cases}$

5. $\displaystyle\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$ where $f(x)=\sqrt{x}$
   HINT : Rationalize the numerator and simplify.
6. $\displaystyle\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}$ where $f(x)=-x^2$

Find the slope of the tangent line to the graph of f at the given point.

7. $f(x)=3x+2$ at $x=1$
8. $f(x)=3-4x$ at $x=-2$
9. $f(x)=x^2$ at $x=3$
10. $f(x)=|x|$ at $x=1$
11. $f(x)=x|x|$ at $x=0$
    HINT: Consider the left and right derivatives separately.
12. $f(x)=\begin{cases} 1 & x\geq 0 \\ 0 & x<0\end{cases}$ (at $x=0$. Remember Heaviside's function?)

Determine whether or not the following functions have a derivative at the indicated point. Explain.

13. $f(x)=\begin{cases} 1 & x\geq 0 \\ -1 & x<0\end{cases}$ at $x=0$
14. $f(x)=\sqrt{x+1}$ at $x=-1$
    HINT: Graphing this function may help.
15. $f(x)=|x^2|$ at $x=0$
16. $f(x)=\sqrt{6-2x}$ at $x=1$
17. $f(x)=\frac{1}{x^2}$ at $x=1$
18. A function f is defined by

$$f(x)=\begin{cases} x & 0\leq x<1 \\ x+2 & 1\leq x<2 \\ 8-x^2 & 2\leq x<3.\end{cases}$$

a) What is $f'(1)$? Explain.

b) Does $f'(2)$ exist? Explain.

c) Evaluate $f'(\frac{5}{2})$.

NOTES:

3.2 Working with Derivatives

By now you know how to find the derivative of a given function (and you can actually check to see whether or not it **has** a derivative at a given point). You also understand the relationship between the derivative and the slope of a tangent line to a given curve (otherwise go to Section 3.1).

Sometimes it is useful to define the derivative $f'(a)$ of a given function at $x = a$ as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.1}$$

provided the (two-sided) limit exists and is finite. Do you see why this definition is equivalent to

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}?$$

Simply replace the symbol "h" by "$x - a$" and simplify. As $h \to 0$ it is necessary that $x - a \to 0$ or $x \to a$.

Notation
When a given function f has a derivative at $x = a$ we say that "f is differentiable at $x = a$" or briefly "f is differentiable at a."

If f is differentiable at every point x of a given interval, **I**, we say that "f is differentiable on **I**."

Example 69. The function f defined by $f(x) = x^2$ is differentiable everywhere on the real line (i.e., at each real number) and its derivative at x is given by $f'(x) = 2x$.

Example 70. **The Power Rule**. The function g defined by $g(x) = x^n$ where $n \geq 0$ is any given integer is differentiable at every point x. If $n < 0$ then it is differentiable everywhere except at $x = 0$. Show that its derivative is given by

$$\frac{d}{dx}\, x^n = nx^{n-1}.$$

Solution We need to recall the *Binomial Theorem:* This says that

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n$$

for some integer n whenever $n \geq 1$, (there are $(n+1)$ terms in total). From this we get the well-known formulae

$$\begin{aligned} (x+h)^2 &= x^2 + 2xh + h^2, \\ (x+h)^3 &= x^3 + 3x^2h + 3xh^2 + h^3, \\ (x+h)^4 &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4. \end{aligned}$$

OK, by definition (and the Binomial Theorem), for $h \neq 0$,

$$\begin{aligned} \frac{g(x+h) - g(x)}{h} &= \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ &= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}. \end{aligned}$$

Since $n \geq 1$ it follows that (because the limit of a sum is the sum of the limits),

$$\begin{aligned}
\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} &= \lim_{h\to 0}(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}) \\
&= nx^{n-1} + \lim_{h\to 0}\left(\frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}\right) \\
&= nx^{n-1} + 0 \\
&= nx^{n-1}.
\end{aligned}$$

Thus $g'(x)$ exists and $g'(x) = nx^{n-1}$.

Remark! Actually, more is true here. It is the case that **for every number** 'a' (integer or not), but a is NOT a variable like 'x, $\sin x$, ...',

$$\boxed{\frac{d}{dx}x^a = ax^{a-1} \text{ if } x > 0.}$$

This formula is useful as it gives a simple expression for the derivative of any power of the independent variable, in this case, 'x'.

QUICKIES

a) $f(x) = x^3$; $f'(x) = 3x^{3-1} = 3x^2$
b) $f(t) = \frac{1}{t} = t^{-1}$, so $f'(t) = (-1)t^{-2} = -\frac{1}{t^2}$
c) $g(z) = \frac{1}{z^2} = z^{-2}$, so $g'(z) = (-2)z^{-3} = -\frac{2}{z^3}$
d) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, so $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
e) $f(x) = x^{-\frac{2}{3}}$; $f'(x) = -\frac{2}{3}x^{-\frac{5}{3}}$
f) $f(x) =$ constant, $f'(x) = 0$

Quick summary

Notation:

1. By cf we mean the function whose values are given by

$$(cf)(x) = cf(x)$$

where c is a constant.

2. By the symbols $f + g$ we mean the function whose values are given by

$$(f+g)(x) = f(x) + g(x)$$

A function f is said to be **differentiable at the point a** if its derivative $f'(a)$ exists there. This is equivalent to saying that both the left- and right-hand derivatives exist at a and are equal. A function f is said to be **differentiable everywhere** if it is differentiable at every point a of the real line.

For example, the function f defined by the absolute value of x, namely, $f(x) = |x|$, is differentiable at every point except at $x = 0$ where $f'_-(0) = -1$ and $f'_+(0) = 1$. On the other hand, the function g defined by $g(x) = x|x|$ is differentiable everywhere. Can you show this ?

Properties of the Derivative

Let f, g be two differentiable functions at x and let c be a constant. Then cf, $f \pm g$, fg are all differentiable at x and

a) $$\frac{d}{dx}(cf) = c\frac{df}{dx} = cf'(x), \qquad \text{c is a constant.}$$

b) $$\begin{aligned} \frac{d}{dx}(f \pm g) &= \frac{df}{dx} \pm \frac{dg}{dx}, \qquad \textbf{Sum/Difference Rule} \\ &= f'(x) \pm g'(x) \end{aligned}$$

c) $$\frac{d}{dx}(fg) = f'(x)\ g(x) + f(x)\ g'(x), \qquad \textbf{Product Rule}$$

d) If for some x, the value $g(x) \neq 0$ then $\frac{f}{g}$ is differentiable at x and

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'(x)\ g(x) - f(x)\ g'(x)}{g^2(x)}, \qquad \textbf{Quotient Rule}$$

where all the derivatives are evaluated at the point 'x'. Hints to the *proofs* or *verification* of these basic Rules may be found at the end of this section. They are left to the reader as a Group Project.

Note that the formula

$$\frac{d}{dx}(fg) = \frac{df}{dx}\frac{dg}{dx}$$

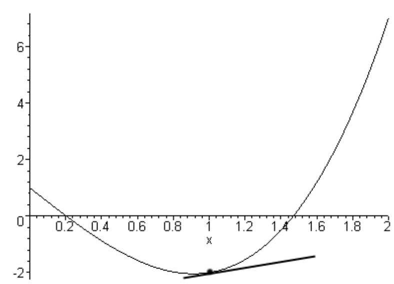
is **NOT TRUE** in general. For example, if $f(x) = x$, $g(x) = 1$, then $f(x)g(x) = x$ and so $(fg)'(x) = 1$. On the other hand, $f'(x)g'(x) = 0$, and so this formula cannot be true.

Example 71. Find the derivative, $f'(x)$ of the function f defined by $f(x) = 2x^3 - 5x + 1$. What is its value at $x = 1$?

Solution We use Example 70 and Properties (a) and (b) to see that

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^3) + \frac{d}{dx}(-5x) + \frac{d}{dx}(1) = 2\frac{d}{dx}(x^3) + (-5)\cdot\frac{d}{dx}(x) + 0 \\ &= 2\cdot 3x^2 + (-5)\cdot 1 \\ &= 6x^2 - 5. \end{aligned}$$

So, $f'(x) = 6x^2 - 5$ and thus the derivative evaluated at $x = 1$ is given by $f'(1) = 6\cdot(1)^2 - 5 = 6 - 5 = 1$.



The tangent line at $x = 1$ to the curve $f(x)$ defined in Example 71.

Figure 39.

Example 72. Given that $f(x) = \sqrt[3]{x} + \sqrt[3]{2} - 1$ find $f'(x)$ at $x = -1$.

Solution We rewrite all "roots" as powers and then use the Power Rule. So,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\sqrt[3]{x} + \sqrt[3]{2} - 1\right) \\ &= \frac{d}{dx}\left(x^{1/3} + 2^{1/3} - 1\right) = \frac{d}{dx}\left(x^{1/3}\right) + \frac{d}{dx}\left(2^{1/3} - 1\right) \\ &= \frac{1}{3}x^{\frac{1}{3}-1} + 0 - 0 \\ &= \frac{1}{3}x^{-\frac{2}{3}}. \end{aligned}$$

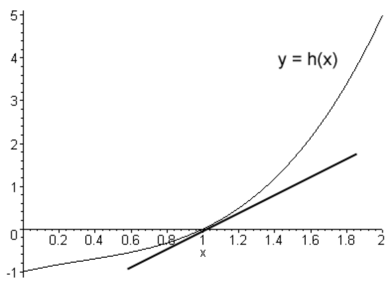
Finally, $f'(-1) = \frac{1}{3}(-1)^{-\frac{2}{3}} = \frac{1}{3}\left(\sqrt[3]{-1}\right)^{-2} = \frac{1}{3}(-1)^{-2} = -\frac{1}{3}$.

Example 73. Find the slope of the tangent line to the curve defined by the function $h(x) = (x^2+1)(x-1)$ at the point $(1,0)$ on its graph.

Solution Using the geometrical interpretation of the derivative (cf., Table 3.1), we know that this slope is equal to $h'(1)$. So, we need to calculate the derivative of h and then evaluate it at $x = 1$. Since h is made up of two functions we can use the Product Rule (Property (c), above). To this end we write $f(x) = (x^2+1)$ and $g(x) = x-1$. Then $h(x) = f(x)g(x)$ and we want $h'(x)$. So, using the Product Rule we see that

$$\begin{aligned} h'(x) &= \frac{d}{dx}f(x)g(x) = f'(x)\ g(x) + f(x)\ g'(x) \\ &= \frac{d}{dx}\left(x^2+1\right)\cdot(x-1) + (x^2+1)\cdot\frac{d}{dx}(x-1) \\ &= (2x+0)\cdot(x-1) + (x^2+1)\cdot(1-0) = 2x(x-1)+x^2+1 \\ &= 3x^2-2x+1. \end{aligned}$$

The required slope is now given by $h'(1) = 3-2+1 = 2$. See Figure 40.



The graph of the function h and its tangent line at $x = 1$. The slope of this straight line is equal to 2

Figure 40.

Example 74. Find the equation of the tangent line to the curve defined by

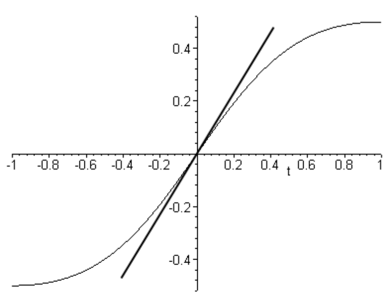
$$h(t) = \frac{t}{t^2+1}$$

at the point $(0,0)$ on its graph.

Solution Since the function h is a quotient of two functions we may use the Quotient Rule, (d). To this end, let $f(t) = t$ and $g(t) = t^2+1$. The idea is that we have to find $h'(t)$ at $t = 0$ since this will give the slope of the tangent line at $t = 0$, and then use the general equation of a line in the form $y = mx+b$ in order to get the actual equation of our tangent line passing through $(0,0)$. OK, now

$$\begin{aligned} h'(t) &= \frac{f'(t)\ g(t) - f(t)\ g'(t)}{g^2(t)} \\ &= \frac{(1)\cdot(t^2+1)-(t)\cdot(2t)}{(t^2+1)^2} \\ &= \frac{1-t^2}{(t^2+1)^2}. \end{aligned}$$

Next, it is clear that $h'(0) = 1$ and so the tangent line must have the equation $y = x+b$ for an appropriate point (x,y) on it. But $(x,y) = (0,0)$ is on it, by hypothesis. So, we set $x = 0, y = 0$ in the general form, solve for b, and conclude that $b = 0$. Thus, the required equation is $y = x+0 = x$, *i.e.*, $y = x$, see Figure 41.



The tangent line $y = x$ through $(0,0)$ for the function h in Example 74.

Figure 41.

Example 75. At which points on the graph of $y = x^3+3x$ does the tangent line have slope equal to 9?

Solution This question is not as direct as the others, above. The idea here is to find the expression for the derivative of y and then set this expression equal to 9 and then solve for x. Now, $y'(x) = 3x^2+3$ and so $9 = y'(x) = 3x^2+3$ implies that $3x^2 = 6$ or $x = \pm\sqrt{2}$. Note the *two* roots here. So there are two points on the required graph where the slope is equal to 9. The $y-$coordinates are then given by setting $x = \pm\sqrt{2}$ into the expression for y. We find the points $(\sqrt{2}, 5\sqrt{2})$ and $(-\sqrt{2}, -5\sqrt{2})$, since $(\sqrt{2})^3 = 2\sqrt{2}$.

Example 76. If $f(x) = (x^2-x+1)(x^2+x+1)$ find $f'(0)$ and $f'(1)$.

Solution Instead of using the Product Rule we can simply expand the product noting that $f(x) = (x^2 - x + 1)(x^2 + x + 1) = x^4 + x^2 + 1$. So, $f'(x) = 4x^3 + 2x$ by the Power Rule, and thus, $f'(0) = 0$, $f'(1) = 6$.

Exercise Set 11.

Find the derivative of each of the following functions using any one of the Rules above: Show specifically which Rules you are using at each step. There is no need to simplify your final answer.

Example: If $f(x) = \dfrac{x^{0.3}}{x+1}$, then

$$\begin{aligned} f'(x) &= \frac{D(x^{0.3})(x+1) - x^{0.3}D(x+1)}{(x+1)^2}, \quad \text{by the Quotient Rule,} \\ &= \frac{(0.3)x^{-0.7}\dot{(}x+1) - x^{0.3}\dot{(}1)}{(x+1)^2}, \quad \text{by the Power Rule with } a = 2/3 \\ &= \frac{(0.3)x^{-0.7}\dot{(}x+1) - x^{0.3}}{(x+1)^2}. \end{aligned}$$

1. $f(x) = x^{1.5}$
2. $f(t) = t^{-2}$
3. $g(x) = 6$
4. $h(x) = x^{\frac{2}{3}}$
5. $k(t) = t^{\frac{1}{5}}$
6. $f(x) = 4.52$
7. $f(t) = t^4$
8. $g(x) = x^{-3}$
9. $f(x) = x^{-1}$
10. $f(x) = x^{\pi}$
11. $f(t) = t^2 - 6$
12. $f(x) = 3x^2 + 2x - 1$
13. $f(t) = (t-1)(t^2+4)$
14. $f(x) = \sqrt{x}\left(3x^2 + 1\right)$
15. $f(x) = \dfrac{x^{0.5}}{2x+1}$
16. $f(x) = \dfrac{x-1}{x+1}$
17. $f(x) = \dfrac{x^3 - 1}{x^2 + x - 1}$
18. $f(x) = \dfrac{x^{\frac{2}{3}}}{\sqrt{x} + 3x^{\frac{3}{4}}}$

Group Project on Differentiation

Prove the Differentiation Rules in Section 3.2 using the definition of the derivative as a limit, the limit properties in Table 2.4, and some basic algebra. Assume throughout that f and g are differentiable at x and $g(x) \neq 0$. In order to prove the Properties proceed as follows using the hints given:

1. **Property a)**

   Show that for any real number c, and $h \neq 0$, we have

   $$(cf)'(x) = c \times \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

   and complete the argument.
2. **Property b) The Sum/Difference Rule:** Show that for a given x and $h \neq 0$,

   $$\frac{(f+g)(x+h)-(f+g)(x)}{h} = \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}.$$

   Then use Table 2.4, a) and the definition of the derivatives.
3. **Property c) The Product Rule:** Show that for a given x and $h \neq 0$,

   $$\frac{(fg)(x+h)-(fg)(x)}{h} = g(x)\frac{f(x+h)-f(x)}{h} + f(x+h)\frac{g(x+h)-g(x)}{h}.$$

   Then use Table 2.4 e),the definition of the derivatives, and the continuity of f at x.
4. **Property d) The Quotient Rule:** First, show that for a given x and any h,

   $$\left(\frac{f}{g}\right)(x+h) - \left(\frac{f}{g}\right)(x) = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}.$$

   Next, rewrite the previous expression as

   $$\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} = \frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)},$$

   and then rewrite it as,

   $$\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)} = \frac{(f(x+h)-f(x))g(x)-f(x)(g(x+h)-g(x))}{g(x+h)g(x)}.$$

   Now, let $h \to 0$ and use Table 2.4, d) and e), the continuity of g at x, and the definition of the derivatives.

Suggested Homework Set 8. *Problems 1, 3, 6, 8, 18*

NOTES:

3.3 The Chain Rule

This section is about a method that will enable you to find the derivative of complicated looking expressions, with some speed and simplicity. After a few examples you'll be using it without much thought ... It will become very natural. Many examples in nature involve variables which depend upon other variables. For example, the speed of a car depends on the amount of gas being injected into the carburator, and this, in turn depends on the diameter of the injectors, etc. In this case we could ask the question: "How does the speeed change if we vary the size of the injectors only ?" and leave all the other variables the same. We are then led naturally to a study of the *composition* (not the same as the product), of various functions and their derivatives.

We recall the **composition of two functions**, (see Chapter 1), and the **limit-definition of the derivative** of a given function from Section 3.2. First, let's see if we can discover the form of the Rule that finds the derivative of the composition of two functions in terms of the individual derivatives. That is, we want an explicit Rule for finding

$$\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx} f(g(x)),$$

in terms of f and $g(x)$.

We assume that f and g are both differentiable at some point that we call x_0 (and so g is also *continuous* there). Furthermore, we must assume that the range of g is contained in the domain of f (so that the composition *makes sense*). Now look at the quantity

$$k(x) = f(g(x)),$$

which is just shorthand for this composition. We want to calculate $k'(x_0)$. So, we need to examine the expression

$$\frac{k(x_0+h)-k(x_0)}{h} = \frac{f(g(x_0+h))-f(g(x_0))}{h},$$

and see what happens when we let $h \to 0$. Okay, now let's **assume that g is not identically a constant function near $x = x_0$**. This means that $g(x) \neq g(x_0)$ for any x in a small interval around x_0. Now,

$$\begin{aligned} \frac{k(x_0+h)-k(x_0)}{h} &= \frac{f(g(x_0+h))-f(g(x_0))}{h} \\ &= \frac{f(g(x_0+h))-f(g(x_0))}{g(x_0+h)-g(x_0)} \cdot \frac{g(x_0+h)-g(x_0)}{h}. \end{aligned}$$

As $h \to 0$, $g(x_0+h) \to g(x_0)$ because g is continuous at $x = x_0$. Furthermore,

$$\frac{g(x_0+h)-g(x_0)}{h} \to g'(x_0),$$

since g is differentiable at the point $x = x_0$. Lastly,

$$\frac{f(g(x_0+h))-f(g(x_0))}{g(x_0+h)-g(x_0)} \to f'(g(x_0))$$

since f is differentiable at $x = g(x_0)$ (use definition 3.1 with $x = g(x_0+h)$ and $a = g(x_0)$ to see this). It now follows by the theory of limits that

$$\begin{aligned}
\lim_{h\to 0}\frac{k(x_0+h)-k(x_0)}{h} &= \lim_{h\to 0}\frac{f(g(x_0+h))-f(g(x_0))}{g(x_0+h)-g(x_0)}\cdot\frac{g(x_0+h)-g(x_0)}{h},\\
&= \lim_{h\to 0}\left(\frac{f(g(x_0+h))-f(g(x_0))}{g(x_0+h)-g(x_0)}\right)\times\\
&\quad\times\cdot\lim_{h\to 0}\left(\frac{g(x_0+h)-g(x_0)}{h}\right),\\
&= f'(g(x_0))\cdot g'(x_0),\\
&= k'(x_0).
\end{aligned}$$

In other words we can believe that

$$k'(x_0)=f'(g(x_0))\cdot g'(x_0),$$

and this is the formula we wanted. It's called the **Chain Rule**.

> **The Chain Rule: Summary**
>
> Let f, g be two differentiable functions with g differentiable at x and $g(x)$ in the domain of f'. Then $y = f\circ g$ is differentiable at x and
>
> $$\frac{d}{dx}(f\circ g)(x)=\frac{d}{dx}f(g(x))=f'(g(x))\cdot g'(x)$$

The Chain Rule also says

$$\boxed{Df(\Box)=f'(\Box)\ D\Box},$$

where "Df = df/dx = f'(x). You can read this as: "Dee of f of box is f prime box dee box". We call this the **Box formulation of the Chain Rule.**

Let's see what this means. When the composition $(f\circ g)$ is defined (and the range of g is contained in the domain of f') then $(f\circ g)'$ exists and

$$\begin{aligned}
\frac{d}{dx}(f\circ g)(x) &= \frac{df}{dx}(g(x))\cdot g'(x)\\
\underbrace{\frac{d}{dx}f(g(x))}_{\text{derivative of composition}} &= \underbrace{f'(g(x))\ \cdot g'(x)}_{\text{derivative of } f \text{ at } g(x)\ \cdot\ \text{derivative of } g \text{ at } x}
\end{aligned}$$

In other words, the derivative of a composition is found by differentiating the *outside* function first, (here, f), evaluating its derivative, (here f'), at the *inside* function, (here, $g(x)$), and finally multiplying this number, $f'(g(x))$, by the derivative of g at x.

The **Chain Rule** is one of the most useful and important rules in the theory of differentiation of functions as it will allow us to find the derivative of very complicated-looking expressions with ease. For example, using the Chain Rule we'll be able to show that

$$\frac{d}{dx}(x+1)^3=3(x+1)^2.$$

Without using the Chain Rule, the alternative is that we have to expand

$$(x+1)^3=x^3+3x^2+3x+1$$

using the Binomial Theorem and then use the Sum Rule along with the Power Rule to get the result which, incidentally, is identical to the stated one since

$$\frac{d}{dx}(x^3+3x^2+3x+1)=3x^2+6x+3=3(x+1)^2.$$

$\boxed{\text{An easy way to remember the Chain Rule is as follows:}}$

Replace the symbol "$g(x)$" by our box symbol, $\Box$. Then the Chain Rule says that

$$\underbrace{\frac{d}{dx} f(\Box)}_{\text{derivative of a composition}} = \underbrace{f'(\Box)\cdot \Box'}_{\text{derivative of } f \text{ at } \Box \,\cdot\, \text{derivative of } \Box \text{ at } x}$$

Symbolically, it can be shortened by writing that

$$\boxed{Df(\Box) = f'(\Box)\; D\Box, \qquad \text{The Chain Rule}}$$

where the $\Box$ may represent (or even *contain*) any other function(s) you wish. In words, it can be remembered by saying that the

$$\boxed{\text{Derivative of f of Box is f prime Box dee-Box}}$$

like a famous brand name for "sneakers", (*i.e.* 'dee-Box').

Consequences of the Chain Rule!

It's **NOT TRUE** that

$$Df(g(x)) = f'(x)\, g'(x),$$

Let g be a differentiable function with $g(x) \neq 0$. Then $\frac{1}{g}$ is differentiable and by the Quotient Rule,

1. $\displaystyle \frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-1}{(g(x))^2}\cdot g'(x)$, or,
2. $\displaystyle \frac{d}{dx}(g(x))^a = a(g(x))^{a-1}\cdot g'(x)$ The Generalized Power Rule

whenever a is a real number and $g(x) > 0$. This Generalized Power Rule follows easily from the Chain Rule, above, since we can let $f(x) = x^a$, $g(x) = \Box$. Then the composition $(f \circ g)(x) = g(x)^a = \Box^a$. According to the Chain Rule,

$$\frac{d}{dx} f(\Box) = f'(\Box)\cdot \Box'.$$

But, by the ordinary Power Rule, Example 70, we know that $f'(x) = ax^{a-1}$. Okay, now since $f'(\Box) = a\Box^{a-1}$, and $\Box' = g'(x)$, the Chain Rule gives us the result.

An easy way to remember these formulae, once and for all, is by writing

$$\boxed{\begin{array}{rcll} D\;\Box^{power} & = & power\cdot \Box^{(power)-1}\cdot D\;\Box & \textbf{Generalized Power Rule} \\ D\left(\dfrac{1}{\Box}\right) & = & \dfrac{-1}{(\Box)^2}\cdot D\;\Box, & \textbf{Reciprocal Rule} \end{array}}$$

where $\Box$ may be some differentiable function of x, and we have used the modern notation "D" for the derivative with respect to x. Recall that the **reciprocal** of *something* is, by definition, " 1 divided by that something".

The Chain Rule can take on different forms. For example, let $y = f(u)$ and assume that the variable u is itself a function of another variable, say x, and we write this as $u = g(x)$. So $y = f(u)$ and $u = g(x)$. So y must be a function of x and it is reasonable to expect that y is a *differentiable* function of x if certain additional conditions on f and g are imposed. Indeed, let y be a differentiable function of u

and let u be a differentiable function of x. Then y is, in fact, a differentiable function of x. Now the question is:

"How does y vary with x?" The result looks like this ...

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &\text{or} \\ y'(x) &= f'(u)\cdot g'(x) \end{aligned}$$

where **we must replace all occurrences of the symbol 'u' in the above by the symbol '$g(x)$' after the differentiations are made.**

Sophie Germain, 1776-1831, was the second of three children of a middle-class Parisian family. Somewhat withdrawn, she never married, and by all accounts lived at home where she worked on mathematical problems with a passion. Of the many stories which surround this gifted mathematician, there is this one ... Upon the establishment of the *École Polytechnique* in 1795, women were not allowed to attend the lectures so Sophie managed to get the lecture notes in mathematics by befriending students. She then had some great ideas and wrote this big essay called a *memoire* and then submitted it (under a male name) to one of the great French mathematcians of the time, **Joseph Lagrange**, 1736-1813, for his advice and opinion. Lagrange found much merit in the work and wished to meet its creator. When he did finally meet her he was delighted that the work had been written by a woman, and went on to introduce her to the great mathematicians of the time. She won a prize in 1816 dealing with the solution of a problem in two-dimensional harmonic motion, yet remained a lone genius all of her life.

Example 77. Let f be defined by $f(x) = 6x^{\frac{1}{2}} + 3$. Find $f'(x)$.

Solution We know $f(x) = 6x^{\frac{1}{2}} + 3$. So, if we let $x = \square$ we get

$$\begin{aligned} f'(x) &= 6\frac{d}{dx}\square^{1/2} + \frac{d}{dx}3 \text{ (by Properties (a) and (b))}, \\ &= 6\cdot\frac{1}{2}\square^{-1/2} + 0, \\ &= 3x^{-1/2} \\ &= \frac{3}{\sqrt{x}}. \end{aligned}$$

Example 78. Let g be defined by $g(t) = t^5 - 4t^3 - 2$. What is $g'(0)$, the derivative of g evaluated at $t = 0$?

Solution

$$\begin{aligned} g'(t) &= \frac{d}{dt}(t^5) - 4\frac{d}{dt}(t^3) - \frac{d}{dt}2 \quad \text{(by Property (b))} \\ &= 5t^4 - 4(3)t^2 - 0 \quad \text{(Power Rule)} \\ &= 5t^4 - 12t^2. \end{aligned}$$

But $g'(0)$ is $g'(t)$ with $t = 0$, right? So, $g'(0) = 5(0)^4 - 12(0)^2 = 0$.

Example 79. Let y be defined by $y(x) = (x^2 - 3x + 1)(2x + 1)$. Evaluate $y'(1)$.

Solution Let $f(x) = x^2 - 3x + 1$, $g(x) = 2x + 1$. Then $y(x) = f(x)g(x)$ and we want $y'(x)$... So, we can use the **Product Rule** (or you can multiply the polynomials out, collect terms and then differentiate each term). Now,

$$\begin{aligned} y'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (2x - 3 + 0)(2x + 1) + (x^2 - 3x + 1)(2 + 0) \\ &= (2x - 3)(2x + 1) + 2(x^2 - 3x + 1), \quad \text{so,} \\ \\ y'(1) &= (2(1) - 3)(2(1) + 1) + 2((1)^2 - 3(1) + 1) \\ &= -5. \end{aligned}$$

Example 80. Let y be defined by $y(x) = \frac{x^2+4}{x^3-4}$. Find the slope of the tangent line to the curve $y = y(x)$ at $x = 2$.

Solution We write $y(x) = \frac{f(x)}{g(x)}$ where $f(x) = x^2 + 4$, $g(x) = x^3 - 4$. We also need $f'(x)$ and $g'(x)$, since the Quotient Rule will come in handy here.

The next Table may be useful as we always need these 4 quantities when using the Quotient Rule:

$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
x^2+4	$2x$	x^3-4	$3x^2$

Now, by the Quotient Rule,

$$\begin{aligned} y'(x) &= \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2} \\ &= \frac{2x(x^3-4)-(x^2+4)(3x^2)}{(x^3-4)^2}. \end{aligned}$$

No need to simplify here. We're really asking for $y'(2)$, right? Why? Think "slope of tangent line $\Rightarrow$ derivative". Thus,

$$\begin{aligned} y'(2) &= \frac{4(4)-8(12)}{16} \\ &= -5 \end{aligned}$$

and the required slope has value -5.

$\boxed{\textbf{Example 81.}}$ Let $y(x)=x^2-\dfrac{6}{x-4}$. Evaluate $y'(0)$.

Solution Now

$$\begin{aligned} y'(x) &= \frac{d}{dx}(x^2)-6\frac{d}{dx}(\frac{1}{x-4}) \\ &\quad \text{(where we used Property (a), the Power Rule, and Consequence 1.)} \\ &= 2x-6\left[\frac{-1}{(x-4)^2}\frac{d}{dx}(x-4)\right] \\ &\quad \text{since } \frac{d}{dx}(\frac{1}{\boxed{\text{x-4}}}) = \frac{d}{dx}\left(\frac{1}{\square}\right) = \left(\frac{-1}{\square^2}\right)\square' \\ &\quad \text{where } \square = (x-4). \end{aligned}$$

All right, now

$$y'(x) = 2x+\frac{6}{(x-4)^2},$$

and so

$$\begin{aligned} y'(0) &= 2(0)+\frac{6}{(-4)^2}, \\ &= \frac{3}{8}. \end{aligned}$$

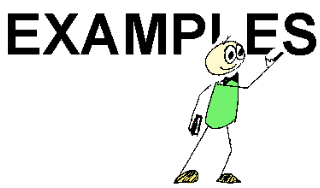
$\boxed{\textbf{Example 82.}}$ Let y be defined by $y(x)=\dfrac{4-x^2}{x^2-2x-3}$. Evaluate $y'(x)$ at $x=1$.

Solution Write $y(x)=\dfrac{f(x)}{g(x)}$. We need $y'(1)$, right ? OK, now we have the table . . .

$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
$4-x^2$	$-2x$	x^2-2x-3	$2x-2$

$$\begin{aligned} y'(1) &= \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} \\ &= \frac{(-2)(1-2-3) - (3)(2-2)}{(1-2-3)^2} \\ &= \frac{8-0}{16} \\ &= \frac{1}{2}. \end{aligned}$$

Example 83. Let y be defined by $y(x) = \dfrac{1.6}{(x+1)^{100}}$. Evaluate $y'(0)$.



Solution Write $y(x) = (1.6)(x+1)^{-100} = 1.6f(x)^{-100}$ where $f(x) = x+1$ (or, replace $f(x)$ by $\square$). Now use Property (a) and Consequence (2) to find that

$$\begin{aligned} y'(x) &= (1.6)(-100)f(x)^{-101} \cdot f'(x) \\ &= -160(x+1)^{-101} \cdot (1) \\ &= \frac{-160}{(x+1)^{101}}, \\ \text{so} \quad y'(0) &= -160. \end{aligned}$$

On the other hand, you could have used Consequence (1) to get the result... For example, write $y(x)$ as $y(x) = \frac{1.6}{f(x)}$ where $f(x) = (x+1)^{100}$. Then

$$\begin{aligned} y'(x) &= \frac{-1.6}{(f(x))^2} \cdot f'(x) \qquad \text{(by Consequence (1))} \\ &= \frac{-1.6}{(x+1)^{200}}(100(x+1)^{99}(1)) \qquad \text{(by Consequence (2))} \\ &= \frac{-160}{(x+1)^{200}}(x+1)^{99} \\ &= \frac{-160}{(x+1)^{101}} \\ \text{and so } y'(0) &= -160, \text{ as before.} \end{aligned}$$

Example 84. Let $y = u^5$ and $u = x^2 - 4$. Find $y'(x)$ at $x = 1$.

Solution Here $f(u) = u^5$ and $g(x) = x^2 - 4$. Now $f'(u) = 5u^4$ by the Power Rule and $g'(x) = 2x$... So,

$$\begin{aligned} y'(x) &= f'(u) \cdot g'(x) \\ &= 5u^4 \cdot 2x \\ &= 10xu^4 \\ &= 10x(x^2-4)^4, \quad \text{since } u = x^2 - 4. \end{aligned}$$

At $x = 1$ we get

$$\begin{aligned} y'(1) &= 10(1)(-3)^4 \\ &= 810. \end{aligned}$$

Since this value is 'large' for a slope the actual tangent line is very 'steep', close to 'vertical' at $x = 1$.

SHORTCUT

Write $y = \Box^5$, then $y' = 5\ \Box^4\ \Box'$, by the Generalized Power Rule. Replacing the $\Box$ by $x^2 - 4$ we find, $y' = 5\ (x^2-4)^4\ 2x = 10x(x^2-4)^4$, as before.

The point is, you don't have to memorize another formula. The "Box" formula basically gives all the different variations of the Chain Rule.

Example 85. Let $y = u^3$ and $u = (x^2 + 3x + 2)$. Evaluate $y'(x)$ at $x = 0$ and interpret your result geometrically.

Solution The Rule of Thumb is:

Whenever you see a function raised to the power of some number (NOT a variable), then put everything "between the outermost parentheses", so to speak, in a box, $\Box$. The whole thing then looks like just a box raised to some power, and you can use the box formulation of the Chain Rule on it.

Chain Rule approach: Write $y = u^3$ where $u = x^2 + 3x + 2$. OK, now $y = f(u)$ and $u = g(x)$ where $f(u) = u^3$ and $g(x) = x^2 + 3x + 2$. Then the Chain Rule gives

$$\begin{aligned} y'(x) &= f'(u)g'(x) \\ &= 3u^2 \cdot (2x+3) \\ &= 3(x^2+3x+2)^2\ (2x+3). \end{aligned}$$

Since $u = x^2 + 3x + 2$, we have to replace each u by the original $x^2 + 3x + 2$. Don't worry, you don't have to simplify this. Finally,

$$\begin{aligned} y'(0) &= 3(3)(2)^2 \\ &= 36 \end{aligned}$$

and this is the slope of the tangent line to the curve $y = y(x)$ at $x = 0$.

Power Rule/Box approach: Write $y = \Box^3$ where $\Box = x^2 + 3x + 2$ and $a = 3$. Then

$$\begin{aligned} y'(x) &= 3\Box^2 \cdot \Box' \\ &= 3(x^2+3x+2)^2(2x+3) \end{aligned}$$

and so $y'(0) = 36$, as before.

Example 86. Let y be defined by $y(x) = (x+2)^2(2x-1)^4$. Evaluate $y'(-2)$.

Solution We have a product and some powers here. So we expect to use a combination of the Product Rule and the Power Rule. OK, we let $f(x) = (x+2)^2$ and $g(x) = (2x-1)^4$, use the Power Rule on f, g, and make the table:

$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
$(x+2)^2$	$2(x+2)$	$(2x-1)^4$	$4(2x-1)^3(2)$.

Using the Product Rule,

$$\begin{aligned} y'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 2(x+2)(2x-1)^4 + 8(x+2)^2(2x-1)^3 \end{aligned}$$

Finally, it is easy to see that $y'(-2) = 0$.

Example 87. Find an expression for the derivative of $y = \sqrt{f(x)}$ where $f(x) > 0$ is differentiable.

Solution OK, this 'square root' is really a power so we think "Power Rule". We can speed things up by using boxes, so write $\Box = f(x)$. Then, the Generalized Power Rule gives us,

$$\begin{aligned} y'(x) &= \frac{d}{dx}\Box^{\frac{1}{2}} \\ &= \frac{1}{2}\Box^{\frac{1}{2}-1} \cdot \Box' \quad \text{(PowerRule)} \\ &= \frac{1}{2}\Box^{-\frac{1}{2}}\Box' \\ &= \frac{1}{2\Box^{\frac{1}{2}}}\Box' \\ &= \frac{\Box'}{2\sqrt{\Box}} \\ &= \frac{f'(x)}{2\sqrt{f(x)}} \end{aligned}$$

SNAPSHOTS

Example 88. $f(x) = (x^{2/3} + 1)^2$, $f'(x)$?

Solution Let $\Box = (x^{2/3} + 1) = x^{2/3} + 1$. So, $f(x) = \Box^2$ and

$$\begin{aligned} f'(x) &= \overbrace{(2) \cdot (x^{2/3} + 1)^1}^{D(\Box^2)} \cdot \overbrace{(2/3) \cdot x^{-1/3}}^{D(\Box)} \\ &= \frac{4}{3} \cdot (x^{2/3} + 1) \cdot x^{-1/3} \\ &= \frac{4}{3} \cdot (x^{1/3} + x^{-1/3}). \end{aligned}$$

Example 89. $f(x) = \sqrt{\sqrt{x} + 1}$. Evaluate $f'(x)$.

Solution Let $\Box = (\sqrt{x} + 1) = x^{1/2} + 1$. Then $f(x) = \sqrt{\Box}$ so

$$\begin{aligned} f'(x) &= \overbrace{(1/2) \cdot (x^{1/2} + 1)^{-1/2}}^{D(\sqrt{\Box})} \cdot \overbrace{(1/2) \cdot x^{-1/2}}^{D(\Box)} \\ &= \frac{1}{4} \cdot (x^{1/2} + 1)^{-1/2} \cdot x^{-1/2} \\ &= \frac{1}{4\sqrt{x \cdot (1 + \sqrt{x})}} \end{aligned}$$

Example 90. $f(x) = \dfrac{\sqrt{x}}{\sqrt{1 + x^2}}$. Find $f'(1)$.

Solution Simplify this first. Note that

$$\frac{\sqrt{x}}{\sqrt{1 + x^2}} = \sqrt{\frac{x}{1 + x^2}} = \Box^{1/2}$$

where $\Box = \dfrac{x}{1+x^2}$. So $f(x) = \sqrt{\Box}$, and

$$\begin{aligned} f'(x) &= \overbrace{(1/2)\cdot(\frac{x}{1+x^2})^{-1/2}}^{D(\sqrt{\Box})} \cdot \overbrace{\frac{(x^2+1)\cdot 1-(x)\cdot(2x)}{(1+x^2)^2}}^{D(\Box)} \\ &= (1/2)\cdot(\frac{x}{1+x^2})^{-1/2}\cdot\frac{1-x^2}{(1+x^2)^2} \\ &= \frac{x^{-1/2}\cdot(1-x^2)}{2\ (1+x^2)^{3/2}}, \end{aligned}$$

where we used the Generalized Power Rule to get $D(\sqrt{\Box})$ and the Quotient Rule to evaluate $D(\Box)$. So, $f'(1) = 0$.

Example 91. $f(x) = \pi \cdot \left(\dfrac{1}{x}\right)^{-2.718}$, where $\pi = 3.14159....$ Find $f'(1)$.

Solution Simplify this first, in the sense that you can turn negative exponents into positive ones by taking the reciprocal of the expression, right? In this case, note that $(1/x)^{-2.718} = x^{2.718}$. So the question now asks us to find the derivative of $f(x) = \pi \cdot x^{2.718}$. The Power Rule gives us $f'(x) = (2.718)\cdot\pi\cdot x^{1.718}$. So, $f'(1) = (2.718)\cdot\pi = 8.53882$.

Example 92. Find an expression for the derivative of $y = f(x^3)$ where f is differentiable.

Solution This looks mysterious but it really isn't. If you don't see an 'x' for the variable, replace all the symbols between the outermost parentheses by '$\Box$'. Then $y = f(\Box)$ and you realize quickly that you need to differentiate a composition of two functions. This is where the Chain Rule comes into play. So,

$$\begin{aligned} y'(x) &= Df(\Box) \\ &= f'(\Box)\cdot D\Box \\ &= f'(x^3)\cdot\frac{d}{dx}(x^3) \text{ (because } \Box = x^3) \\ &= f'(x^3)\cdot 3x^2 \end{aligned}$$

The Generalized Power Rule takes the form

$$\frac{d}{dx}\Box^r = r\ \Box^{r-1}\frac{d}{dx}\Box,$$

where the box symbol, $\Box$, is just another symbol for some differentiable function of x.

So, we have shown that any function f for which $y = f(x^3)$ has a derivative $y'(x) = 3x^2 f'(x^3)$ which is the desired expression. Remember that $f'(x^3)$ means that you find the derivative of f, and every time you see an x you replace it by x^3.

OK, but what does this $f'(x^3)$ really mean?

Let's look at the function f, say, defined by $f(x) = (x^2+1)^{10}$. Since $f(\Box) = (\Box^2+1)^{10}$ it follows that $f(x^3) = ((x^3)^2+1)^{10} = (x^6+1)^{10}$, where we replaced $\Box$ by x^3 (or you put x^3 IN the box, remember the *Box method*?).

The point is that this new function $y = f(x^3)$ has a derivative given by

$$y'(x) = 3x^2\cdot f'(x^3),$$

which means that we find $f'(x)$, replace each one of the x's by x^3, and simplify (as much as possible) to get $y'(x)$. Now, we write $f(\Box) = \Box^{10}$, where $\Box = x^2+1$. The Generalized Power Rule gives us

$$\begin{aligned} f'(x) &= D(\Box^{10}) = (10)\Box^9\ (d\Box) = (10)(x^2+1)^9\ (2x), \\ &= (20x)(x^2+1)^9.\ \textit{So,} \\ y'(x) &= 3x^2\cdot f'(x^3) = (3x^2)\ (20\ x^3)\ (x^6+1)^9 \\ &= 60\ x^5(x^6+1)^9. \end{aligned}$$

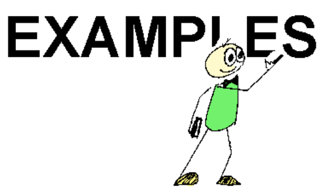
Example 92 represents a, so-called, **transformation of the independent variable** (since the original 'x' is replaced by 'x^3') and such transformations appear within the context of **differential equations** where they can be used to simplify very difficult looking differential equations to simpler ones.

A Short Note on Differential Equations

More importantly though, examples like the last one appear in the study of **differential equations** which are equations which, in some cases called **linear**, look like polynomial equations

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 = 0$$

and each x is replaced by a symbol '$D = \frac{d}{dx}$' where the related symbol $D^n = \frac{\mathrm{d}^n}{dx^n}$ means the operation of taking the n^{th} derivative. This symbol, $D = \frac{d}{dx}$, has a special name: it's called a **differential operator** and its domain is a collection of functions while it range is also a collection of functions. In this sense, the concept of an operator is more general than that of a function. Now, the symbol D^2 is the derivative of the derivative and it is called the **second derivative**; the derivative of the second derivative is called the **third derivative** and denoted by D^3, and so on. The coefficients a_m above are usually given functions of the independent variable, x.



Symbolically, we write these **higher-order derivatives** using Leibniz's notation:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = y''(x)$$

for the **second derivative of y**,

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2} = y'''(x)$$

for the **third derivative of y**, and so on. These higher order derivatives are very useful in determining the graphs of functions and in studying a 'function's behavior'. We'll be seeing them soon when we deal with **curve sketching**.

Example 93. Let f be a function with the property that $f'(x) + f(x) = 0$ for every x. We'll meet such functions later when we discuss Euler's constant, $e \approx 2.71828...$, and the corresponding exponential function.

Show that the new function y defined by $y(x) = f(x^3)$ satisfies the differential equation $y'(x) + 3x^2y(x) = 0$.

Solution We use Example 92. We already know that, by the Chain Rule, $y(x) = f(x^3)$ has its derivative given by $y'(x) = 3x^2 \cdot f'(x^3)$. So,

$$\begin{aligned} y'(x) + 3x^2y(x) &= 3x^2 \cdot f'(x^3) + 3x^2 f(x^3) \\ &= 3x^2(f'(x^3) + f(x^3)) \end{aligned}$$

But $f'(x) + f(x) = 0$ means that $f'(\triangle) + f(\triangle) = 0$, right? (Since it is true for any 'x' and so for any symbol '$\triangle$'). Replacing $\triangle$ by x^3 gives $f'(x^3) + f(x^3) = 0$ as a consequence, and the conclusion now follows ...

The function y defined by $y(x) = f(x^3)$, where f is any function with $f'(x) + f(x) = 0$, satisfies the equation $y'(x) + 3x^2y(x) = 0$.

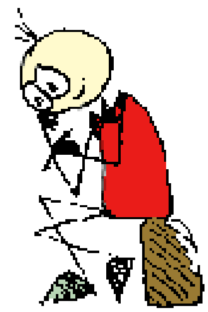
Example 94. Find the second derivative $f''(x)$ given that $f(x) = (2x+1)^{101}$. Evaluate $f''(-1)$.

Solution We can just use the Generalized Power Rule here. Let $\square = 2x+1$. Then $\square' = 2$ and so $f'(x) = 101 \cdot \square^{100} \cdot \square' = 101 \cdot \square^{100} \cdot 2 = 202 \cdot \square^{100}$. Doing this one more time, we find $f''(x) = (202) \cdot (100) \cdot 2 \cdot \square^{99} = 40,400 \cdot \square^{99} = 40,400 \cdot (2x+1)^{99}$.

Finally, since $(-1)^{\text{odd number}} = -1$, we see that $f''(-1) = 40,400 \cdot (-1)^{99} = -40,400$.

Example 95. Find the second derivative $f''(x)$ of the function defined by $f(x) = (1+x^3)^{-1}$. Evaluate $f''(0)$.

Solution Use the Generalized Power Rule again. Let $\square = x^3+1$. Then $\square' = 3x^2$ and so $f'(x) = (-1) \cdot \square^{-2} \cdot \square' = (-1) \cdot \square^{-2} \cdot (3x^2) = -(3x^2) \cdot \square^{-2} = -(3x^2) \cdot (1+x^3)^{-2}$. To find the derivative of THIS function we can use the Quotient Rule. So,



$$\begin{aligned} f''(x) &= -\{\frac{(1+x^3)^2 \cdot (6x) - (3x^2) \cdot (2)(1+x^3)^1 (3x^2)}{(1+x^3)^4}\} \\ &= \frac{-6x}{(1+x^3)^2} + \frac{18x^4}{(1+x^3)^3} \\ &= \frac{6x \cdot (2x^3-1)}{(1+x^3)^3} \end{aligned}$$

It follows that $f''(0) = 0$.

Exercise Set 12.

Find the indicated derivatives.

1. $f(x) = \pi,\ f'(x) = ?$
2. $f(t) = 3t-2,\ f'(0) = ?$
3. $g(x) = x^{\frac{2}{3}},\ g'(x) = ?$ at $x = 1$
4. $y(x) = \sqrt{(x-4)^3},\ y'(x) = ?$
5. $f(x) = \dfrac{1}{\sqrt{x^5}},\ f'(x) = ?$
6. $g(t) = \sqrt[3]{t^2+t-2},\ g'(t) = ?$
7. $f(x) = 3x^2,\ \dfrac{d^2 f}{dx^2} = ?$
8. $f(x) = x(x+1)^4,\ f'(x) = ?$
9. $y(x) = \dfrac{x^2-x+3}{\sqrt{x}},\ y'(1) = ?$
10. $y(t) = (t+2)^2(t-1),\ y'(t) = ?$
11. $f(x) = 16x^2(x-1)^{\frac{2}{3}},\ f'(x) = ?$
12. $y(x) = (2x+3)^{105},\ y'(x) = ?$
13. $f(x) = \sqrt{x}+6,\ f'(x) = ?$
14. $f(x) = x^3-3x^2+3x-1,\ f'(x) = ?$
15. $y(x) = \dfrac{1}{x} + \sqrt{x^2-1},\ y'(x) = ?$
16. $f(x) = \dfrac{1}{1+\sqrt{x}},\ f''(x) = ?$
17. $f(x) = (x-1)^2 + (x-2)^3,\ f''(0) = ?$
18. $y(x) = (x+0.5)^{-1.324},\ y''(x) = ?$

19. Let f be a differentiable function for every real number x. Show that $\frac{d}{dx}f(x^2) = 2xf'(x^2)$.

20. Let g be a differentiable function for every x with $g(x) > 0$. Show that $\frac{d}{dx}\sqrt[3]{g(x)} = \frac{g'(x)}{3\sqrt[3]{g(x)^2}}$.

21. Let f be a function with the property that f is differentiable and

$$f'(x) + f(x) = 0.$$

Show that $y = f(x^2)$ satisfies the differential equation

$$y'(x) + 2xy(x) = 0.$$

22. Let $y = f(x)$ and assume f is differentiable for each x in $(0, 1)$. Assume that f has an inverse function, F, defined on its range, so that $f(F(x)) = x$ for every $x, 0 < x < 1$. Show that F has a derivative satisfying the equation $F'(x) = \frac{1}{f'(F(x))}$ at each $x, 0 < x < 1$.
(**Hint:** Differentiate both sides of $f(F(x)) = x$.)

23. Let $y = t^3$ and $t = \sqrt{u} + 6$. Find $\frac{dy}{du}$ when $u = 9$.

24. Find the equation of the tangent line to the curve $y = (x^2 - 3)^8$ at the point $(x, y) = (2, 1)$.

25. Given $y(x) = f(g(x))$ and that $g'(2) = 1, g(2) = 0$ and $f'(0) = 1$. What is the value of $y'(2)$?

26. Let $y = r + \frac{2}{r}$ and $r = 3t - 2\sqrt{t}$. Use the Chain Rule to find an expression for $\frac{dy}{dt}$.

27. **Hard**. Let $f(x) = \sqrt{x + \sqrt{x}}$. Evaluate $f'(9)$. If $x = t^2$, what is $\frac{df}{dt}$?

28. Use the definition of $\sqrt{x^2}$ as $\sqrt{x^2} = |x|$ for each x, to show that the function $y = |x|$ has a derivative whenever $x \neq 0$ and $y'(x) = \frac{x}{|x|}$ for $x \neq 0$.

29. **Hard** Show that if f is a differentiable at the point $x = x_0$ then f is continuous at $x = x_0$. (Hint: You can try a proof by 'contradiction', that is you assume the conclusion is false and, using a sequence of logically correct arguments, you deduce that the original claim is false as well. Since, generally speaking, a statement in mathematics cannot be both true **and** false, (aside from **undecidable statements**) it follows that the conclusion has to be true. So, assume f is not continuous at $x = x_0$, and look at each case where f is discontinuous (unequal one-sided limits, function value is infinite, etc.) and, **in each case**, derive a contradiction.)

Alternately, you can prove this directly using the methods in the *Advanced Topics* chapter. See the Solution Manual for yet another method of showing this.

Suggested Homework Set 9. *Do problems 4, 10, 16, 23, 25, 27*

Web Links

For more information and applications of the Chain Rule see:

www.math.hmc.edu/calculus/tutorials/chainrule/
www.ugrad.math.ubc.ca/coursedoc/math100/notes/derivative/chainap.html
www.khanacademy.org/math/calculus/v/the-chain-rule

NOTES:

3.4 Implicit Functions and Their Derivatives

You can imagine the variety of different functions in mathematics. So far, all the functions we've encountered had this one thing in common: You could write them as $y = f(x)$ (or $x = F(y)$) in which case you know that x is the independent variable and y is the dependent variable. We know which variable is which. Sometimes it is not so easy to see "which variable is which" especially if the function is written as, say,

$$x^2 - 2xy + \tan(xy) - 2 = 0.$$

What do we do? Can we solve for either one of these variables at all? And if we can, do we solve for x in terms of y, or y in terms of x? Well, we don't always "have to" solve for any variable here, and we'll still be able to find the derivative so long as we agree on which variable x, *or* y is the independent one. Actually, Newton was the first person to perform an *implicit differentiation.* Implicit functions appear very often in the study of **general solutions** of differential equations. We'll see later on that the general solution of a *separable* differential equation is usually given by an implicit function. Other examples of implicit functions include the equation of closed curves in the xy-plane (circles, squares, ellipses, etc. to mention a few of the common ones).

In his *Method of Fluxions*, (1736), Isaac Newton was one of the first to use the procedure of this section, namely, *implicit differentiation.* He used his brand of derivatives though, things he called *fluxions* and he got into big trouble because they weren't well defined. In England, one famous philosopher by the name of **Bishop Berkeley** criticized Newton severely for his inability to actually explain what these fluxions really were. Nevertheless, Newton obtained the right answers (according to our calculations). *What about Leibniz?* Well, even Leibniz got into trouble with his, so-called, *differentials* because he really couldn't explain this stuff well, either! His nemesis in this case was one **Bernard Nieuwentijt** of Amsterdam (1694). Apparently, neither Berkeley nor Nieuwentijt could put the Calculus on a rigorous foundation either, so, eventually the matter was dropped. It would take another 150 years until Weierstrass and others like him would come along and make sense out of all this Calculus business with rigorous definitions (like those in the Advanced Topics chapter).

Review

You should review the **Chain Rule** and the **Generalized Power Rule** in the preceding section. A mastery of these concepts and the usual rules for differentiation will make this section much easier to learn.

We can call our usual functions **explicit** because their values are given explicitly (i.e., we can write them down) by solving one of the variables in terms of the other. This means that for each value of x there is only one value of y. But this is the same as saying that y is a function of x, right? An equation involving two variables, say, x, y, is said to be an **explicit relation** if one can solve for y (or x) uniquely in terms of x (or y).

Example 96. For example, the equation $2y = 2x^6 - 4x$ is an explicit relation because we can easily solve for y in terms of x. In fact, it reduces to the rule $y = x^6 - 2x$ which defines a function $y = f(x)$ where $f(x) = x^6 - 2x$. Another example is given by the function y whose values are given by $y(x) = x + \sqrt{x}$ whose values are easily calculated: Each value of x gives a value of $y = x + \sqrt{x}$ and so on, and y can be found directly using a calculator. Finally, $3x + 6 - 9y^2 = 0$ also defines an explicit relation because now we can solve for x in terms of y and find $x = 3y^2 - 2$.

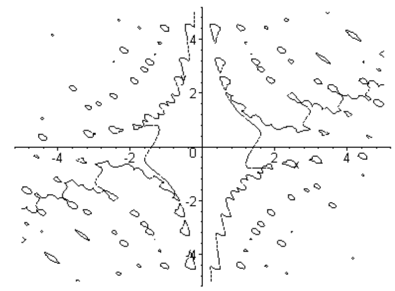
In the same spirit we say that an equation involving two variables, say, x, y, is said to be an **implicit relation** if it is *not explicit.*

Example 97. For example, the relation defined by the rule $y^5 + 7y = x^3 \cos x$ is implicit. Okay, you can isolate the y, but what's left still involves y and x, right? The equation defined by $x^2y^2 + 4\sin(xy) = 0$ also defines an implicit relation.

Such implicit relations are useful because they usually define a *curve* in the xy-plane, a curve which is not, generally speaking, the graph of a function. In fact, you can probably believe the statement that a "closed curve" (like a circle, ellipse, etc.) cannot be the graph of a function. Can you show why? For example, the circle defined by the implicit relation $x^2 + y^2 = 4$ is not the graph of a unique function,

(think of the, so-called, Vertical Line Test for functions).

So, if y is 'obscured' by some complicated expression as in, say, $x^2 - 2xy + tan(xy) - 2 = 0$, then it is not easy to solve for 'y' given a value of 'x'; in other words, it would be very difficult to isolate the y's on one side of the equation and group the x's together on the other side. In this case y is said to be defined implicitly or y is an **implicit function** of x. By the same token, x may be considered an implicit function of y and it would equally difficult to solve for x as a function of y. Still, it is possible to draw its graph by looking for those points x, y that satisfy the equation, see Figure 42.



An approximate plot of the implicit relation $x^2 - 2xy + \tan(xy) - 2 = 0$. This plot fails the "vertical line test" and so it cannot be the graph of a function.

Figure 42.

Other examples of functions defined implicitly are given by:

$(x-1)^2 + y^2 = 16$ A circle of radius 4 and center at $(1, 0)$.

$\frac{(x-2)^2}{9} + \frac{(y-6)^2}{16} = 1$ An ellipse 'centered' at $(2, 6)$.

$(x-3)^2 - (y-4)^2 = 5$ A hyperbola.

OK, so how do we find the derivative of such 'functions' defined implicitly?

1. Assume, say y, is a differentiable function of x, (or x is a differentiable function of y).
2. Write $y = y(x)$ (or $x = x(y)$) to show the dependence of y on x, (even though we really don't know what it 'looks like').
3. Differentiate the relation/expression which defines y **implicitly** with respect to x (or y - *this expression is a curve in the xy-plane.)*
4. *Solve for the derivative* $\frac{dy}{dx}$ **explicitly**, *yes, explicitly!*

Note: It can be shown that the 4 steps above always **produce an expression for $\frac{dy}{dx}$ which can be solved explicitly**. In other words, even though y is given implicitly, the function $\frac{dy}{dx}$ is explicit, that is, given a point $P(x, y)$ on the defining curve described in (3) *we can actually solve* for the term $\frac{dy}{dx}$.

This note is based on the assumption that we already know that y *can* be written as a differentiable function of x. This assumption isn't obvious, and involves an important result called the *Implicit Function Theorem* which we won't study here but which can be found in books on *Advanced Calculus*. One of the neat things about this implicit function theorem business is that it tells us that, under certain conditions, *we can always solve for the derivative* dy/dx even though we can't solve for y! Amazing, isn't it?

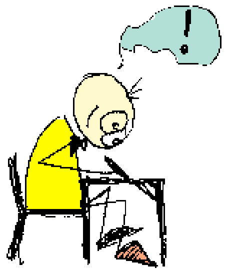
Example 98. Find the derivative of y with respect to x, that is, $\frac{dy}{dx}$, when x, y are related by the expression $xy - y^2 = 6$.

Solution We assume that y is a differentiable function of x so that we can write $y = y(x)$ and y is differentiable. Then the relation between x, y above really says that

$$xy(x) - y(x)^2 = 6.$$

OK, since this is true for all x under consideration (the x's were not specified, so don't worry) it follows that we can take the derivative of both sides and still get equality, *i.e.*

$$\frac{d}{dx}(xy(x) - y(x)^2) = \frac{d}{dx}(6).$$



Now, $\frac{d}{dx}(6) = 0$ since the derivative of a constant is always 0 and

$$\begin{aligned}\frac{d}{dx}(xy(x) - y(x)^2) &= \frac{d}{dx}\ xy(x) - \frac{d}{dx}\ y(x)^2 \\ &= \left[x\frac{dy}{dx} + y(x)\frac{d(x)}{dx}\right] - 2y(x)\frac{dy}{dx} \\ &= [x - 2y(x)]\frac{dy}{dx} + y(x)\end{aligned}$$

where we used a combination of the Product Rule and the Generalized Power Rule (see Example 87 for a similar argument). So, we have

$$[x - 2y(x)]\frac{dy}{dx} + y(x) = \frac{d}{dx}(6) = 0,$$

and solving for $\frac{dy}{dx}$ we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{-y(x)}{x - 2y(x)}. \\ &= \frac{y(x)}{2y(x) - x}.\end{aligned}$$

OK, so we have found $\frac{dy}{dx}$ in terms of x and $y(x)$ that is, since $y = y(x)$,

$$\frac{dy}{dx} = \frac{y}{2y - x}$$

provided x and y are related by the original expression $xy - y^2 = 6$ which describes a curve in the xy-plane. This last display then describes the values of the derivative $y'(x)$ along this curve for a given point P(x,y) on it.

To find the slope of the tangent line to a point P(x_0, y_0) on this curve we calculate

$$\frac{dy}{dx} = \frac{y_0}{2y_0 - x_0}$$

where $x_0y_0 - y_0^2 = 6$, that's all. So, for example the point $(7, 1)$ is on this curve because $x_0 = 7, y_0 = 1$ satisfies $x_0y_0 - y_0^2 = 6$. You see that the derivative at this point $(7, 1)$ is given by

$$\frac{dy}{dx} = \frac{1}{2(1) - 7} = -\frac{1}{5}$$

Example 99. Let $x^3 + 7x = y^3$ define an implicit relation for x in terms of y. Find $x'(1)$.

Solution We'll assume that x can be written as a differentiable function of y.We take the derivative of both sides (with respect to y this time!). We see that

$$3x^2\frac{dx}{dy} + 7\frac{dx}{dy} = 3y^2,$$

since

$$\frac{d}{dy}x^3 = 3x^2\frac{dx}{dy}$$

by the Generalized Power Rule. We can now solve for the expression dx/dy and find a formula for the derivative, namely,

$$\frac{dx}{dy} = \frac{3y^2}{3x^2 + 7}.$$

Now we can find the derivative easily at any point (x, y) on the curve $x^3 + 7x = y^3$. For instance, the derivative at the point $(1, 2)$ on this curve is given by substituting the values $x = 1, y = 2$ in the formula for the derivative just found, so that

$$\frac{dx}{dy} = \frac{(3)(2)^2}{(3)(1)^2 + 7} = \frac{12}{10} = \frac{6}{5}.$$

For a geometrical interpretation of this derivative, see Figure 43 on the next page.

Example 100. Find the slope of the tangent line to the curve $y = y(x)$ given implicitly by the relation $x^2 + 4y^2 = 5$ at the point $(-1, 1)$.

Solution First, you should always check that the given point $(-1, 1)$ is **on** this curve, otherwise, there is nothing to do! Let $x_0 = -1, y_0 = 1$ and $P(x_0, y_0) = P(-1, 1)$. We see that $(-1)^2 + 4(1)^2 = 5$ and so the point $P(-1, 1)$ is on the curve.

Since we want the slope of a tangent line to the curve $y = y(x)$ at $x = x_0$, we need to find it's derivative $y'(x)$ and evaluate it at $x = x_0$.

OK, now

$$\begin{aligned}
\frac{d}{dx}(x^2 + 4y(x)^2) &= \frac{d}{dx}(5) \\
2x + \underbrace{4\,\frac{d}{dx}(y(x)^2)} &= 0 \\
2x + 4\;2y(x) \cdot y'(x) &= 0 \\
y'(x) &= -\frac{2x}{8y(x)} = -\frac{x}{4y(x)}, \quad (\text{if } y(x) \neq 0)
\end{aligned}$$

and this gives the value of the derivative, $y'(x)$ at any point (x, y) on the curve, that is $y' = -\frac{x}{4y}$ where (x, y) is on the curve (remember $y = y(x)$). It follows that at $(-1, 1)$, this derivative is equal to

$$y'(-1) = (-1)\frac{(-1)}{4(1)} = \frac{1}{4}.$$

Example 101. A curve in the xy-plane is given by the set of all points (x, y) satisfying the equation $y^5 + x^2y^3 = 10$. Find $\frac{dx}{dy}$ at the point $(x, y) = (-3, 1)$.

Solution Verify that $(-3, 1)$ is, indeed, on the curve. This is true since $1^5 + (-3)^2(1)^3 = 10$, as required. Next, we assume that $x = x(y)$ is a differentiable function of y. Then

$$\begin{aligned}
\frac{d}{dy}(y^5 + x^2y^3) &= \frac{d}{dy}(10) \\
5y^4 + 2x(y)\;x'(y)\;y^3 + x(y)^2(3y^2) &= 0,
\end{aligned}$$

(where we used the Power Rule and the Product Rule). Isolating the term $x'(y) = \frac{dx}{dy}$ gives us the required derivative,

$$\frac{dx}{dy} = \frac{-3x^2y^2 - 5y^4}{2xy^3}.$$

When $x = -3, y = 1$ so,

$$\frac{dx}{dy} = \frac{-27 - 5}{-6} = \frac{16}{3}.$$

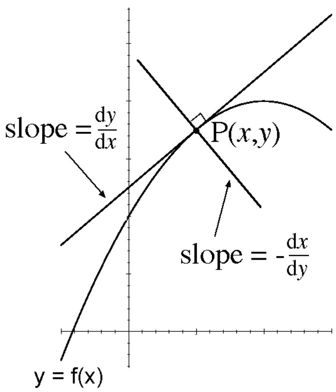


Figure 43. *Geometric meaning of* $\frac{dx}{dy}$

Remark If we were to find $\frac{dy}{dx}$ at $(-3, 1)$ we would obtain $\frac{3}{16}$; the **reciprocal** of $\frac{dx}{dy}$. Is this a coincidence? No. It turns out that if y is a differentiable function of x and x is a differentiable function of y then their derivatives are related by the relation

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

if $\frac{dx}{dy} \neq 0$, at the point $P(x, y)$ under investigation. This is another consequence of the Implicit Function Theorem and a result on Inverse Functions.

O.K., we know what $\frac{dy}{dx}$ means geometrically, right ? Is there some geometric meaning for $\frac{dx}{dy}$? Yes, the value of $\frac{dx}{dy}$ at $P(x, y)$ on the given curve is equal to the **negative of the slope of the line perpendicular to the tangent line through P.** For example, the equation of the tangent line through $P(-3, 1)$ in Example 101 is given by $y = (3x + 25)/16$, while the equation of the line perpendicular to this tangent line and through P is given by $y = (16x + 51)/3$. This last (perpendicular) line is called the **normal line** through P. See Figure 43.

Exercise Set 13.

Use implicit differentiation to find the required derivative.

1. $x^2 + xy + y^2 = 1$, $\frac{dy}{dx}$ at $(1, 0)$
2. $2xy^2 - y^4 = x^3$, $\frac{dy}{dx}$ and $\frac{dx}{dy}$
3. $\sqrt{x+y} + xy = 4$, $\frac{dy}{dx}$ at $(16, 0)$
4. $x - y^2 = 4$, $\frac{dy}{dx}$
5. $x^2 + y^2 = 9$, $\frac{dy}{dx}$ at $(0, 3)$

Find the equation of the tangent line to the given curve at the given point.

6. $2y^2 - x^2 = 1$, at $(-1, -1)$
7. $2x = xy + y^2$, at $(1, 1)$
8. $x^2 + 2x + y^2 - 4y - 24 = 0$, at $(4, 0)$
9. $(x + y)^3 - x^3 - y^3 = 0$, at $(1, -1)$

Suggested Homework Set 10. *Problems 1, 2, 4, 7, 9*

Web Links

For more examples on implicit differentiation see:

archives.math.utk.edu/visual.calculus/3/implicit.7/index.html
www.ugrad.math.ubc.ca/coursedoc/math100/notes/derivative/implicit.html
(the above site requires a Java-enabled browser)

NOTES:

3.5 Derivatives of Trigonometric Functions

Our modern world runs on electricity. In these days of computers, space travel and robots we need to have a secure understanding of the basic laws of electricity and its uses. In this realm, electric currents both alternating (as in households), and direct (as in a flashlight battery), lead one to the study of sine and cosine functions and their interaction. For example, how does an electric current vary over time? We need its 'rate of change' with respect to time, and this can be modeled using its derivative.

An integrated circuits board

In another vein, so far we've encountered the derivatives of many different types of functions; polynomials, rational functions, roots of every kind, and combinations of such functions. In many applications of mathematics to physics and other physical and natural sciences we need to study combinations of trigonometric functions and other 'changes', the shapes of their graphs and other relevant data. In the simplest of these applications we can mention the study of **wave phenomena**. In this area we model incoming or outgoing waves in a fluid (such as a lake, tea, coffee, etc.) as a combination of sine and cosine waves, and then study how these waves change over time. Well, to study how these waves change over time we need to study their derivatives, right? This, in turn, means that we need to be able to find the derivatives of the sine and cosine functions and that's what this section is all about.

There are two fundamental limits that we need to recall here from an earlier chapter, namely

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \tag{3.2}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0, \tag{3.3}$$

Let's also recall some fundamental trigonometric identities in Table 3.4.

All angles, A, B and x are in **radians** in the Table above, and this is customary in calculus.

Recall that 1 *radian* $= \frac{180}{\pi}$ *degrees.*

I1	$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
I2	$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
I3	$\sin^2 x + \cos^2 x = 1$
I4	$\sec^2 x - \tan^2 x = 1$
I5	$\csc^2 x - \cot^2 x = 1$
I6	$\cos 2x = \cos^2 x - \sin^2 x$
I7	$\sin 2x = 2\sin x \cos x$
I8	$\cos^2 x = \frac{1+\cos 2x}{2}$
I9	$\sin^2 x = \frac{1-\cos 2x}{2}$

Table 3.4: Useful Trigonometric Identities

The first result is that the derivative of the sine function is the cosine function, that is,

$$\boxed{\frac{d}{dx}\sin x = \cos x.}$$

This is not too hard to show; for example, assume that $h \neq 0$. Then

$$\begin{aligned}\frac{\sin(x+h)-\sin x}{h} &= \frac{\sin x\cos h + \cos x \sin h - \sin x}{h}, \quad \text{(by I1)}\\ &= \sin x\,\frac{\cos h - 1}{h} + \cos x\,\frac{\sin h}{h}, \text{(re-arranging terms)}\end{aligned}$$

Now we use a limit theorem from Chapter 2: Since the last equation is valid for each $h \neq 0$ we can pass to the limit and find

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} &= \sin x \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \lim_{h\to 0}\left(\frac{\sin h}{h}\right)\\ &= (\sin x)\cdot(0) + (\cos x)\cdot(1), \quad \text{(by (3.3) and (3.2))}\\ &= \cos x.\end{aligned}$$

A similar derivation applies to the next result;

$$\boxed{\frac{d}{dx}\cos x = -\sin x}$$

For example,

$$\begin{aligned}\frac{\cos(x+h)-\cos x}{h} &= \frac{\cos x\cos h - \sin x \sin h - \cos x}{h}, \quad \text{(by I2)}\\ &= \cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h}, \quad \text{(re} - \text{arranging terms)}\end{aligned}$$

As before, since this last equation is valid for each $h \neq 0$ we can pass to the limit and find

$$\begin{aligned}\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} &= \cos x \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) - \sin x \lim_{h\to 0}\left(\frac{\sin h}{h}\right)\\ &= (\cos x)\cdot(0) - (\sin x)(1), \quad \text{(by (3.3) and (3.2))}\\ &= -\sin x.\end{aligned}$$

Since these two limits define the derivative of each trigonometric function we get the boxed results, above.

OK, now that we know these two fundamental derivative formulae for the sine and cosine functions we can derive all the other such formulae (for $\tan, \cot, \sec$, and $\csc$) using basic properties of derivatives.

For example, let's show that

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$$

or, since $\frac{1}{\cos^2 x} = \sec^2 x$, we get

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

as well. To see this we use the Quotient Rule and recall that since $\tan x = \frac{\sin x}{\cos x}$,

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \quad \text{(by definition)} \\
&= \frac{\cos x \frac{d}{dx}(\sin x) - (\frac{d}{dx}\cos x)\sin x}{\cos^2 x} \quad \text{(Quotient Rule)} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \quad \text{(just derived above)} \\
&= \frac{1}{\cos^2 x} \text{ (by I3).}
\end{aligned}$$

By imitating this argument it's not hard to show that

$$\boxed{\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x},}$$

or, equivalently,

$$\boxed{\frac{d}{dx}\cot x = -\csc^2 x}$$

a formula which we leave to the reader as an **exercise**, as well.

There are two more formulae which need to be addressed, namely, those involving the derivative of the secant and cosecant functions. These are:

$$\boxed{\begin{aligned}\frac{d}{dx}\sec x &= \sec x \tan x \\ \frac{d}{dx}\csc x &= -\csc x \cot x\end{aligned}}$$

Each can be derived using the Quotient Rule. Now armed with these formulae and the Chain Rule we can derive formulae for derivatives of very complicated looking functions, see Table 3.5.

Example 102. Find the derivative of f where $f(x) = \sin^2 x + 6x$.

Solution The derivative of a sum is the sum of the derivatives. So

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(6x) \\
&= \frac{d}{dx}(\sin x)^2 + 6
\end{aligned}$$

Now let $\square = \sin x$. We want $\frac{d}{dx}\square^2$ so we'll need to use the Generalized Power Rule here... So,

$$\begin{aligned}
\frac{d}{dx}(\sin x)^2 &= \frac{d}{dx}\square^2 \\
&= 2\square^1\frac{d\square}{dx} \quad \text{(Power Rule)} \\
&= 2\square\square' \\
&= 2\ (\sin x)\ (\cos x), \quad \text{(since } \square' = \cos x) \\
&= \sin 2x, \quad \textit{(by I7)}
\end{aligned}$$

The final result is $f'(x) = 6 + \sin 2x$.

Joseph Louis (Comte de) Lagrange, 1736 -1813, was born in Torino, Italy and died in Paris, France. His main contributions to mathematics were in the fields of analysis where he studied analytical and celestial mechanics, although he excelled in everything that he studied. In 1766, Lagrange became the successor of Euler in the Berlin Academy of Science, and during the next year he was awarded the first of his many prizes for his studies on the irregularities of the motion of the moon. He helped to found the Academy of Science in Torino in 1757, and the École Polytechnique in 1795. He also helped to create the first commission on *Weights and Measures* and was named to the *Legion d'Honneur* by Napoleon and elevated to Count in 1808.

Example 103. Evaluate $\dfrac{d}{dx}\sqrt{1+\cos x}$ at $x = 0$.

Solution We write $f(x) = \sqrt{1+\cos x}$ and convert the root to a power (always do this so you can use the Generalized Power Rule).

We get $f(x) = (1+\cos x)^{\frac{1}{2}} = \Box^{\frac{1}{2}}$ if we set $\Box = 1+\cos x$ so that we can put the original function into a more recognizable form. So far we know that

$$f(x) = \sqrt{1+\cos x} = \Box^{\frac{1}{2}}$$

So, by the Power Rule, we get

$$f'(x) = \frac{1}{2}\Box^{-\frac{1}{2}}\Box'$$

where $\Box'$ is the derivative of $1+\cos x$ (**without** the root), *i.e.*

$$\begin{aligned}
\Box' &= \frac{d}{dx}(1+\cos x)\\
&= \frac{d}{dx}(1) + \frac{d}{dx}(\cos x)\\
&= 0 - \sin x\\
&= -\sin x
\end{aligned}$$

Combining these results we find

$$\begin{aligned}
f'(x) &= \frac{1}{2}(1+\cos x)^{-\frac{1}{2}}(-\sin x)\\
&= -\frac{\sin x}{2\sqrt{1+\cos x}}
\end{aligned}$$

after simplification. At $x = 0$ we see that

$$\begin{aligned}
f'(0) &= -\frac{\sin 0}{2\sqrt{1+\cos 0}} = -\frac{0}{2\sqrt{2}}\\
&= 0
\end{aligned}$$

which is what we are looking for.

Example 104. Evaluate $\dfrac{d}{dx}\left(\dfrac{\cos x}{1+\sin x}\right)$.

Solution Write $f(x) = \cos x$, $g(x) = 1+\sin x$. We need to find the derivative of the quotient $\frac{f}{g}$ and so we can think about using the Quotient Rule.

Now, recall that

$$\frac{d}{dx}(f(g(x))) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

In our case,

$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
$\cos x$	$-\sin x$	$1+\sin x$	$\cos x$

Combining these results we find, (provided $1 + \sin x \neq 0$),

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right) &= \frac{(-\sin x)(1+\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2} \\
&= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} \\
&= -\frac{1+\sin x}{(1+\sin x)^2} \qquad \text{(by I3)} \\
&= -\frac{1}{1+\sin x}.
\end{aligned}$$

Example 105. Let the function be defined by $f(t) = \dfrac{3}{\sin(t)}$. Evaluate $f'(\frac{\pi}{4})$.

Solution OK, we have a constant divided by a function so it looks like we should use the Power Rule (or the Quotient Rule, either way you'll get the same answer). So, let's write $f(t) = 3\Box^{-1}$ where $\Box = \sin(t)$ then

$$f'(t) = (-1) \cdot 3 \cdot \Box^{-2}\Box'$$

by the Generalized Power Rule. But we still need $\Box'$, right? Now $\Box = \sin(t)$, so $\Box' = \cos t$. Combining these results we find

$$\begin{aligned}
f'(t) &= -3(\sin t)^{-2}(\cos t) \\
&= -3\frac{\cos t}{(\sin t)^2}.
\end{aligned}$$

Note that this last expression is also equal to $-3\csc t \cot t$. At $t = (\frac{\pi}{4})$, (which is 45 degrees expressed in radians), $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ and so

$$\begin{aligned}
f'(\frac{\pi}{4}) &= -3\frac{(\frac{1}{\sqrt{2}})}{(\frac{1}{\sqrt{2}})^2} = -3\frac{1}{\sqrt{2}} \cdot \frac{(\sqrt{2})^2}{1}, \\
&= -3\sqrt{2}.
\end{aligned}$$

Note: Notice that we could have written $f(t) = \frac{3}{\sin(t)}$ as $f(t) = 3\csc t$ and use the derivative formula for $\csc t$ mentioned above. This would give $f'(t) = -3\csc t \cot t$ and we could then continue as we did above.

Example 106. Let's look at an example which can be solved in two different ways. Consider the implicit relation

$$y + \sin^2 y + \cos^2 y = x.$$

We want $y'(x)$.

The easy way to do this is to note that, by trigonometry (I3), $\sin^2 y + \cos^2 y = 1$ regardless of the value of y. So, the original relation is really identical to $y + 1 = x$. From this we observe that $dy/dx = 1$.

But what if you didn't notice this identity? Well, we differentiate both sides as is the case whenever we use implicit differentiation. The original equation really means

$$y + \{\sin(y)\}^2 + \{\cos(y)\}^2 = x.$$

Use of the Generalized Power Rule then gives us,

$$\frac{dy}{dx} + 2\{\sin(y)\}^1 \frac{d}{dx}\sin(y) + 2\{\cos(y)\}^1 \frac{d}{dx}\cos(y) = 1,$$

Derivatives of Trigonometric Functions: Summary

Let $\Box$ denote any differentiable function, and D denote the operation of differentiation. Then

$D\sin\Box = \cos\Box \cdot D\Box$	$D\cos\Box = -\sin\Box \cdot D\Box$
$D\tan\Box = \sec^2\Box \cdot D\Box$	$D\cot\Box = -\csc^2\Box \cdot D\Box$
$D\sec\Box = \sec\Box \cdot \tan\Box \cdot D\Box$	$D\csc\Box = -\csc\Box \cdot \cot\Box \cdot D\Box$

Table 3.5: Derivatives of Trigonometric Functions

or

$$\frac{dy}{dx} + 2\{\sin(y)\}^1\cos(y)\frac{dy}{dx} + 2\{\cos(y)\}^1(-\sin(y))\frac{dy}{dx} \quad = \quad 1.$$

But the second and third terms cancel out, and we are left with

$$\frac{dy}{dx} = 1,$$

as before. Both methods do give the same answer as they should.

Example 107. Evaluate the following derivatives using the rules of this Chapter and Table 3.5.

a) $f(x) = \sin(2x^2+1)$
b) $f(x) = \cos 3x \sin\sqrt{x}$
c) $f(t) = (\cos 2t)^2$, at $t = 0$
d) $f(x) = \cos(\sin x)$ at $x = 0$
e) $h(t) = \dfrac{t}{\sin 2t}$ at $t = \pi/4$

Solution **a)** Replace the stuff between the outermost brackets by a box, $\Box$. We want $D\sin\Box$, right? Now, since $\Box = 2x^2+1$, we know that $D\Box = 4x$, and so Table 3.5 gives

$$\begin{aligned} D\sin\Box &= \cos\Box \cdot D\Box \\ &= \cos(2x^2+1)\cdot 4x \\ &= 4x\cos(2x^2+1). \end{aligned}$$

b) We use a combination of the Product Rule and Table 3.5. So,

$$\begin{aligned} f'(x) &= D(\cos 3x)\cdot\sin\sqrt{x} + \cos(3x)\cdot D\sin\sqrt{x} \\ &= (-3\cdot\sin(3x))\cdot(\sin\sqrt{x}) + \cos(3x)\cdot\frac{\cos(\sqrt{x})}{2\sqrt{x})}, \end{aligned}$$

since $D\sin\sqrt{x} = \cos(\sqrt{x})\cdot D(\sqrt{x}) = \cos(\sqrt{x})\cdot((1/2)\ x^{-1/2}) = \cos(\sqrt{x})/(2\sqrt{x})$.

c) Let $\Box = \cos 2t$. The Generalized Power Rule comes to mind, so, use of Table 3.5 shows that

$$\begin{aligned} f'(t) &= 2\Box\cdot D\Box, \\ &= (2\cdot\cos 2t)\cdot(-2\cdot\sin 2t) \\ &= -4\cos 2t\sin 2t \\ &= -2\sin 4t, \qquad \text{(where we use Table 3.4, (I7), with } x = 2t\text{).} \end{aligned}$$

So $f'(0) = -2\sin(0) = 0$.

d) We need to find the derivative of something that looks like $\cos\Box$. So, let $\Box = \sin x$. We know that $D\Box = D\sin x = \cos x$, and once again, Table 3.5 shows that

$$\begin{aligned} f'(x) &= -\sin\Box \cdot D\Box, \\ &= -\sin(\sin x)\cdot\cos x, \\ &= -\cos x \cdot \sin(\sin x). \end{aligned}$$

So $f'(0) = -\cos(0)\cdot\sin(\sin(0)) = -1 \cdot sin(0) = 0$.

e) We see something that looks like a quotient so we should be using the Quotient Rule, right? Write $f(t) = t$, $g(t) = \sin 2t$. We need to find the derivative of the quotient $\frac{f}{g}$. Now, recall that this Rule says that (replace the x's by t's)

$$\frac{d}{dt}(f(g(t)) = \frac{f'(t)g(t) - f(t)g'(t)}{g(t)^2}.$$

In our case,

$f(t)$	$f'(t)$	$g(t)$	$g'(t)$
t	1	$\sin 2t$	$2\cos 2t$

Substituting these values into the Quotient Rule we get

$$\begin{aligned} \frac{d}{dt}(f(g(t)) &= \frac{1\cdot\sin 2t - t\cdot 2\cos 2t}{(\sin 2t)^2}, \\ &= \frac{\sin 2t - 2t\cos 2t}{(\sin 2t)^2}. \end{aligned}$$

At $\pi/4$, $\sin(\pi/4) = \sqrt{2}/2$, so, $\sin(2\cdot\pi/4) = 1$, $\cos(2\cdot\pi/4) = 0$, and the required derivative is equal to 1.

NOTES:

Exercise Set 14.

Evaluate the derivative of the functions whose values are given below, at the indicated point.

1. $\sin\sqrt{x},\ at\ x=1$
2. $\sec(2x)\cdot\sin x$
3. $\sin x\cos x,\ at\ x=0$
4. $\dfrac{\cos x}{1-\sin x}$
5. $\sqrt{1+\sin t},\ at\ t=0$
6. $\sin(\cos(x^2))$
7. $x^2\cdot\cos 3x$
8. $x^{2/3}\cdot\tan(x^{1/3})$
9. $\cot(2+x+\sin x)$
10. $(\sin 3x)^{-1}$
11. $\dfrac{x+1}{\sin x},\ at\ x=\pi/2$
12. $\sin(2x^2)$
13. $\sin^2 x,\ at\ x=\pi/4$
14. $\cot(3x-2)$
15. $\dfrac{2x+3}{\sin x}$
16. $\cos(x\cdot\sin x)$
17. $\sqrt{x}\cdot sec(\sqrt{x})$
18. $\csc(x^2-2)\cdot\sin(x^2-2),\quad x^2\neq 2.$
19. $\cos^2(x-6)+\csc(2x)$
20. $(\cos 2x)^{-2}$

21. Let y be defined by

$$y(x)=\begin{cases}\dfrac{\sin x}{\tan x} & x\neq 0\\ 1 & x=0\end{cases},$$

a) Show that y is continuous at $x=0$,
b) Show that y is differentiable at $x=0$ and,
c) Conclude that $y'(0)=0$.

Suggested Homework Set 11. *Do problems 1, 4, 6, 13, 20*

NOTES:

3.6 Important Results About Derivatives

This section is about things we call **theorems**. Theorems are truths about things mathematical ... They are statements which can be substantiated (or proved) using the language of mathematics and its underlying logic. It's not always easy to prove something, whether it be mathematical or not. The point of a 'proof' is that it makes everything you've learned 'come together', so to speak, in a more logical, coherent fashion.

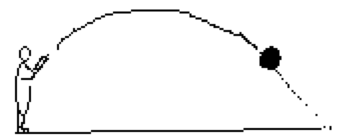


Figure 44.

The results here form part of the cornerstones of basic Calculus. One of them, the **Mean Value Theorem** will be used later when we define the, so-called, **antiderivative** of a function and the **Riemann integral**.

We will motivate this first theorem by looking at a sample real life situation.

A ball is thrown upwards by an outfielder during a baseball game. It is clear to everyone that the ball will reach a maximum height and then begin to fall again, hopefully in the hands of an infielder. Since the motion of the ball is 'smooth' (not 'jerky') we expect the trajectory produced by the ball to be that of a differentiable function (remember, there are no 'sharp corners' on this flight path). OK, now since the trajectory is differentiable (as a function's graph) there must be a (two-sided) derivative at the point where the ball reaches its maximum, right? What do you think is the value of this derivative? Well, look at an idealized trajectory... it has to be mainly 'parabolic' (because of gravity) and it looks like the path in the margin.

Tangent lines to the left (respectively, right) of the point where the maximum height is reached have positive (respectively, negative) slope and so we expect the tangent line to be horizontal at M (the point where the maximum value is reached). This is the key point, **a horizontal tangent line means a 'zero derivative' mathematically**. Why? Well, you recall that the derivative of f at a point x is the slope of the tangent line at the point $P(x, f(x))$ on the graph of f. Since a horizontal line has zero slope, it follows that the derivative is also zero.

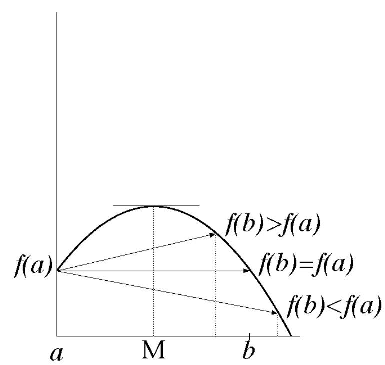


Figure 45.

OK, now let's translate all this into the language of mathematics. The curve has an equation $y = f(x)$ and the ball leaves the hand of the outfielder at a point a with a height of $f(a)$ (meters, feet, ... we won't worry about units here). Let's say that the ball needs to reach 'b' at a height $f(b)$, where $f(b) = f(a)$, above the ground. The fastest way of doing this, of course, is by throwing the ball in a straight line path from point to point (see Figure 45), but this is not realistic! If it were, the tangent line along this flight path would still be horizontal since $f(a) = f(b)$, right!?

So, the ball can't really travel in a 'straight line' from a to b, and will always reach a 'maximum' in our case, a maximum where necessarily $y'(M) = 0$, as there is a horizontal tangent line there, see Figure 45. OK, now let's look at all possible (differentiable) curves from $x = a$ to $x = b$, starting at height $f(a)$ and ending at height $f(b) = f(a)$, (as in Figure 46). We want to know "Is there always a point between a and b at which the curve reaches its maximum value ?"

A straight line from $(a, f(a))$ to $(b, f(b))$, where $f(b) = f(a)$, is one curve whose maximum value is the same everywhere, okay? And, as we said above, this is necessarily horizontal, so this line is the same as its tangent line (for each point x between a and b). As can be seen in Figure 46, all the 'other' curves seem to have a maximum value at some point between a and b and, when that happens, there is a horizontal tangent line there.

It looks like we have discovered something here: If f is a differentiable function on

an interval $I = (a, b)$ (recall $(a, b) = \{x : a < x < b\}$) and $f(a) = f(b)$ then $f'(c) = 0$ for some c between a and b; *at least one* c, but there may be more than one. Actually, this mathematical statement is true! The result is called **Rolle's Theorem** and it is named after **Michel Rolle**, (1652-1719), a French mathematician.

Of course we haven't 'proved' this theorem of Rolle but it is believable! Its proof can be found in more advanced books in **Analysis**, a field of mathematics which includes Calculus.)

We will state it here for future reference, though:

Rolle's Theorem (1691)

Let f be a continuous function on $[a, b]$ and let f be differentiable at each point in (a, b). If $f(a) = f(b)$, then there is at least one point c between a and b at which $f'(c) = 0$.

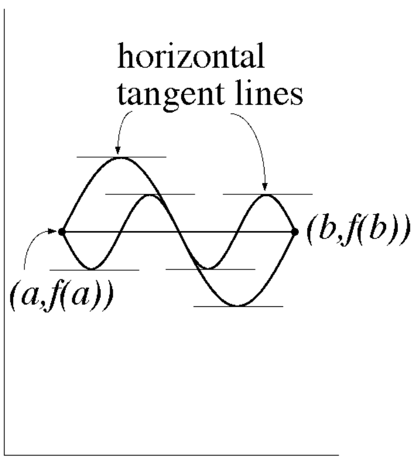


Figure 46.

Remark

1. Remember that the point c, whose existence is guaranteed by the theorem, is not necessarily unique. There may be lots of them... but there is always **at least** one. Unfortunately, **the theorem doesn't tell us where it is** so we need to rely on graphs and other techniques to find it.
2. Note that whenever the derivative is zero it seems that the graph of the function has a 'peak' or a 'sink' at that point. In other words, such points appear to be related to where the graph of the function has a **maximum** or **minimum** value. This observation is very important and will be very useful later when we study the general problem of sketching the graph of a general function.

Example 108. Show that the function whose values are given by $f(x) = \sin(x)$ on the interval $[0, \pi]$ satisfies the assumptions of Rolle's Theorem. Find the required value of c explicitly.

Solution We know that 'sin' as well as its derivative, 'cos', are continuous everywhere. Also, $\sin(0) = 0 = \sin(\pi)$. So, if we let $a = 0, b = \pi$, we see that we can apply Rolle's Theorem and find that $y'(c) = 0$ where c is somewhere in between 0 and π. So, this means that $\cos c = 0$ for some value of c. This is true! We can choose $c = \pi/2$ and see this c exactly.

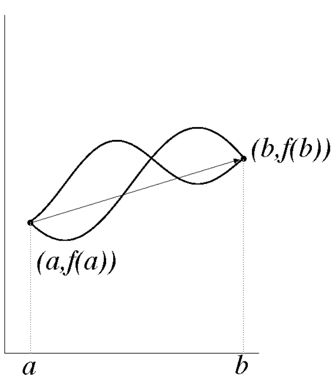


Figure 47.

We have seen Rolle's Theorem in action. Now, let's return to the case where the baseball goes from $(a, f(a))$ to $(b, f(b))$ but where $f(a) \neq f(b)$ (players of different heights!). **What can we say in this case?**

Well, we know that there is the straight line path from $(a, f(a))$ to $(b, f(b))$ which, unfortunately, does not have a zero derivative *anywhere* as a curve (see Figure 47). But look at all possible curves going from $(a, f(a))$ to $(b, f(b))$. This is only a thought experiment, OK? They are differentiable (let's assume this) and they bend this way and that as they proceed from their point of origin to their destination. Look at **how** they turn and compare this to the straight line joining the origin and destination. It looks like **you can always find a tangent line** to any one of these curves **which is parallel to the line joining $(a, f(a))$ to $(b, f(b))$!** (see Figure 48). It's *almost* like Rolle's Theorem (graphically) but it is **not** Rolle's Theorem because $f(a) \neq f(b)$. Actually, if you think about it a little, you'll see that it's more general than Rolle's Theorem. It has a different name ... and it too is a

true mathematical statement! We call it the **Mean Value Theorem** and it says the following:

Mean Value Theorem

Let f be continuous on the interval $a \leq x \leq b$ and differentiable on the interval $a < x < b$. Then there is a point c between a and b at which

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Remark The number on the left of the equation, namely, $\frac{f(b)-f(a)}{b-a}$ is really the slope of the line pointing from $(a, f(a))$ to $(b, f(b))$. Moreover, $f'(c)$ is the slope of the tangent line through some point $(c, f(c))$ on the graph of f. Since **these slopes are equal**, the corresponding lines must be parallel, which is what we noticed above.

Example 109. Show that the function whose values are given by $f(x) = \cos 2x$ on the interval $[0, \pi/2]$ satisfies the assumptions of the Mean Value Theorem. Show that there is a value of c such that $\sin 2c = 2/\pi$.

Solution Here, '$\cos 2x$' as well as its derivative, '$-2\sin 2x$', are continuous everywhere. Also, $\cos(0) = 1$ and $\cos(\pi) = -1$. So, if we let $a = 0, b = \pi/2$, we see that we can apply the Mean Value Theorem and find that $y'(c) = 0$ where c is somewhere in between 0 and $\pi/2$. This means that $-2\sin 2c = -4/\pi$, or for some value of c, we must have $\sin 2c = 2/\pi$. We may not know what this value of c is,

exactly, but it does exist! In fact, in the next section we'll show you how to find this value of c using **inverse trigonometric functions**.

Applications

Example 110. Let y be continuous in the interval $a \leq x \leq b$ and a differentiable function on an interval (a, b) whose derivative is equal to zero at each point x, $a < x < b$. Show that $y(x) = \textit{constant}$ for each x, $a < x < b$. *i.e.* If $y'(x) = 0$ for all x then the values $y(x)$ are equal to one and the same number (or, y is said to be a **constant function**).

Solution This is one very nice application of the Mean Value Theorem. OK, let t be any point in (a, b). Since y is continuous at $x = a$, $y(a)$ is finite. Re-reading the statement of this example shows that all the assumptions of the Mean Value Theorem are satisfied. So, the quotient

$$\frac{y(t)-y(a)}{t-a} = y'(c)$$

where $a < c < t$ is the conclusion. But whatever c is, we know that $y'(c) = 0$ (by hypothesis, *i.e.* at each point x the derivative at x is equal to 0). It follows that $y'(c) = 0$ and this gives $y(t) = y(a)$. But now look, t can be changed to some **other** number, say, t^*. We do the same calculation once again and we get

$$\frac{y(t^*)-y(a)}{t^*-a} = y'(c^*)$$

where now $a < c^* < t^*$, and c^* is generally different from c. Since $y'(c^*) = 0$ (again, by hypothesis) it follows that $y(t^*) = y(a)$ as well. OK, but all this means that $y(t) = y(t^*) = y(a)$. So, we can continue like this and repeat this argument for

every possible value of t in (a, b), and every time we do this we get that $y(t) = y(a)$. It follows that for **any** choice of t, $a < t < b$, we must have $y(t) = y(a)$. In other words, we have actually proved that $y(t) = constant$ $(= y(a))$, for t in $a < t < b$. Since y is continuous at each endpoint a, b, it follows that $y(b)$ must also be equal to $y(a)$. Finally, we see that $y(x) = y(a)$ for every x in $[a, b]$.

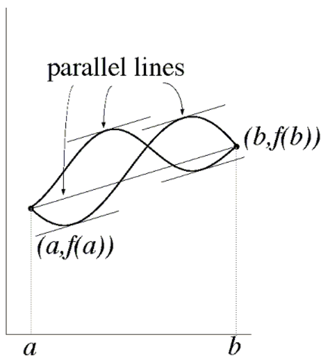


Figure 48.

Example 111. The function defined by $y = |x|$ has $y(-1) = y(1)$ but yet $y'(c) \neq 0$ for any value of c. **Explain why this doesn't contradict Rolle's Theorem.**

Solution **All** the assumptions of a theorem need to be verified **before** using the theorem's conclusion. In this case, the function f defined by $f(x) = |x|$ has no derivative at $x = 0$ as we saw earlier, and so the assumption that f *be differentiable over* $(-1, 1)$ is not true since it is not differentiable at $x = 0$. So, we can't use the Theorem at all. This just happens to be one of the many functions that doesn't satisfy the conclusion of this theorem. You can see that there's no contradiction to Rolle's Theorem since it doesn't apply.

Example 112. The function f is defined by

$$f(x) = \begin{cases} -x, & -2 \leq x \leq 0 \\ 1 - x, & 0 < x \leq 3. \end{cases}$$

In this example, the function f is defined on $[-2, 3]$ by the 2 curves in the graph and $\frac{f(3)-f(-2)}{3-(-2)} = -\frac{4}{5}$ but there is no value of c, $-2 < c < 3$ such that $f'(c) = -\frac{4}{5}$, because $f'(c) = -1$ at every c except $c = 0$ where $f'(0)$ is not defined. **Does this contradict the Mean Value Theorem?**

Solution No, there is no contradiction here. Once again, all the assumptions of the Mean Value Theorem must be verified before proceeding to its conclusion. In this example, the function f defined above is not continuous at $x = 0$ because its left-hand limit at $x = 0$ is $f_-(0) = 0$, while its right-hand limit, $f_+(0) = 1$. Since these limits are different f is not continuous at $x = 0$. Since f is not continuous at $x = 0$, it cannot be continuous on **all** of $[-2, 3]$. So, we can't apply the conclusion. So, there's nothing wrong with this function or Rolle's Theorem.

Example 113. Another very useful application of the Mean Value Theorem/Rolle's Theorem is in the theory of differential equations which we spoke of earlier.

Let y be a differentiable function for each x in $(a, b) = \{x : a < x < b\}$ and continuous in $[a, b] = \{x : a \leq x \leq b\}$. Assume that y has the property that for every number x in (a, b),

$$\frac{dy}{dx} + y(x)^2 + 1 = 0.$$

Show that this function y cannot have two zeros (or roots) in the interval $[a, b]$.

Solution Use Rolle's Theorem and show this result by assuming the contrary. This is called a **a proof by contradiction**, remember? Assume that, if possible, there are two points A, B in the interval $[a, b]$ where $y(A) = y(B) = 0$. Then, by Rolle's Theorem, there exists a point c in (A, B) with $y'(c) = 0$. Use this value of c in the equation above. This means that

$$\frac{dy}{dx}(c) + y(c)^2 + 1 = 0,$$

Summary

Rolle's Theorem

Let f be continuous at each point of a closed interval $[a, b]$ and differentiable at each point of (a, b). If $f(a) = f(b)$, then there is a point c between a and b at which $f'(c) = 0$.

Remark Don't confuse this result with Bolzano's Theorem (Chapter 2). Bolzano's Theorem deals with the existence of a **root** of a continuous function f while Rolle's Theorem deals with the existence of a root of the *derivative* of a function.

Mean Value Theorem

Let f be continuous on the interval $a \le x \le b$ and differentiable on the interval $a < x < b$. Then there is a point c between a and b at which $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Table 3.6: Rolle's Theorem and the Mean Value Theorem

right? Now, since $y'(c) = 0$, it follows that $y(c)^2 + 1 = 0$. But $y(c)^2 \ge 0$. So, this is an impossibility, it can never happen. This last statement is the contradiction. The original assumption that *there are two points* A, B *in the interval* $[a, b]$ *where* $y(A) = y(B) = 0$ must be false. So, there can't be 'two' such points. It follows that y cannot have two zeros in $a \le x \le b$.

Remark This is a really interesting aspect of most differential equations: We **really don't know what '$y(x)$' looks like** either explicitly or implicitly but still, we can get some information about its graph! In the preceding example we showed that $y(x)$ could not have two zeros, for example. This sort of analysis is part of an area of differential equations called **"qualitative analysis"**.

Note The function y defined by $y(x) = \tan(c - x)$ where c is any fixed number, has the property that $\frac{dy}{dx} + y(x)^2 + 1 = 0$. If $c = \pi$, say, then $y(x) = \tan(\pi - x)$ is such a function whose graph is reproduced in Figure 49.

Note that this function has 'lots' of zeros! Why does this graph not contradict the result of Example 113? It's because on this interval, $[0, \pi]$ the function f is not defined at $x = \pi/2$ (so it not continuous on $[0, \pi]$), and so Example 113 does not apply.

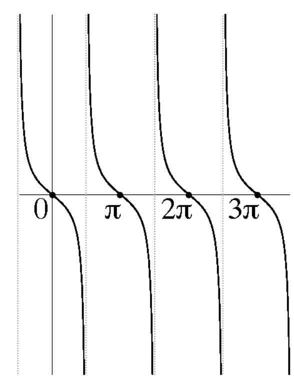


Figure 49.

Example 114. In a previous example we saw that if y is a differentiable function on $[a, b]$ and $y'(x) = 0$ for all x in (a, b) then $y(x)$ must be a constant function.

The same ideas may be employed to show that if y is a twice differentiable function on (a, b) (*i.e.* the derivative itself has a derivative), y and y' are each continuous on $[a, b]$ and if $y''(x) = 0$ for each x in (a, b) then $y(x) = mx + b$, for each $a < x < b$, for some constants m and b. That is, y must be a linear function.

Solution We apply the Mean Value Theorem to the function y' first. Look at the interval $[a, x]$. Then $\frac{y'(x)-y'(a)}{x-a} = y''(c)$ where $a < c < x$. But $y''(c) = 0$ (regardless of the value of c) so this means $y'(x) = y'(a) = constant$. Since x can be any

Intermediate Value Theorem

Let f be continuous at each point of a closed interval $[a, b] = \{x : a \leq x \leq b\}$. Assume,

1. $f(a) \neq f(b)$
2. z is a point between the numbers $f(a)$ and $f(b)$.

Then there is at least one value of c between a and b such that $f(c) = z$

Remark This result is very useful in finding the **root** of certain equations, or the points of intersection of two or more curves in the plane.

Bolzano's Theorem

Let f be **continuous on a closed interval** $[a, b]$ (i.e., at each point x in $[a, b]$).

If $f(a)f(b) < 0$, then there is at least one point c between a and b such that $f(c) = 0$. In other words there is at least one root of f in the interval (a, b).

Table 3.7: Main Theorems about Continuous Functions

number, $x > a$, and the *constant* above does not 'change' (it is equal to $y'(a)$), it follows that $y'(x) = y'(a)$ for **any** x in (a, b).

Now, apply the Mean Value Theorem to y, NOT y' ... Then

$$\frac{y(x) - y(a)}{x - a} = y'(c)$$

where $a < c < x$ (**not** the same c as before, though). But we know that $y'(c) = y'(a)$ (from what we just proved) so this means that $y(x) - y(a) = y'(a)(x - a)$ or

$$\begin{aligned} y(x) &= y'(a)(x - a) + y(a) \\ &= mx + b \end{aligned}$$

if we chose $m = y'(a)$ and $b = y(a) - ay'(a)$. That's all!

Two other big theorems of elementary Calculus are the Intermediate Value Theorem and a special case of it called Bolzano's Theorem, both of which we saw in our chapter on Limits and Continuity. We recall them here.

Bernhard Bolzano, 1781-1848, Czechoslovakian priest and mathematician who specialized in Analysis where he made many contributions to the areas of limits and continuity and, like Weierstrass, he produced a method (1850) for constructing a continuous function which has no derivative anywhere! He helped to establish the tenet that mathematical truth should rest on rigorous proofs rather than intuition.

Example 115. Show that there is a root of the equation $f(x) = 0$ in the interval $[0, \pi]$, where $f(x) = x \sin x + \cos x$.

Solution OK, what's this question about? The key words are 'root' and 'function' and at this point, basing ourselves on the big theorems above, we must be dealing with an application of **Bolzano's Theorem**, you see? (Since it deals with roots of functions, see Table 3.7.) So, let $[a, b] = [0, \pi]$ which means that $a = 0$ and $b = \pi$. Now the function whose values are given by $x \sin x$ is continuous (as it is the product of two continuous functions) and since '$\cos x$' is continuous it follows that $x \sin x + \cos x$ is continuous (as the sum of continuous functions is, once again continuous). Thus f is continuous on $[a, b] = [0, \pi]$.

What about $f(0)$? Well, $f(0) = 1$ (since $0 \sin 0 + \cos 0 = 0 + (+1) = +1$).

And $f(\pi)$? Here $f(\pi) = -1$ (since $\pi \sin \pi + \cos \pi = 0 + (-1) = -1$).

So, $f(\pi) = -1 < 0 < f(0) = 1$ which means that $f(a) \cdot f(b) = f(0) \cdot f(\pi) < 0$. So, **all** the hypotheses of Bolzano's theorem are satisfied. This means that the conclusion follows, that is, there is a point c between 0 and π so that $f(c) = 0$.

Remark Okay, but 'where' is the root of the last example?

Well, we need more techniques to solve this problem, and there is one, very useful method, called **Newton's method** which we'll see soon, (named after the same Newton mentioned in Chapter 1, one of its discoverers.)

Exercise Set 15.

Use Bolzano's theorem to show that each of the given functions has a root in the given interval. **Don't forget to verify the assumption of continuity in each case.** You may want to use your calculator.

In the first few exercises show 1) that the function is continuous, and 2) that there are two points a, b inside the given interval with $f(a)f(b) < 0$. Then use Bolzano's Theorem.

1. $y(x) = 3x - 2,\ 0 \le x \le 2$
2. $y(x) = x^2 - 1,\ -2 \le x \le 0$
3. $y(x) = 2x^2 - 3x - 2,\ 0 \le x \le 3$
4. $y(x) = \sin x + \cos x,\ 0 \le x \le \pi$
5. $y(x) = x \cos x + \sin x,\ 0 \le x \le \pi$
6. The function y has the property that y is three-times differentiable in (a, b) and continuous in $[a, b]$. If $y'''(x) = 0$ for all x in (a, b) show that $y(x)$ is of the form $y(x) = Ax^2 + Bx + C$ for a suitable choice of A, B, and C.
7. The following function y has the property that $\frac{dy}{dx} + y(x)^4 + 2 = 0$ for x in (a, b). Show that $y(x)$ cannot have two zeros in the interval $[a, b]$.
8. Use the Mean Value Theorem to show that $\sin x \le x$ for any x in the interval $[0, \pi]$.
9. Use Rolle's Theorem applied to the sine function on $[0, \pi]$ to show that the cosine function must have a root in this interval.
10. Apply the Mean Value Theorem to the sine function on $[0, \pi/2]$ to show that $x - \sin x \le \frac{\pi}{2} - 1$. Conclude that if $0 \le x \le \frac{\pi}{2}$, then $0 \le x - \sin x \le \frac{\pi}{2} - 1$.
11. Use a calculator to find that value c in the conclusion of the Mean Value Theorem for the following two functions:
    a) $f(x) = x^2 + x - 1,\ [a, b] = [0, 2]$
    b) $g(x) = x^2 + 3,\ [a, b] = [0, 1]$

    Hint In (a) calculate the number $\frac{f(b)-f(a)}{b-a}$ explicitly. Then find $f'(c)$ as a function of c, and, finally, solve for c.
12. An electron is shot through a 1 meter wide plasma field and its time of travel is recorded at 0.3×10^{-8} seconds on a timer at its destination. Show, using the Mean Value Theorem, that its velocity at some point in time had to **exceed the speed of light** in that field given approximately by 2.19×10^8 m/sec. (**Note** This effect is actually observed in nature!)

Suggested Homework Set 12. *Do problems 1, 3, 6, 8, 11*

3.7 Inverse Functions

One of the most important topics in the theory of functions is that of the **inverse of a function**, a function which is NOT the same as the reciprocal (or 1 divided by the function). Using this new notion of an inverse we are able to 'back-track' in a sense, the idea being that we interchange the domain and the range of a function when defining its inverse and points in the range get associated with the point in the domain from which they arose. These inverse functions are used everywhere in Calculus especially in the topic of finding the **area between two curves**, or calculating the **volume of a solid of revolution** two topics which we will address later. The two main topics in Calculus namely, differentiation and integration of functions, are actually related. In the more general sense of an **inverse of an operator**, these operations on functions are *almost* inverses of one another. Knowing how to manipulate and find inverse functions is a necessity for a thorough understanding of the methods in Calculus. In this section we will learn what they are, how to find them, and how to sketch them.

Review

You should be completely familiar with Chapter 1, and especially how to find the composition of two functions using the 'box' method or any other method.

We recall the notion of the **composition of two functions** here: Given two functions, f, g where the range of g is contained in the domain of f, (i.e., R=Ran(g) $\subseteq$ Dom(f)=D) we define the **composition of f and g**, denoted by the symbol $f \circ g$, a new function whose values are given by $(f \circ g)(x) = f(g(x))$ where x is in the domain of g (denoted briefly by D).

Example 116. Let $f(x) = x^2 + 1$, $g(x) = x - 1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution Recall the box methods of Chapter 1. By definition, since $f(x) = x^2 + 1$ we know that $f(\square) = \square^2 + 1$. So,

$$f(\boxed{\text{g(x)}}) = \boxed{\text{g(x)}}^2 + 1 = \boxed{\text{(x-1)}}^2 + 1 = (x-1)^2 + 1 = x^2 - 2x + 2.$$

On the other hand, when the same idea is applied to $(g \circ f)(x)$, we get $(g \circ f)(x) = x^2$.

Note: This shows that the operation of **composition is not commutative**, that is, $(g \circ f)(x) \neq (f \circ g)(x)$, in general. The point is that **composition is not the same as multiplication**.

Let f be a given function with domain, D=Dom(f), and range, R=Ran(f). We say that the function **F is the inverse of f** if all these four conditions hold:

$$\begin{aligned}
&Dom(F) = Ran(f) \\
&Dom(f) = Ran(F) \\
&(F \circ f)(x) = x, \quad \text{for every x in Dom(f)} \\
&(f \circ F)(x) = x, \quad \text{for every x in Dom(F)}
\end{aligned}$$

Thus, the inverse function's domain is R. The inverse function of f is usually written f^{-1} whereas the reciprocal function of f is written as $\frac{1}{f}$ so that $(\frac{1}{f})(x) = \frac{1}{f(x)} \neq f^{-1}(x)$. This is the source of much confusion!

Example 117. Find the composition of the functions f, g where $f(x) = 2x+3$, $g(x) = x^2$, and show that $(f \circ g)(x) \neq (g \circ f)(x)$.

Solution Using the box method or any other method we find

$$(f \circ g)(x) = f(g(x)) = 2g(x) + 3 = 2x^2 + 3$$

while

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 = (2x+3)^2 = 4x^2 + 12x + 9$$

So we see that

$$(f \circ g)(x) \neq (g \circ f)(x)$$

as the two expressions need to be exactly the same for equality.

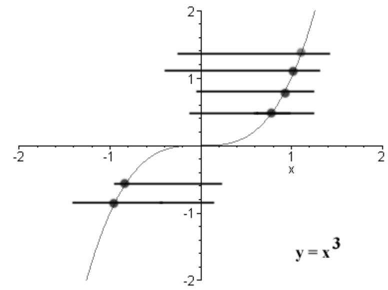


Figure 50.

Example 118. Show that the functions f, F defined by $f(x) = 2x + 3$ and $F(x) = \frac{x-3}{2}$ are inverse of one another. That is, show that F is the inverse of f and f is the inverse of F.

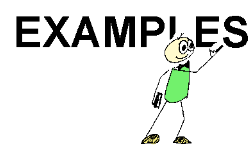
Solution As a check we note that Dom(F) = Ran(f) = $(-\infty, \infty)$ and

$$f(F(x)) = 2F(x) + 3 = 2(\frac{x-3}{2}) + 3 = x,$$

which means that $(f \circ F)(x) = x$. On the other hand, Dom(f) = Ran(F) = $(-\infty, \infty)$ and

$$F(f(x)) = \frac{f(x) - 3}{2} = \frac{(2x+3) - 3}{2} = x,$$

which now means that $(F \circ f)(x) = x$. So, by definition, these two functions are inverse functions of one another.



How can we tell if a given function has an inverse function?. In order that two functions f, F be inverses of one another it is necessary that each function be **one-to-one** on their respective domains. This means that the **only** solution of the equation $f(x) = f(y)$ (resp. $F(x) = F(y)$) is the solution $x = y$, whenever x, y are in Dom(f), (resp. Dom(F)). The simplest geometrical test for deciding whether a given function is one-to-one is the so-called *Horizontal Line Test*. Basically, one looks at the graph of the given function on the xy-plane, and if every horizontal line through the range of the function intersects the graph at only one point, then the function is one-to-one and so it has an inverse function, see Figure 50. The moral here is "Not every function has an inverse function, only those that are one-to-one!"

Example 119. Show that the function $f(x) = x^2$ has no inverse function if we take its domain to be the interval $[-1, 1]$.

Solution This is because the Horizontal Line Test shows that every horizontal line through the range of f intersects the curve at two points (except at (0,0), see Figure 51). Since the Test fails, f is not one-to-one and this means that f cannot have an inverse. Can you show that this function **does have an inverse** if its domain is restricted to the smaller interval $[0, 1]$?

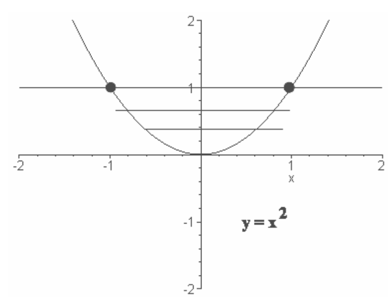


Figure 51.

Example 120. Find the form of the inverse function of the function f defined by $f(x) = 2x + 3$, where x is real.

How to find the inverse of a function

1.
   - Write $y = f(x)$
   - Solve for x in terms of y
   - Then $x = F(y)$ where F is the inverse.

2.
   - Interchange the x's and y's.
   - Solve for the symbol y in terms of x.
   - This gives $y = F(x)$ where F is the inverse.

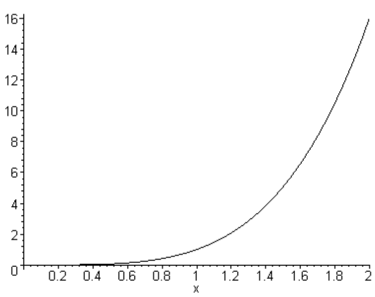
It follows that the graph of the inverse function, F, is obtained by reflecting the graph of f about the line $y = x$. More on this later.

Table 3.8: How to Find the Inverse of a Function

Solution Use Table 3.8. Write $y = f(x) = 2x + 3$. We solve for x in terms of y. Then

$$y = 2x + 3 \text{ } means \text{ } x = \frac{y-3}{2} = F(y).$$

Now interchange x and y. So the inverse of f is given by F where $F(x) = \dfrac{x-3}{2}$.



The graph of $f(x) = x^4$. If $x \geq 0$ this function is one-to-one.It is not true that f is one-to-one if the domain of f contains negative points, since in this case there are horizontal lines that intersect the graph in TWO points.

Figure 52.

Example 121. $f(x) = x^4$, $x \geq 0$, what is its inverse function $F(x)$ (also denoted by $f^{-1}(x)$) ?

Solution Let's use Table 3.8, once again. Write $y = x^4$. From the graph of f (Figure 52) we see that it is one-to-one if $x \geq 0$. Solving for x in terms of y, we get $x = \sqrt[4]{y}$ since x is real, and $y \geq 0$. So $f^{-1}(y) = F(y) = \sqrt[4]{y}$ or $f^{-1}(x) = F(x) = \sqrt[4]{x}$ is the inverse function of f.

Example 122. If $f(x) = x^3 + 1$, what is its inverse function, $f^{-1}(x)$?

Solution We solve for x in terms of y, as usual. Since $y = x^3 + 1$ we know $y - 1 = x^3$ or $x = \sqrt[3]{y-1}$ (and y can be *any* real number here). Interchanging x and y we get $y = \sqrt[3]{x-1}$, or $f^{-1}(x) = \sqrt[3]{x-1}$, or $F(x) = \sqrt[3]{x-1}$

The **derivative of the inverse function** f^{-1} of a given function f is related to the derivative of f by means of the next formula

$$\frac{dF}{dx}(x) = \frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(F(x))} \tag{3.4}$$

where the symbol $f'(f^{-1}(x))$ means that the derivative of f is evaluated at the point $f^{-1}(x)$, where x is given. **Why?**

The simplest reason is that the Chain Rule tells us that since $f(F(x)) = x$ we can differentiate the composition on the left using the Box Method (with $F(x)$ in the box...). By the Chain Rule we know that

$$Df(\square) = f'(\square) \cdot D(\square).$$

Applying this to our definition of the inverse of f we get

$$\begin{aligned} x &= f(F(x)) \\ Dx &= Df(F(x)) = Df(\square) \\ 1 &= f'(\square) \cdot D(\square) \\ &= f'(F(x)) \cdot F'(x). \end{aligned}$$

Now solving for the symbol $F'(x)$ in the last display (because this is what we want) we obtain

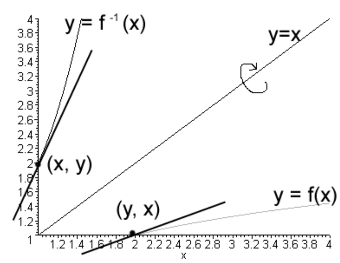
$$F'(x) = \frac{1}{f'(F(x))},$$

where $F(x) = f^{-1}(x)$ is the inverse of the original function $f(x)$. This proves our claim.

Another, more geometrical, argument proceeds like this: Referring to Figure 53 in the margin let (x, y) be a point on the graph of $y = f^{-1}(x)$. We can see that the tangent line to the graph of f has equation $y = mx + b$ where m, its slope, is also the derivative of f at the point in question (*i.e.*, $f'(y)$). On the other hand, its reflection is obtained by interchanging x, y, and so the equation of its counterpart (on the other side of $y = x$) is $x = my + b$. Solving for y in terms of x in this one we get $y = \frac{x}{m} - \frac{b}{m}$. This means that *it* has slope equal to the reciprocal of the first one. Since these slopes are actually derivatives this means that

$$(f^{-1})'(x) = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

since our point (x, y) lies on the graph of the inverse function, $y = f^{-1}(x)$.



The two tangents are reflections of one another, and so their slopes must be the reciprocal of one another (see the text).

Figure 53.

Example 123. For Example 121 above, what is $\frac{df^{-1}}{dx}(16)$? i.e., the derivative of the inverse function of f at $x = 16$?

Solution Using Equation 3.4, we have

$$(f^{-1})'(16) = \frac{1}{f'(f^{-1}(16))}$$

But $f^{-1}(x) = \sqrt[4]{x}$ means that $f^{-1}(16) = \sqrt[4]{16} = 2$. So, $(f^{-1})'(16) = \frac{1}{f'(2)}$ and now we need $f'(2)$. But $f(x) = x^4$, so $f'(2) = 4(2)^3 = 32$. Finally, we find that $(f^{-1})'(16) = \frac{1}{32}$.

Example 124. A function f with an inverse function denoted by F has the property that $F(0) = 1$ and $f'(1) = 0.2$. Calculate the value of $F'(0)$.

Solution We don't have much given here but yet we can actually find the answer as follows: Since

$$F'(0) = \frac{1}{f'(F(0))}$$

by (3.4) with $x = 0$, we set $F(0) = 1$, and note that $f'(F(0)) = f'(1)$. But since $f'(1) = 0.2$ we see that

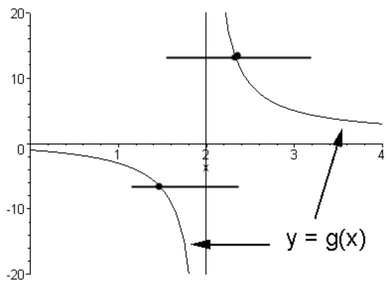
$$F'(0) = \frac{1}{f'(F(0))} = \frac{1}{f'(1)} = \frac{1}{0.2} = 5.$$

Example 125. Let g be a function defined by

$$g(x) = \frac{x+2}{x-2}$$

with $Dom(g) = \{x : x \neq 2\}$. Show that g has an inverse function, G, find its form, and describe its Domain and Range.

Solution Let's denote its inverse function by G. The first question you should be asking yourself is: "How do we know that there *is* an inverse function at all?" In other words, we have to show that the graph of g satisfies the *Horizontal Line Test* mentioned above (see Fig. 50), or, in other words, g is one-to-one. To do this we can do one of two things: We can either draw the graph as in Fig. 54 (if you have that much patience), or check the condition algebraically by showing that if $g(x) = g(y)$ then $x = y$ must be true (for any points x, y in the domain of g). Since the graph is already given in the margin we are done, but let's look at this using the algebraic test mentioned here.



The graph of the function g in Example 125. Note that any horizontal line intersects the graph of g in only one point! This means that g is one-to-one on its domain. The vertical line across the point $x = 2$ is called a vertical asymptote (a line on which the function becomes infinite). More on this in Chapter 5.

Figure 54.

In order to prove that g is one-to-one algebraically, we have to show that if $g(x) = g(y)$ then $x = y$. Basically, we use the definitions, perform some algebra, simplify and see if we get $x = y$ at the end. If we do, we're done. Let's see.

We assume that $g(x) = g(y)$ (here y is thought of as an independent variable, just like x). Then, by definition, this means that

$$\frac{x+2}{x-2} = \frac{y+2}{y-2}$$

for $x, y \neq 2$. Multiplying both sides by $(x-2)(y-2)$ we get $(x+2)(y-2) = (y+2)(x-2)$. Expanding these expressions we get

$$xy + 2y - 2x - 4 = yx + 2x - 2y - 4$$

from which we easily see that $x = y$. That's all. So g is one-to-one. Thus, its inverse function G exists.

Next, to find its values, $G(x)$, we replace all the x's by y's and solve for y in terms of x, (cf., Table 3.8). Replacing all the x's by y's (and the only y by x) we get

$$x = \frac{y+2}{y-2}.$$

Multiplying both sides by $(y-2)$ and simplifying we get

$$y = \frac{2x+2}{x-1}.$$

This is $G(x)$.

Its domain is $Dom(G) = \{x : x \neq 1\} = Ran(g)$ while its range is given by $Ran(G) = Dom(g) = \{x : x \neq 2\}$ by definition of the inverse.

Now that **we know how to find the form of the inverse** of a given (one-to-one) function, the natural question is: "What does it look like?". Of course, it is simply another one of those graphs whose shape may be predicted by means of existing computer software or by the old and labor intensive method of finding the critical points of the function, the asymptotes, etc. So, *why worry about the graph of an inverse function?* Well, one reason is that the **graph of an inverse function is related to the graph of the original function** (that is, the one for which it is the inverse). How? Let's have a look at an example.

Example 126. Let's look at the graph of the function $f(x) = \sqrt{x}$, for x in $(0, 4)$, and its inverse, the function, $F(x) = x^2$, for x in $(0, 2)$, Figure 55.

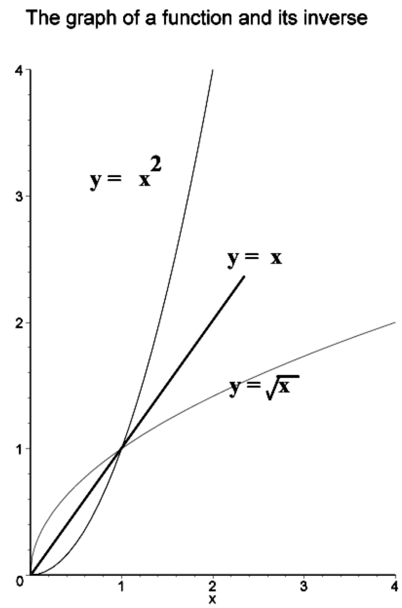


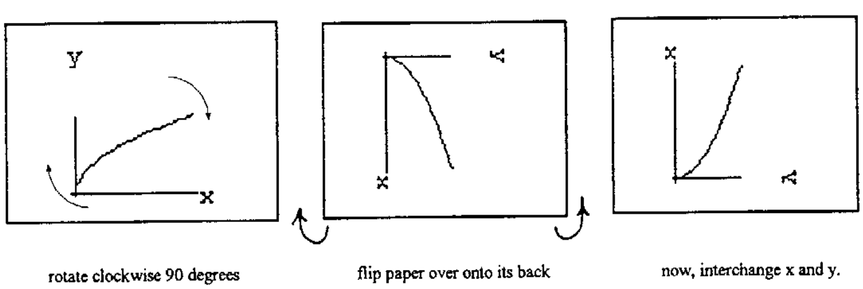
Figure 55. *The graphs of $y = x^2$ and its inverse, $y = \sqrt{x}$ superimposed on one another*

When we study these graphs carefully, we note, by definition of the inverse function, that the **domain and the range are interchanged.** So, this means that if we interchanged the x-axis (on which lies the domain of f) and the y-axis, (on which we find the range of f) we would be in a position to graph the inverse function of f. This graph of the inverse function is simply the reflection of the graph for $y = \sqrt{x}$, about the line $y = x$. Try it out ! Better still, check out the following experiment!

EXPERIMENT:

1. Make a copy of the graph of $f(x) = \sqrt{x}$, below, by tracing it onto some **tracing paper** (so that you can see the graph from both sides). Label the axes, and fill in the domain and the range of f by thickening or thinning the line segment containing them, or, if you prefer, by colouring them in.
2. Now, **turn the traced image around, clockwise, by 90 degrees** so that the x-axis is vertical (but pointing down) and the y-axis is horizontal (and pointing to the right).
3. Next, **flip the paper over onto its back** without rotating the paper! **What do you see? The graph of the inverse function** of $f(x) = \sqrt{x}$, that is, $F(x) = x^2$.

REMARK This technique of making the graph of the inverse function by rotating the original graph clockwise by 90 degrees and then flipping it over always works! You will **always** get the graph of the inverse function on the back side (verso), as if it had been sketched on the x and y axes as usual (once you interchange x and y). Here's a visual summary of the construction ...



Why does this work? Well, there's some *Linear Algebra* involved. (The author's module entitled *The ABC's of Calculus: Module on Inverse Functions* has a thorough explanation!)

We summarize the above in this

RULE OF THUMB. We can always find the graph of the inverse function by applying the above construction to the original graph

or, equivalently,

by reflecting the original graph of f about the line $y = x$ and eliminating the original graph.

Example 127. We sketch the graphs of the function f, and its inverse, F, given by $f(x) = 7x+4$ and $F(x) = \frac{x-4}{7}$, where $Dom(f) = \Re$, where $\Re = (-\infty, +\infty)$. The graphs of f and its inverse superimposed on the same axes are shown in Figure 56.

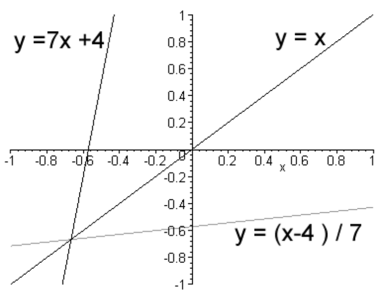


Figure 56. *The graphs of $f(x) = 7x + 4$ and its inverse $F(x) = \frac{x-4}{7}$ superimposed on one another.*

NOTE THAT if you are **given the graph of the inverse function**, $F(x)$, of a function $f(x)$, you can **find the graph of** $f(x)$ by applying the preceding "rule of thumb" with f and F interchanged. Furthermore, the inverse of the inverse function of a function f (so, we're looking for the inverse) is f itself. Why? Use the definition of the inverse! We know that $F(f(x)) = x$, for each x in $Dom(f)$, and $x = f(F(x))$ for each x in $Dom(F)$; together, these relations say that "F is the inverse of f". If we interchange the symbols 'F' and 'f' in this equation we get the **same** equation with the interpretation "f is the inverse of F", which is what we wanted!

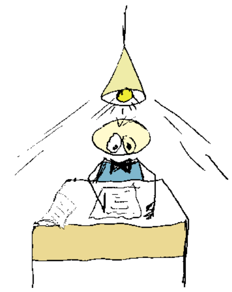
Exercise Set 16.

Sketch the graphs of the following functions and their inverses. Don't forget to indicate the domain and the range of each function.

1. $f(x) = 4 - x^2, \quad 0 \le x \le 2$
2. $g(x) = (x-1)^{-1}, \quad 1 < x < \infty$
3. $f(z) = 2 - z^3, \quad -\infty < z < \infty$
4. $h(x) = \sqrt{5+2x}, \quad -\frac{5}{2} \le x < \infty$
5. $f(y) = (2+y)^{\frac{1}{3}}, \quad -2 < y < \infty$
6. Let f be a function with domain $D = \Re$. Assume that f has an inverse function, F, defined on $\Re$ (another symbol for the real line) also.
   (i) Given that $f(2) = 0$, what is $F(0)$?
   (ii) If $F(6) = -1$, what is $f(-1)$?
   (iii) Conclude that the only solution of $f(x) = 0$ is $x = 2$.
   (iv) Given that $f(-2) = 8$, what is the solution y, of $F(y) = -2$? Are there any other solutions ??
   (v) We know that $f(-1) = 6$. Are there any other points, x, such that $f(x) = 6$?
7. Given that f is such that its inverse F exists, $f'(-2.1) = 4$, $F(-1) = -2.1$, find the value of the derivative of F at $x = -1$.

Find the form of the inverse of the given functions on the given domain and determine the Domain and the Range of the inverse function. Don't forget to show that each is one-to-one first.

8. $f(x) = x, \qquad -\infty < x < +\infty$
9. $f(x) = \dfrac{1}{x}, \qquad x \neq 0$
10. $f(x) = x^3, \qquad -\infty < x < +\infty$
11. $f(t) = 7t + 4, \qquad 0 \leq t \leq 1$
12. $g(x) = \sqrt{2x+1}, \qquad x \geq -\dfrac{1}{2}$
13. $g(t) = \sqrt{1-4t^2}, \qquad 0 \leq t \leq \dfrac{1}{2}$
14. $f(x) = \dfrac{2+3x}{3-2x}, \qquad x \neq \dfrac{3}{2}$
15. $g(y) = y^2 + y, \qquad -\dfrac{1}{2} \leq y < +\infty$



Suggested Homework Set 13. *Work out problems 3, 5, 6, 8, 12, 15.*

Web Links

More on Inverse Functions at:

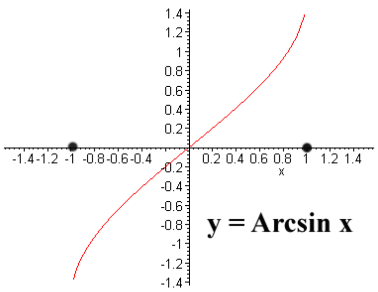
library.thinkquest.org/2647/algebra/ftinvers.htm
(requires a Java-enabled browser)
www.khanacademy.org/math/algebra/algebra-functions/v/function-inverse-example-1
www.math.duke.edu/education/ccp/materials/intcalc/inverse/index.html

NOTES:

3.8 Inverse Trigonometric Functions

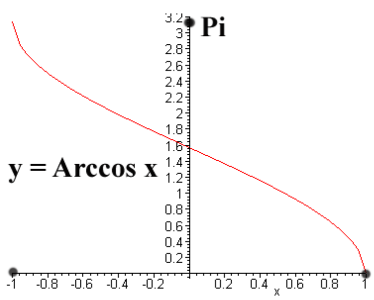
When you think of the graph of a trigonometric function you may have the general feeling that it's very *wavy*. In this case, the Horizontal Line Test should fail as horizontal lines through the range will intersect the graph quite a lot! So, how can they have an inverse? The only way this can happen is by making the domain 'small enough'. It shouldn't be surprising if it has an inverse on a suitable interval. So, every trigonometric function has an inverse on a suitably defined interval.

At this point we introduce the notion of the **inverse of a trigonometric function**. The graphical properties of the sine function indicate that it has an inverse when $\mathrm{Dom}(\sin) = [-\pi/2, \pi/2]$. Its inverse is called the **Arcsine function** and it is defined for $-1 \le x \le 1$ by the rule that



$$y = \underline{\mathrm{A}}\mathrm{rc}\underline{\sin}(\underline{x}) \text{ means that } y \text{ is an } \underline{\text{a}}\text{ngle whose } \underline{\text{sine}} \text{ is } \underline{x}.$$

Since $\sin(\pi/2) = 1$, it follows that $\mathrm{Arcsin}(1) = \pi/2$. The cosine function with $\mathrm{Dom}(\cos) = [0, \pi]$ also has an inverse and it's called the **Arccosine function**, This Arccosine function is defined for $-1 \le x \le 1$, and its rule is given by $y = \mathrm{Arccos}(x)$ which means that y is an angle whose cosine is x. Thus, $\mathrm{Arccos}(1) = 0$, since $\cos(0) = 1$. Finally, the tangent function defined on $(-\pi/2, \pi/2)$ has an inverse called the **Arctangent function** and it's defined on the interval $(-\infty, +\infty)$ by the statement that $y = \mathrm{Arctan}(x)$ only when y is an angle in $(-\pi/2, \pi/2)$ whose tangent is x. In particular, since $\tan(\pi/4) = 1$, $\mathrm{Arctan}(1) = \pi/4$. The remaining inverse trigonometric functions can be defined by the relations $y = \mathrm{Arccot}(x)$, the **Arccotangent function**, which is defined only when y is an angle in $(0, \pi)$ whose cotangent is x (and x is in $(-\infty, +\infty)$). In particular, since $\cot(\pi/2) = 0$, we see that $\mathrm{Arccot}(0) = \pi/2$. Furthermore, $y = \mathrm{Arcsec}(x)$, the **Arcsecant function**, only when y is an angle in $[0, \pi]$, different from $\pi/2$, whose secant is x (and x is outside the open interval $(-1, 1)$). In particular, $\mathrm{Arcsec}(1) = 0$, since $\sec(0) = 1$. Finally, $y = \mathrm{Arccsc}(x)$, the **Arccosecant function**, only when y is an angle in $[-\pi/2, \pi/2]$, different from 0, whose cosecant is x (and x is outside the open interval $(-1, 1)$). In particular, since $\csc(\pi/2) = 1$, $\mathrm{Arccsc}(1) = \pi/2$.



NOTE: sin, cos are defined for all x (**in radians**) but this is **not true** for their 'inverses', arcsin (or Arcsin), arccos (or Arccos). Remember that the inverse of a function is always defined on the **range** of the original function.

Example 128. Evaluate Arctan(1).

Solution By definition, we are looking for an angle <u>in radians</u> whose tangent is 1. So $y = \mathrm{Arctan}(1)$ means $\tan y = 1$ or $y = \dfrac{\pi}{4}$.

Function	Domain	Range
$y = \mathrm{Arcsin}\, x$	$-1 \le x \le +1$	$-\frac{\pi}{2} \le y \le +\frac{\pi}{2}$
$y = \mathrm{Arccos}\, x$	$-1 \le x \le +1$	$0 \le y \le \pi$
$y = \mathrm{Arctan}\, x$	$-\infty < x < +\infty$	$-\frac{\pi}{2} < y < +\frac{\pi}{2}$
$y = \mathrm{Arccot}\, x$	$-\infty < x < +\infty$	$0 < y < \pi$
$y = \mathrm{Arcsec}\, x$	$\mid x \mid \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$y = \mathrm{Arccsc}\, x$	$\mid x \mid \ge 1$	$-\frac{\pi}{2} \le y \le +\frac{\pi}{2},\ y \ne 0$

Table 3.9: The Inverse Trigonometric Functions

Example 129. Evaluate Arcsin($\frac{1}{2}$).

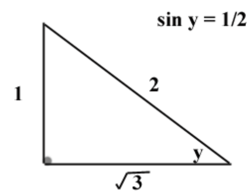


Figure 57.

Solution By definition, we are looking for an angle in <u>radians</u> whose sine is $\dfrac{1}{2}$. So $y = \text{Arcsin}(\dfrac{1}{2})$ means $\sin y = \dfrac{1}{2}$ or $y = \dfrac{\pi}{6}$ (see Figure 57).

Example 130. Evaluate Arccos($\frac{1}{\sqrt{2}}$).

Solution By definition, we seek an angle in <u>radians</u> whose cosine is $\dfrac{1}{\sqrt{2}}$. So $y = \text{Arccos}(\dfrac{1}{\sqrt{2}})$ means $\cos y = \dfrac{1}{\sqrt{2}}$ or $y = \dfrac{\pi}{4}$.

Example 131. Evaluate Arcsec($\sqrt{2}$).

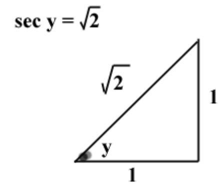


Figure 58.

Solution By definition,we are looking for an angle in radians whose secant is $\sqrt{2}$. So $y = \text{Arcsec}(\sqrt{2})$ means $\sec y = \sqrt{2}$ $(= \frac{\sqrt{2}}{1})$. The other side has length $s^2 = (\sqrt{2})^2 - 1^2 = 2 - 1 = 1$. So $s = 1$. Therefore, the $\triangle$ is isosceles and $y = \dfrac{\pi}{4}$ (see Figure 58).

Example 132. Find the value of $\sin(\text{Arccos}(\frac{\sqrt{2}}{2}))$.

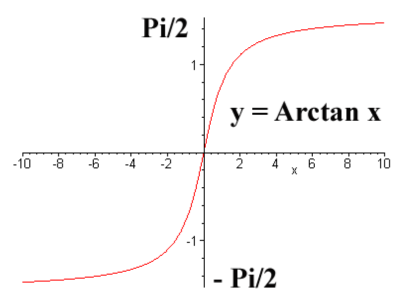
Solution Let $y = \text{Arccos}(\dfrac{\sqrt{2}}{2})$ then $\cos y = \dfrac{\sqrt{2}}{2}$. But we want $\sin y$. So, since $\cos^2 y + \sin^2 y = 1$, we get

$$\sin y = \pm\sqrt{1 - \cos^2 y} = \pm\sqrt{1 - \frac{1}{2}} = \pm\frac{1}{\sqrt{2}}.$$

Hence

$$\sin(\text{Arccos}(\frac{\sqrt{2}}{2})) = \frac{1}{\sqrt{2}} \;(= \frac{\sqrt{2}}{2}).$$

Example 133. Find $\sec(\text{Arctan}(-\frac{1}{2}))$.



Solution Now $y = \text{Arctan}(-\dfrac{1}{2})$ means $\tan y = -\dfrac{1}{2}$ but we want $\sec y$. Since $\sec^2 y - \tan^2 y = 1$ this means $\sec y = \pm\sqrt{1 + \tan^2 y} = \pm\sqrt{1 + \frac{1}{4}} = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{2}$. Now we use Table 3.10, above.

Now, if we have an angle whose tangent is $-\frac{1}{2}$ then the angle is either in II or IV. But the angle must be in the interval $(-\frac{\pi}{2}, 0)$ of the domain of definition $(-\frac{\pi}{2}, \frac{\pi}{2})$ of tangent. Hence it is in IV and so its secant is > 0. Thus, $\sec y = \sqrt{5}/2$ and we're done.

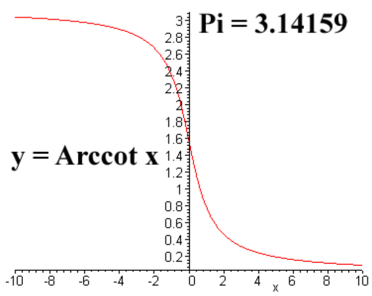
Example 134. Find the sign of $\sec(\text{Arccos}(\frac{1}{2}))$.

Solution Let $y = \text{Arccos}(\dfrac{1}{2}) \Rightarrow \cos y = \dfrac{1}{2} > 0$, therefore y is in I or IV. By definition,

Other Method: Signs of Trigonometric Functions

Quadrant	sin	cos	tan	cot	sec	csc
I	+	+	+	+	+	+
II	+	-	-	-	-	+
III	-	-	+	+	-	-
IV	-	+	-	-	+	-

Table 3.10: Signs of Trigonometric Functions



$\text{Arccos}(\frac{1}{2})$ is in $[0, \pi]$. Therefore y is in I or II, but this means that y must be in I. So, $\sec y > 0$ by Table 3.10, and this forces $\sec\left(\text{Arccos}\left(\frac{1}{2}\right)\right) = \sec y > 0$.

Example 135. Determine the sign of the number $\csc(\text{Arcsec}(2))$.

Solution Let $y = \text{Arcsec}(2)$. Then $\sec y = 2 > 0$. Therefore y is in I or IV. By definition, Arcsec(2) is in I or II. Therefore y is in I, and from the cosecant property, $\csc y > 0$ if y is in $[0, \frac{\pi}{2})$. So, $\csc(\text{Arcsec}(2)) = \csc y > 0$.

Example 136. Find the sign of $\tan(\text{Arcsin}(-\dfrac{1}{2}))$.

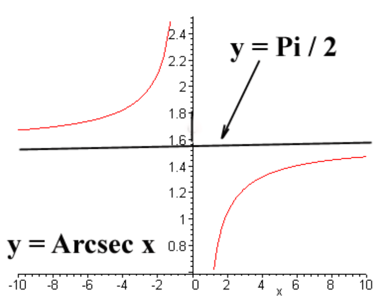
Solution Let $y = \text{Arcsin}(-\dfrac{1}{2})$. Then $\sin y = -\frac{1}{2} \Rightarrow y$ in III or IV. By definition of Arcsin; But, $y = \text{Arcsin}(-\dfrac{1}{2})$ must be in I or IV. Therefore y is in IV. So $\tan(\text{Arcsin}(-\dfrac{1}{2})) < 0$ (because $\tan < 0$ in IV).

CAREFUL!!

Many authors of Calculus books use the following notations for the inverse trigonometric functions:

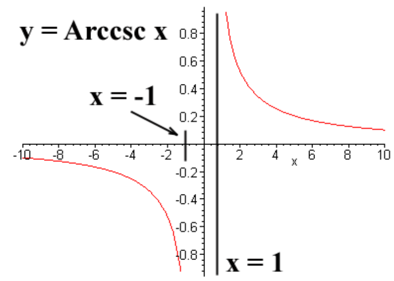
$$\text{Arcsin}\ x \Longleftrightarrow \sin^{-1} x$$
$$\text{Arccos}\ x \Longleftrightarrow \cos^{-1} x$$
$$\text{Arctan}\ x \Longleftrightarrow \tan^{-1} x$$
$$\text{Arccot}\ x \Longleftrightarrow \cot^{-1} x$$
$$\text{Arcsec}\ x \Longleftrightarrow \sec^{-1} x$$
$$\text{Arccsc}\ x \Longleftrightarrow \csc^{-1} x$$

The reason we try to avoid this notation is because it makes too many readers associate it with the reciprocal of those trigonometric functions and not their inverses. The reciprocal and the inverse are really different! Still, **you should be able to use both notations interchangeably**. It's best to know what the notation means, first.



NOTE: The inverse trigonometric functions we defined here in Table 3.9, are called the **principal branch** of the inverse trigonometric function, and we use the notation with an upper case letter '**A**' for **A**rcsin, etc. to emphasize this. **Non-principal**

branches of these inverse trigonometric functions begin with the lower case letter. Thus, $\arctan x$ refers to a non-principal branch of the inverse tangent function, i.e., one whose range is something other than $(-\pi/2, \pi/2)$, something like $(\pi/2, 3\pi/2)$ or $(3\pi/2, 5\pi/2)$. In the same vein, $\arcsin x$, $\arccos x$, etc. denote non-principal branches of the arcsine and arccosine functions, repectively. Just about everything you ever wanted know about the basic theory of principal and non-principal branches of the inverse trigonometric functions may be found in the author's *Module on Inverse Functions* in the series *The ABC's of Calculus*, The Nolan Company, Ottawa, 1994.



Finally, we emphasize that since these functions are *inverses* then for any symbol, $\Box$, representing some point in the domain of the corresponding inverse function (see Table 3.9), we always have

$$\begin{array}{rclcrcl}
\sin(\text{Arcsin } \Box) &=& \Box & \qquad & \cos(\text{Arccos } \Box) &=& \Box \\
\tan(\text{Arctan } \Box) &=& \Box & & \cot(\text{Arccot } \Box) &=& \Box \\
\sec(\text{Arcsec } \Box) &=& \Box & & \csc(\text{Arccsc } \Box) &=& \Box
\end{array}$$

the same equalities holding, in fact, for *any branch* of the inverse functions and not just only the principal one. (So we can replace the upper case "A" by lower case "a" above.)

On the other hand, the relationship between principal and non-principal branches of the main inverse trigonometric functions is given by,

$$\boxed{\begin{array}{c} \text{Arcsin}\,(\Box) = \ (-1)^m \arcsin(\Box) + m\pi, \qquad \text{Arccos}\,(\Box) = \ \pm\arccos(\Box) + 2m\pi, \\ \text{Arctan}\,(\Box) = \ \arctan(\Box) + m\pi \end{array}}$$

where m is an integer that may be positive, negative, or zero and that depends on the branch being considered. Another way of writing this is,

$$\boxed{\begin{array}{c} \arcsin(\sin \Box) = \ (-1)^m \Box + m\pi, \qquad \arccos(\cos \Box) = \ \pm\Box + 2m\pi, \\ \arctan(\tan \Box) = \ \Box + m\pi \end{array}}$$

where m is as before. (This number m differs from case-to-case, of course).

Example 137. Find the value of $\text{Arcsin}\,(\sin(2\pi/3))$.

Solution If x lies in $[-\pi/2, \pi/2]$, then it is clear from the definition of the principal branch that $\text{Arcsin}\,(\sin x) = x$. But here $x = 2\pi/3$ and $2\pi/3$ is NOT in the interval $[-\pi/2, \pi/2]$. So, what do we do? Well, we calculate the value of $\sin 2\pi/3$ which has to be some number in the interval $[-1, 1]$. When we apply the Arcsin function to this number $\sin 2\pi/3 = \sqrt{3}/2$ we HAVE to get an ANGLE in $[-\pi/2, \pi/2]$ (because that is how the main branch of the arcsine function is defined). So what is $\text{Arcsin}\,(\sqrt{3}/2)$? Remember, we are looking for an angle in the interval $[-\pi/2, \pi/2]$ whose sine is $\sqrt{3}/2$. But then, it is easy to see that $\sin(\pi/3) = \sqrt{3}/2$ which means that $\text{Arcsin}\,(\sqrt{3}/2) = \pi/3$. In conclusion,

$$\text{Arcsin}\,(\sin(2\pi/3)) = \pi/3,$$

not quite what you expect, right?

Remark: Observe that Example 137 is an instance of the case $m = 1$ in the first formula in the box above Example 137.

Example 138. Find the value of $\text{Arccos}\,(\cos(13\pi/4))$.

Solution Recall that whenever x lies in $[0, \pi]$, then the definition of the principal branch of the arccosine function tells us that $\text{Arccos}\,(\cos x) = x$. Now $x = 13\pi/4$ and of course this number is not in the interval $[0, \pi]$. So, as before, we calculate the value of $\cos 13\pi/4$ which is some number in the interval $[-1, 1]$. When we apply the Arccosine function to $\cos 13\pi/4 = \cos(3\pi + \pi/4) = -\sqrt{2}/2$ we HAVE to get an ANGLE in $[0, \pi]$ (because that is how the main branch of the arccosine function is defined). So what is $\text{Arccos}\,(-\sqrt{2}/2)$? So, we are looking for an angle in the interval $[0, \pi]$ whose cosine is $-\sqrt{2}/2$. Since the cosine is negative the angle must be in either QIII or QIV. Finally, we see that $3\pi/4$ is the required angle since $\cos(3\pi/4) = -\sqrt{2}/2$. Therefore,

$$\text{Arccos}\,(\cos(13\pi/4)) = 3\pi/4,$$

(and there is no misprint here).

Remark: Observe that Example 138 is an instance of the case $m = 2$, with a "$-\Box$", in the second formula in the box above Example 137. (Specifically, $3\pi/4 = -(13\pi/4) + 2 \times 2 \times \pi$.)

Example 139. Find the value of $\text{Arctan}\,(\tan(-16\pi/3))$.

Solution For x in $(-\pi/2, \pi/2)$, the definition of the principal branch of the arctangent function tells us that $\text{Arctan}\,(\tan x) = x$. Since $x = -16\pi/3$, this number is not in the interval $(-\pi/2, \pi/2)$. Next, $\tan(-16\pi/3)$ must be some number in the interval $(-\infty, +\infty)$. In fact,

$$\tan(-16\pi/3) = \tan(-5\pi - \pi/3) = \tan(-\pi/3) = -\tan(\pi/3) = -\sqrt{3}.$$

Finally, $\text{Arctan}\,(-\sqrt{3})$, denotes an angle whose tangent is equal to $-\sqrt{3}$. Since the tangent is negative this angle must be in QII or QIV. It is now easy to see that it must be $-\pi/3$. So,

$$\text{Arctan}\,(\tan(-16\pi/3)) = -\pi/3.$$

Remark: Example 139 is an instance of the case $m = 5$, in the third formula in the box above Example 137. (Specifically, $-\pi/3 = -(16\pi/3) + 5 \times \pi$.)

NOTE: In problems involving *negative angles* we may use the identities $\sin(-\Box) = -\sin(\Box)$, $\cos(-\Box) = \cos(\Box)$ and $\tan(-\Box) = -\tan(\Box)$ to convert said angles to positive ones.

Exercise Set 17.

Evaluate the following expressions.

1. $\sin(\text{Arccos}(0.5))$, 2. $\cos(\text{Arcsin}(0))$, 3. $\sec(\sin^{-1}(\frac{1}{2}))$
4. $\csc(\tan^{-1}(-\frac{1}{2}))$, 5. $\sec(\sin^{-1}\frac{\sqrt{3}}{2})$, 6. $\text{Arcsin}(\tan(-\pi/4))$
7. $\text{Arctan}\,(\tan\pi/4))$, 8. $\sin(\text{Arcsin}\,1)$, 9. $\cos(\cos^{-1}\frac{\sqrt{2}}{2})$
10. $\tan(\arctan(-1))$, 11. $\text{Arcsin}\,(\sin\pi)$, 12. $\text{Arccos}\,(\cos\frac{\pi}{2})$
13. $\text{Arcsin}\,(\sin(-\frac{2\pi}{3}))$, 14. $\text{Arccos}\,(\cos(\frac{5\pi}{4}))$, 15. $\text{Arctan}\,(\tan(\frac{3\pi}{4}))$
16. $\arcsin(\sin(-\frac{\pi}{2}))$, 17. $\arctan(\tan(-\frac{\pi}{4}))$, 18. $\arccos(\cos\pi)$
19. $\text{Arctan}\,(1+\tan\pi)$, 20. $2\text{Arctan}\,(3\tan(3))$, 21. $\text{Arctan}\,(\sqrt{3}+2\tan\pi)$.

Suggested Homework Set 14. *Do problems 1, 7, 9, 10, 12, 16, 18, 19, 21.*

NOTES:

3.9 Derivatives of Inverse Trigonometric Functions

Now that we know what these inverse trigonometric functions are, how do we find the derivative of the inverse of, say, the Arcsine (sin^{-1}) function? Well, we know from equation 3.4 that

$$\frac{dF}{dx}(x) = \frac{1}{f'(F(x))}$$

where $F(x) = f^{-1}(x)$ is the more convenient notation for the inverse of f. Now let $f(x) = \sin x$, and $F(x) = \text{Arcsin}\, x$ be its inverse function. Since $f'(x) = \cos x$, we see that

$$\begin{aligned} \frac{d}{dx}\text{Arcsin}\, x &= \frac{dF}{dx}(x) \\ &= \frac{1}{f'(F(x))} \\ &= \frac{1}{\cos(F(x))} \\ &= \frac{1}{\cos(\text{Arcsin}\, x)} \end{aligned}$$

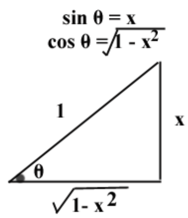


Figure 59.

Now, let $\theta = \text{Arcsin}\, x$, where θ is a lowercase Greek letter pronounced 'thay-ta'. It is used to denote angles. Then, by definition, $\sin\theta = x$, and we're looking for the value of $\cos\theta$, right? But since $\sin^2(\theta) + \cos^2(\theta) = 1$, this means that $\cos\theta = \pm\sqrt{1 - x^2}$. So, which is it? There are two choices, here.

Look at the definition of the Arcsin function in Table 3.9. You'll see that this function is defined only when the domain of the original sin function is restricted to $[-\pi/2, \pi/2]$. But, by definition, $Ran(\text{Arcsin}) = Dom(\sin) = [-\pi/2, \pi/2]$. So, $\cos\theta = \cos \text{Arcsin}\, x \geq 0$ because $\text{Arcsin}\, x$ is in the interval $[-\pi/2, \pi/2]$ and the cos function is either 0 or positive in there. So we must choose the '+' sign. Good. So,

$$\cos(\text{Arcsin}\, x) = \sqrt{1 - x^2}.$$

For another argument, see Figure 59. Finally, we see that

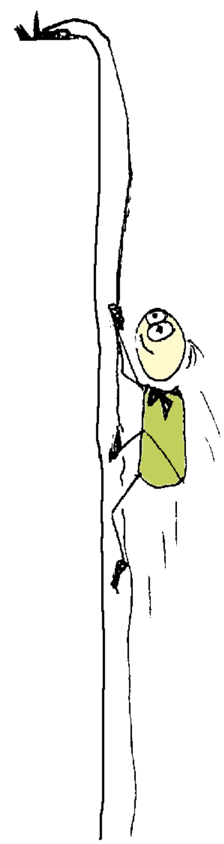
$$\begin{aligned} \frac{d}{dx}\text{Arcsin}\, x &= \frac{1}{\cos(\text{Arcsin}\, x)} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

The other derivatives are found using a similar approach.

$$\begin{array}{ll} \dfrac{d}{dx}\sin^{-1}(u) = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx} & \dfrac{d}{dx}\cos^{-1}(u) = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |\, u \,| < 1, \\ \dfrac{d}{dx}\tan^{-1}(u) = \dfrac{1}{1+u^2}\dfrac{du}{dx} & \dfrac{d}{dx}\cot^{-1}(u) = \dfrac{-1}{1+u^2}\dfrac{du}{dx} \\ \dfrac{d}{dx}\sec^{-1}(u) = \dfrac{1}{|\, u \,|\sqrt{u^2-1}}\dfrac{du}{dx}, & \dfrac{d}{dx}\csc^{-1}(u) = \dfrac{-1}{|\, u \,|\sqrt{u^2-1}}\dfrac{du}{dx}, \ |\, u \,| > 1 \end{array}$$

Table 3.11: Derivatives of Inverse Trigonometric Functions

If we let $u = u(x) = \square$ be any differentiable function, then we can use the basic derivative formulae and derive very general ones using the Chain Rule. In this way we can obtain Table 3.11.



Example 140. Evaluate the derivative of $y = \cos^{-1}(\dfrac{1}{x})$, (or Arccos$(\frac{1}{x})$).

Solution You can use any method here, but it always comes down to the Chain Rule. Let $u = \dfrac{1}{x}$, *then* $\dfrac{du}{dx} = -\dfrac{1}{x^2}$. By Table 3.11,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \\
&= -\frac{1}{\sqrt{1-(\frac{1}{x})^2}}(-\frac{1}{x^2}) = \frac{\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}} \\
&= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{\sqrt{x^2}}{|x|^2\ \sqrt{x^2-1}} \\
&= \frac{|x|}{|x|^2\ \sqrt{x^2-1}} \\
&= \frac{1}{|x|\ \sqrt{x^2-1}}, \quad (|x|>1).
\end{aligned}$$

Example 141. Evaluate the derivative of $y = \cot^{-1}(\sqrt{x})$.

Solution Let $u = \sqrt{x}$, *then* $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. So,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx} \\
&= -\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x}(1+x)}.
\end{aligned}$$

Example 142. If $y = \csc^{-1}(\sqrt{x+1})$, what is $y'(x)$?

Solution Let $u = \sqrt{x+1}$, *then* $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x+1}}$. So,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\csc^{-1}u = \frac{-1}{|\,u\,|\sqrt{u^2-1}}\frac{du}{dx} \\
&= \frac{-1}{\sqrt{x+1}(\sqrt{x+1-1})}\cdot\frac{1}{2\sqrt{x+1}} \\
&= \frac{-1}{2(x+1)\sqrt{x}}.
\end{aligned}$$

NOTES:

Exercise Set 18.

Use Table 3.11 and the Chain Rule to find the derivatives of the functions whose values are given here.

1. $\text{Arcsin}(x^2)$, at $x = 0$
2. $x^2\ \text{Arccos}(x)$
3. $\tan^{-1}(\sqrt{x})$
4. $\text{Arcsin}(\cos x)$
5. $\dfrac{\sin^{-1} x}{\sin x}$
6. $\sqrt{\sec^{-1} x}$
7. $\sin(2\text{Arcsin}\, x)$, at $x = 0$
8. $\cos(\sin^{-1}(4x))$
9. $\dfrac{1}{\text{Arctan} x}$
10. $x^3\ \text{Arcsec}(x^3)$

Suggested Homework Set 15. *Do problems 1, 4, 5, 7, 9*

Web Links

On the topic of Inverse Trigonometric Functions see:

http://www.khanacademy.org/math/trigonometry/

NOTES:

3.10 L'Hospital's Rule

The Big Picture

In Chapter 2 we saw various methods for evaluating limits; from using their properties as continuous functions to possibly applying *extended real numbers* (see the web-site for this one). They all give the same answer, of course. Now that we've mastered the machinery of the derivative we can derive yet another method for handling limits involving the so-called **indeterminate forms**. This method was described by one **Marquis de L'Hospital** (1661-1704), and pronounced 'Lo-pit-al', who in fact wrote the first book ever on Calculus back in 1696. L'Hospital was a student of the famous mathematician **Johann Bernoulli** (1667-1748), who absorbed the methods of Leibniz from the master himself. This 'rule' was likely due to Bernoulli who discovered things faster than he could print them! So, it became known as L'Hospital's Rule because it first appeared in L'Hospital's Calculus book. Actually, most of what you're learning in this book is more than 300 years old so it must be really important in order to survive this long, right?

Review

You should review all the material on limits from Chapter 2. You should be really good in finding derivatives too! The section on **Indeterminate Forms** is particularly important as this method allows you yet another way of evaluating such mysterious looking limits involving '0/0', '∞/∞', etc. Also remember the basic steps in evaluating a limit: Rewrite or simplify or rationalize, and finally evaluate using whatever method (this Rule, extended real numbers, continuity, numerically by using your calculator, and finally, incantations).

We begin by recalling the notion of an indeterminate form. A limit problem of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

is called an **indeterminate form** if the expression $f(a)/g(a)$ is one of the following types:

$$\frac{\infty}{\pm\infty}, \quad \infty - \infty, \quad (\pm\infty)^0, \quad 1^{\pm\infty}, \quad \frac{0}{0}, \quad 0^0$$

Up until now, when you met these forms in a limit you couldn't do much except simplify, rationalize, factor, etc. and then see if the form becomes "determinate". If the numerator and denominator are both differentiable functions with some nice properties, then it is sometimes possible to determine the limit by appealing to **L'Hospital's Rule**. Before we explore this Rule, a few words of caution ...

CAUTION

1. The Rule is about LIMITS
2. The Rule always involves a QUOTIENT of two functions

So, what this means is "**If your limit doesn't involve a quotient of two functions then you can't use the Rule!**" So, if you can't use the Rule, you'll have to **convert your problem** into one where you can use it.

Before describing this Rule, we define the simple notions of a **neighborhood of a point** a. Briefly stated, if a is finite, a neighborhood of a consists of an open interval (see Chapter 1) containing a.

Example 143. The interval $(-1, 0.5)$ is a neighborhood of 0, as is the interval $(-0.02, 0.3)$.

In the same vein, a **left-neighborhood** of $x = a$ consists of an open interval with a as its right endpoint.

Example 144. The interval $(-1, 0)$ is a left-neighborhood of 0, so is $(-3, 0)$, or $(-2.7, 0)$, etc.

Similarly, a **right-neighborhood** of $x = a$ consists of an open interval with a has its left endpoint

Example 145. The interval $(1, 4)$ is a right-neighborhood of 1, so is $(1, 1.00003)$, or $(1, 1000)$, etc

Table showing the likelihood that $\sin x/x \to 1$ as $x \to 0$. **Note that x can be positive or negative so long as $x \to 0$.**

x	$f(x)/g(x)$
.50000	.95885
−.33333	.98158
.25000	.98961
−.20000	.99334
−.12500	.99740
.10000	.99833
.01000	.99998
.00826	.99998
.00250	.99998
−.00111	.99999
−.00010	.99999
.00008	.99999
.00002	.99999
−.00001	.99999
.00001	.99999
...	...
0	1.00000

Figure 60.

Finally, a **punctured neighborhood of** a is an open interval around a without the point a itself. Just think of it as an *open interval with one point missing.*

Example 146. The interval $(-0.5, 0.2)$ without the point 0, is a punctured neighborhood of 0. Also, the interval $(-1, 6)$ without the point 0, is a punctured neighborhood of 0. The statement of L'Hospital's Rule is in Table 3.12.

L'Hospital's Rule

Let f, g be two functions defined and differentiable in a punctured neighborhood of a, where a is finite. If $g'(x) \neq 0$ in this punctured neighborhood of a and $f(a)/g(a)$ is one of $\pm\infty/\infty$, *or* $0/0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (it may be $\pm\infty$).

The Rule also holds if a is replaced by $\pm\infty$, or even if the limits are *one-sided limits* (i.e., limit as x approaches a from the right or left).

Table 3.12: L'Hospital's Rule for Indeterminate Forms of Type 0/0.

Example 147. Use L'Hospital's Rule (Table 3.12) to show that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Solution Here $a = 0$, $f(x) = \sin x, g(x) = x$. The first thing to do is to **check the form!** Is it really an indeterminate form? Yes, because $(\sin 0)/0 = 0/0$.

The next thing to do is to **check the assumptions on the functions**. Both these functions are differentiable around $x = 0$, $f'(x) = \cos x$, and $g'(x) = 1 \neq 0$ in any neighborhood of $x = 0$. So we can go to the next step.. The next step is to **see if**

the the limit of the quotient of the derivatives exists!? Now,

$$\begin{aligned} \lim_{x\to 0} \frac{f'(x)}{g'(x)} &= \lim_{x\to 0} \frac{\cos x}{1} \\ &= \lim_{x\to 0} \cos x \\ &= \cos 0 = 1, \quad \text{since the cosine function is continuous at } x = 0. \end{aligned}$$

So, it *does* exist and consequently so does the original limit (by the Rule) and we have

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Remember the geometric derivation of this limit in Chapter 2? This is much easier, no?

Three Steps to Solving Limit Problems using L'Hospital's Rule.

- What is the 'form'? ($\infty/\infty, 0/0$?)
- Check the assumptions on the quotient.
- Investigate the existence of the limit

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

If this limit exists, then so does the original one and they must be equal!

Example 148. Show that if α is any given real number, then

$$\lim_{x\to 0} \frac{\tan(\alpha x)}{x} = \alpha.$$

Solution

1. What is the form? The form is $\tan(\alpha \cdot 0)/0 = \tan(0)/0 = 0/0$, which is indeterminate.

2. Check the assumptions on the functions. Here $a = 0$ and we have a quotient of the form $f(x)/g(x) = \tan(\alpha x)/x$, where $f(x) = \tan(\alpha x)$ and $g(x) = x$. Then, by the Chain Rule, $f'(x) = \alpha \sec^2(\alpha x)$, while $g'(x) = 1$, which is not zero near $x = 0$. Both functions are differentiable near 0, so there's no problem, we can go to Step 3.

3. Check the existence of the limit of the quotient of the derivatives.

$$\begin{aligned} \lim_{x\to 0} \frac{f'(x)}{g'(x)} &= \lim_{x\to 0} \frac{\alpha \sec^2(\alpha x)}{1} \\ &= \alpha \cdot \lim_{x\to 0} \sec^2(\alpha x) \\ &= \alpha \cdot \sec^2(0), \quad \text{since the secant function is continuous at } x = 0. \\ &= \alpha \cdot 1 = \alpha. \end{aligned}$$

So, this limit *does* exist and consequently so does the original limit (by the Rule) and we have

$$\lim_{x\to 0} \frac{\tan(\alpha x)}{x} = \alpha.$$

Example 149. Evaluate

$$\lim_{x\to -1} \frac{x^2 + 6x + 5}{x^2 - x - 2}$$

Solution

1. What is the form? The form is $((-1)^2 + 6(-1) + 5)/((-1)^2 - (-1) - 2) = 0/0$, which is indeterminate.

2. Check the assumptions on the functions. Here $a = -1$ and we have a quotient of the form $f(x)/g(x) = (x^2+6x+5)/(x^2-x-2)$, where $f(x) = x^2+6x+5$ and $g(x) = x^2-x-2$. A simple calculation gives $f'(x) = 2x+6$, while $g'(x) = 2x-1 \neq 0$, near $x = -1$. Both functions are differentiable near -1, so we go to Step 3.

Table showing the likelihood that $f(x)/g(x) \to -4/3$ as $x \to -1$. **Note that x can be greater than or less than 1 so long as $x \to -1$.**

x	$f(x)/g(x)$
$-.990000$	-1.413793
$-.999917$	-1.340425
$-.999989$	-1.335845
$-.999997$	-1.334609
$-.999999$	-1.334188
...	...
-1	-1.333333
...	...
-1.00003	-1.333307
-1.00250	-1.331391
-1.00500	-1.329451
-1.02000	-1.317881

Figure 61.

3. Check the existence of the limit of the quotient of the derivatives.

$$\begin{aligned}\lim_{x\to -1} \frac{f'(x)}{g'(x)} &= \lim_{x\to -1} \frac{2x+6}{2x-1} \\ &= (-2+6)/(-2-1), \quad \text{continuity at } x=-1. \\ &= 4/(-3) = -4/3.\end{aligned}$$

So, this limit *does* exist and consequently so does the original limit (by the Rule) and

$$\lim_{x\to -1} \frac{x^2+6x+5}{x^2-x-2} = -\frac{4}{3}.$$

We check this out numerically in Figure 61. Note that $-4/3 \approx -1.333333...$

This rule of L'Hospital is not all powerful, you can't use it all the time! Now we look at an example where everything *looks good* at first but you still can't apply the Rule.

Example 150. Evaluate

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x^2}$$

Solution

1. What is the form? The form is $(\sqrt{1}-1)/0 = 0/0$, which is indeterminate.

2. Check the assumptions on the functions. Here $a = 0$ and we have a quotient of the form $f(x)/g(x) = (\sqrt{x+1}-1)/(x^2)$, where $f(x) = \sqrt{x+1}-1$ and $g(x) = x^2$. A simple calculation gives $f'(x) = (2 \cdot \sqrt{x+1})^{-1}$, while $g'(x) = 2x \neq 0$, near $x = 0$. Both functions are differentiable near 0, so we can go to Step 3.

Table showing the likelihood that $f(x)/g(x) \to -\infty$ as $x \to 0^-$.

x	$f(x)/g(x)$
$-.0050000$	-99.87532
$-.0033333$	-149.87520
$-.0016667$	-299.8751
$-.0014290$	-349.8751
$-.0010000$	-499.875
$-.0001000$	-4999.9
$-.0000010$	-500000

Figure 62.

3. Check the existence of the limit of the quotient of the derivatives.

$$\begin{aligned}\lim_{x\to 0} \frac{f'(x)}{g'(x)} &= \lim_{x\to 0} \frac{\frac{1}{2\cdot\sqrt{x+1}}}{2x} \\ &= \lim_{x\to 0} \frac{1}{4x\cdot\sqrt{x+1}},\end{aligned}$$

But this limit *does not exist* because the left and right-hand limits are not equal. In fact,

$$\begin{aligned}\lim_{x\to 0^-} \frac{1}{4x\cdot\sqrt{x+1}} &= -\infty, \quad (\textit{see Figure } 62) \\ \lim_{x\to 0^+} \frac{1}{4x\cdot\sqrt{x+1}} &= +\infty, \quad (\textit{see Figure } 63)\end{aligned}$$

Since the limit condition is not verified, we can't use the Rule.

So, it seems like we're back to where we started. What do we do? Back to the original ideas. We see a square root so we should be rationalizing the numerator, so as to simplify it. So, let $x \neq 0$. Then

$$\begin{aligned}\frac{\sqrt{x+1}-1}{x^2} &= \frac{(\sqrt{x+1}-1)\cdot(\sqrt{x+1}+1)}{x^2(\sqrt{x+1}+1)} \\ &= \frac{(x+1)-1}{x^2(\sqrt{x+1}+1)} \\ &= \frac{x}{x^2\,(\sqrt{x+1}+1)} \\ &= \frac{1}{x\,(\sqrt{x+1}+1)}.\end{aligned}$$

But this quotient *does not have a limit at* 0 because the left and right-hand limits are not equal there. In fact,

$$\lim_{x\to 0^+} \frac{1}{x\,(\sqrt{x+1}+1)} = +\infty, \quad \textit{while}$$
$$\lim_{x\to 0^-} \frac{1}{x\,(\sqrt{x+1}+1)} = -\infty.$$

The conclusion is that the **original limit does not exist**.

Table showing the likelihood that $f(x)/g(x) \to +\infty$ as $x \to 0^+$.

x	$f(x)/g(x)$
.0050000	99.87532
.0033333	149.87520
.0016667	299.8751
.0014290	349.8751
.0010000	499.875
.0001000	4999.9
.0000010	500000

Figure 63.

Example 151. Evaluate the limit

$$\lim_{x\to 1} \frac{3\sqrt[3]{x}-x-2}{3(x-1)^2}.$$

using any method. Verify your guess numerically.

Solution **1. What is the form?** At $x = 1$, the form is $(3-3)/0 = 0/0$, which is indeterminate.

2. Check the assumptions on the functions. Here $a = 1$ and we have a quotient of the form $f(x)/g(x) = (3\sqrt[3]{x}-x-2)/3(x-1)^2$, where $f(x) = 3\sqrt[3]{x}-x-2$ and $g(x) = 3(x-1)^2$. A simple calculation gives $f'(x) = x^{-2/3}-1$, while $g'(x) = 6(x-1) \neq 0$, near $x = 1$. Both functions are differentiable near 1, so we can go to Step 3.

3. Check the existence of the limit of the quotient of the derivatives. Note that, in this case,

$$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{x^{-2/3}-1}{6(x-1)}.$$

which is **still indeterminate** and of the form '0/0'. So we want to apply the Rule to it! This means that we have to check the conditions of the Rule for these functions, too. Well, let's assume you did this already. You'll see that, eventually, this part gets easier the more you do.

So, differentiating the (new) numerator and denominator gives

$$\begin{aligned}\lim_{x\to 1} \frac{x^{-2/3}-1}{6(x-1)} &= \lim_{x\to 1} \frac{-\frac{2}{3}x^{-5/3}}{6} = -\frac{2}{18} \\ &= -\frac{1}{9}.\end{aligned}$$

Table showing the likelihood that $f(x)/g(x) \to -1/9$ as $x \to 1$. Note that $-1/9 \approx -0.111111...$

x	$f(x)/g(x)$
1.00333	−.11089
1.00250	−.1109
1.001667	−.1110
1.001250	−.1111
1.001000	−.111
...	...
1	−.11111...
...	...
.9990000	−.1112
.9950000	−.11143
.9750000	−.11268
.9000000	−.11772

Figure 64.

Sometimes you have to use L'Hospital's Rule more than once in the same problem!

This is really nice! Why? Because the existence of **this** limit means that we can apply the Rule to guarantee that

$$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = -\frac{1}{9},$$

and since **this** limit exists, the Rule can be applied again to get that

$$\lim_{x\to 1} \frac{3\sqrt[3]{x} - x - 2}{3(x-1)^2} = -\frac{1}{9}.$$

See Figure 64 for the numerical evidence supporting this answer. Keep that battery charged!

Example 152. Evaluate

$$\lim_{x\to 0} \frac{\tan 2x - 2x}{x - \sin x.}$$

Table showing the likelihood that $f(x)/g(x) \to 16$ as $x \to 0$.

x	$f(x)/g(x)$
.100000	16.26835
.033333	16.02938
.016667	16.00743
.010101	15.97674
.010000	15.99880
.001042	16.03191
.001031	15.96721
.001020	16.05640
...	...
0.00000	16.00000..
...	...
−.00100	15.99880
−.01000	16.00260
−.05000	16.06627
−.10000	16.26835

Figure 65.

Solution The form is 0/0 and all the conditions on the functions are satisfied. Next, the limit of the **quotient of the derivatives** is given by (remember the Chain Rule and the universal symbol 'D' for a derivative),

$$\lim_{x\to 0} \frac{D(\tan 2x - 2x)}{D(x - \sin x)} = \lim_{x\to 0} \frac{2\cdot sec^2 2x - 2}{1 - \cos x},$$

which is also an indeterminate form of the type 0/0. The conditions required by the Rule about these functions are also satisfied, so we need to check the limit of the **quotient of these derivatives**. This means that we need to check the existence of the limit

$$\lim_{x\to 0} \frac{D\left(2\cdot sec^2 2x - 2\right)}{D\left(1 - \cos x\right)} = \lim_{x\to 0} \frac{8\cdot sec^2(2x)\cdot \tan(2x)}{\sin x},$$

which is yet another indeterminate form of the type 0/0 (because $\tan 0 = 0, \sin 0 = 0$). The conditions required by the Rule about these functions are also satisfied, so we need to check the limit of the **quotient of these new derivatives**. We do the same thing all over again, and we find

$$\begin{aligned}
\lim_{x\to 0} \frac{D(8\cdot sec^2(2x)\cdot \tan(2x))}{D(\sin x)} &= (8)\cdot \lim_{x\to 0} \frac{2\cdot sec^4 2x + 4\cdot sec^2 2x \cdot \tan^2(2x)}{\cos x}, \\
&= \frac{(8)\cdot(2+0)}{1} \\
&= 16.
\end{aligned}$$

Phew, this was a lot of work! See Figure 65 for an idea of how this limit is reached. Here's a shortcut in writing this down:

NOTES:

SHORTCUT

We can write

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan 2x-2x}{x-\sin x} &= \lim_{x\to 0}\frac{D(\tan 2x-2x)}{D(x-\sin x)}, \\
&= \lim_{x\to 0}\frac{2\cdot\sec^2 2x-2}{1-\cos x}, \\
&= \lim_{x\to 0}\frac{D\left(2\cdot\sec^2 2x-2\right)}{D\,(1-\cos x),} \\
&= \lim_{x\to 0}\frac{8\cdot\sec^2(2x)\cdot\tan(2x)}{\sin x}, \\
&= \lim_{x\to 0}\frac{D(8\cdot\sec^2(2x)\cdot\tan(2x))}{D(\sin x)}, \\
&= (8)\cdot\lim_{x\to 0}\frac{2\cdot\sec^4 2x+2\cdot\sec^2 2x\cdot\tan^2(2x)}{\cos x}, \\
&= \frac{(8)\cdot(2+0)}{1}, \\
&= 16,
\end{aligned}$$

if all the limits on the right exist!

WATCH OUT!

Even if ONE of these limits ON THE RIGHT of an equation fails to exist, then we have to STOP and TRY SOMETHING ELSE. If they ALL exist then you have your answer.

NOTES:

SNAPSHOTS

Example 153. Evaluate

$$\lim_{x\to 3}\frac{x^2-4x+3}{x^2+x-12}$$

Solution The form is $0/0$ and all the conditions on the functions are satisfied. Next, the limit of the **quotient of the derivatives** exists and is given by

$$\lim_{x\to 3}\frac{2x-4}{2x+1}=\frac{((2)(3)-4)}{((2)(3)+1)}=\frac{2}{7}.$$

So, by the Rule, the original limit also exists and

$$\lim_{x\to 3}\frac{x^2-4x+3}{x^2+x-12}=\lim_{x\to 3}\frac{2x-4}{2x+1}=\frac{2}{7}.$$

Sometimes a thoughtless application of L'Hospital's Rule gives NO information even though the actual limit may exist! For example, the Sandwich Theorem shows that

$$\lim_{x\to 0+} x\sin\left(\frac{1}{x}\right)=0.$$

However, if we apply the Rule to

$$\lim_{x\to 0+} x\sin\left(\frac{1}{x}\right)=\lim_{x\to 0+}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}},$$

we get

$$\lim_{x\to 0+}\frac{-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)}{-\frac{1}{x^2}}=\lim_{x\to 0+}\cos\left(\frac{1}{x}\right)$$

and this last limit DOES NOT EXIST! The point is that you're not supposed to apply L'Hospital's Rule here. Why? Because $\sin 1/0$ is not an indeterminate form of the type required!

Example 154. Evaluate

$$\lim_{x\to 1}\frac{x^2-2x+1}{x^3-x}.$$

Solution The form is $0/0$ and all the conditions on the functions are satisfied. The limit of the **quotient of the derivatives** exists and is given by

$$\lim_{x\to 1}\frac{2x-2}{3x^2-1}=\frac{0}{2}=0.$$

So, by the Rule, the original limit also exists and

$$\lim_{x\to 1}\frac{x^2-2x+1}{x^3-x}=\lim_{x\to 1}\frac{2x-2}{3x^2-1}=0.$$

Example 155. Evaluate

$$\lim_{x\to 1}\frac{\sin\pi\, x}{x^2-1}$$

Solution The form is $(\sin 0)/0=0/0$ and all the conditions on the functions are satisfied. The limit of the **quotient of the derivatives** exists and is given by

$$\begin{aligned}\lim_{x\to 1}\frac{\pi\cdot\cos(\pi x)}{2x} &= \frac{\pi\cdot\cos(\pi)}{2}\\ &= -\frac{\pi}{2}.\end{aligned}$$

So, by the Rule, the original limit also exists and

$$\begin{aligned}\lim_{x\to 1}\frac{\sin\pi\, x}{x^2-1} &= \lim_{x\to 1}\frac{\pi\cdot\cos(\pi x)}{2x}\\ &= -\frac{\pi}{2}.\end{aligned}$$

L'Hospital's Rule for Limits at Infinity

Suppose that f and g are each differentiable in some interval of the form $M < x < \infty$, $f'(x) \neq 0$ in $M < x < \infty$, and the next two limits exist and

$$\lim_{x \to \infty} f(x) = 0 \qquad \textit{and} \qquad \lim_{x \to \infty} g(x) = 0.$$

Then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

whenever the latter limit (the one on the right) exists. A similar result is true if we replace ∞ by $-\infty$.

Table 3.13: L'Hospital's Rule for Indeterminate Forms of Type 0/0.

Example 156. Evaluate

$$\lim_{x \to 0} \frac{\sin 4x}{\sin 7x}.$$

Solution The form is $(\sin 0)/(\sin 0) = 0/0$ and all the conditions on the functions are satisfied. The limit of the **quotient of the derivatives** exists and is given by

$$\begin{aligned} \lim_{x \to 0} \frac{4 \cdot \cos 4x}{7 \cdot \cos 7x} &= \frac{4 \cdot 1}{7 \cdot 1} \\ &= \frac{4}{7}. \end{aligned}$$

So, by the Rule, the original limit exists and

$$\begin{aligned} \lim_{x \to 0} \frac{\sin 4x}{\sin 7x} &= \lim_{x \to 0} \frac{4 \cdot \cos 4x}{7 \cdot \cos 7x} \\ &= \frac{4}{7}. \end{aligned}$$

At this point we move on to the study of *limits at infinity*. In these cases the Rule still applies as can be shown in theory: Refer to Table 3.13 above for the result. The Rule is used in exactly the same way, although we must be more careful in handling these limits, because they are at '$\pm\infty$', so it may be helpful to review your section on *Extended Real Number Arithmetic*, in Chapter 2, in order to cook up your guesses.

Sometimes these limit problems may be '*in disguise*' so you may have to move things around and get them in the right form (i.e., a quotient) BEFORE you apply the Rule.

Example 157. Compute

$$\lim_{x \to \infty} \frac{1}{x \ \sin(\frac{\pi}{x})}.$$

Solution **Check the form** In this case we have '$1/(\infty \cdot 0)$, which is indeterminate because '$0 \cdot \infty$' is itself indeterminate. So, we can't even *think* about using the Rule. But, if we rewrite the expression as

$$\lim_{x \to \infty} \frac{1/x}{\sin(\frac{\pi}{x})},$$

then the form is of the type '0/0', right? (because $1/\infty = 0$).

Check the assumptions on the functions. This is OK because the only thing that can go wrong with the derivative of $f(x) = 1/x$ is at $x = 0$, so if we choose $M = 1$, say, in Table 3.13, then we're OK. The same argument applies for g.

Check the limit of the quotient of the derivatives. We differentiate the numerator and the denominator to find,

$$\lim_{x \to \infty} \frac{1/x}{sin(\pi/x)} = \lim_{x \to \infty} \frac{-1/x^2}{(-\pi/x^2) \cdot cos(\pi/x)} = \lim_{x \to \infty} \frac{1}{\pi cos(\pi/x)} = \frac{1}{\pi},$$

since all the limits exist, and $\pi/x \to 0,\ as\ x \to \infty$ (which means $\cos(\pi/x) \to \cos 0 = 1$ as $x \to \infty$).

Table showing the likelihood that $(x^2-1)/(x^2+1) \to 1$ as $x \to -\infty$.

x	$f(x)/g(x)$
-20	.99501
-50	.99920
-200	.99995
-300	.99997
$-1,000$	.99998
$-10,000$	.99999
...	...
$-\infty$	1.00000...

Figure 66.

Don't like to tangle with infinity?

No problem. Whenever you see '$x \to \infty$ just let $x = 1/t$ everywhere in the expressions and then let $t \to 0^+$. Suddenly, the limit at ∞ is converted to a one-sided limit at 0. In fact, what happens is this:

$$\begin{aligned} \lim_{x \to +\infty} \frac{f(x)}{g(x)} &= \lim_{t \to 0+} \frac{f'(1/t)}{g'(1/t)} \\ &= \lim_{x \to 0+} \frac{f'(1/x)}{g'(1/x)} \end{aligned}$$

In this way you can find limits at infinity by transforming them to limits at 0.

Example 158. Compute

$$\lim_{x \to -\infty} \frac{x^2-1}{x^2+1}.$$

Solution **Without the Rule** In this case we have ∞/∞, which is indeterminate. Now let's convert this to a problem where the symbol $-\infty$ is converted to 0^-. We let $x = 1/t$. Then,

$$\begin{aligned} \lim_{x \to -\infty} \frac{x^2-1}{x^2+1} &= \lim_{t \to 0^-} \frac{(1/t)^2-1}{(1/t)^2+1}, \\ &= \lim_{t \to 0^-} \frac{1-t^2}{1+t^2}, \\ &= 1, \end{aligned}$$

since both the numerator and denominator are continuous there. Did you notice we didn't use the Rule at all? (see Figure 66 to convince yourself of this limit). This is because the last limit you see here is **not an indeterminate form** at all.

With the Rule The original form is ∞/∞, which is indeterminate. We can also use the Rule because it applies and all the conditions on the functions are met. This

means that, provided all the following limits exist,

$$\begin{aligned}\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} &= \lim_{x\to-\infty} \frac{2x}{2x}, \\ &= \lim_{x\to-\infty} 1, \\ &= 1,\end{aligned}$$

as before.

Example 159. Compute

$$\lim_{x\to+\infty} \sqrt{x+1} - \sqrt{x}.$$

Solution **Check the form** In this case we have $\infty - \infty$, which is indeterminate. You can't use the Rule at all here because the form is not right, we're not dealing with a quotient, but a difference. So, let's convert this to a problem where the symbol $+\infty$ is converted to 0^+. We let $x = 1/t$. Then, (see the margin),

This is a standard trick; for any two positive symbols, $\square, \triangle$ we have $\sqrt{\square} - \sqrt{\triangle} = \frac{\square - \triangle}{\sqrt{\square}+\sqrt{\triangle}}$. Here $\square = (1/t) + 1, \triangle = 1/t$.

$$\begin{aligned}\lim_{x\to+\infty} \sqrt{x+1} - \sqrt{x} &= \lim_{t\to 0^+} \sqrt{(1/t)+1} - \sqrt{1/t}, \\ &= \lim_{t\to 0^+} \frac{(\sqrt{(1/t)+1} - \sqrt{1/t}) \cdot (\sqrt{(1/t)+1} + \sqrt{1/t})}{\sqrt{(1/t)+1} + \sqrt{1/t}}, \\ &= \lim_{t\to 0^+} \frac{(1/t+1) - (1/t)}{\sqrt{(1/t)+1} + \sqrt{1/t}}, \\ &= \lim_{t\to 0^+} \frac{1}{\sqrt{(1/t)+1} + \sqrt{1/t}}, \\ &= 0,\end{aligned}$$

since the denominator is infinite at this point and so its reciprocal is zero.

Once again, did you notice we didn't use the Rule at all? This is because the last limit you see here is **not an indeterminate form** either.

Example 160. Evaluate

$$\lim_{x\to+\infty} x^2 \cdot \sin\frac{1}{x}.$$

Solution **Check the form** In this case we have $(\infty) \cdot (0)$, which is indeterminate. You can't use the Rule at all here because the form is not right again, we're not dealing with a quotient, but a product. So, once again we convert this to a problem where the symbol $+\infty$ is converted to 0^+. We let $x = 1/t$. Then,

Table showing the likelihood that $f(x)/g(x) \to +\infty$ as $x \to +\infty$.

x	$x^2 \cdot \sin(1/x)$
20	19.99167
60	59.99722
200	199.99917
300	299.99944
2000	1999.99992
10,000	9999.9
...	...
$+\infty$	$+\infty$

Figure 67.

$$\begin{aligned}\lim_{x\to+\infty} x^2 \cdot \sin\frac{1}{x} &= \lim_{x\to+\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x^2}}, \\ &= \lim_{t\to 0^+} \frac{\sin t}{t^2},\end{aligned}$$

and this last form *is* indeterminate. So we can use the Rule on it (we checked all the assumptions, right?). Then,

$$\begin{aligned}\lim_{x\to+\infty} x^2 \cdot \sin\frac{1}{x} &= \lim_{t\to 0^+} \frac{\sin t}{t^2}, \\ &= \lim_{t\to 0^+} \frac{\cos t}{2t}, \\ &= +\infty,\end{aligned}$$

since this last form is of the type $\cos 0/\infty = 0$. So, the limit exists and

$$\lim_{x\to+\infty} x^2 \cdot \sin\frac{1}{x} = +\infty,$$

see Figure 67.

Exercise Set 19.

Determine the limits of the following quotients if they exist, using any method. Try to check your answer numerically with your calculator too.

1. $\lim_{x\to 0} \frac{-\sin x}{2x}$
2. $\lim_{t\to 0} \frac{\sin t}{t}$
3. $\lim_{x\to 0} \frac{1-\cos x}{x}$
4. $\lim_{x\to -1} \frac{x^2-1}{x+1}$
5. $\lim_{x\to 1} \frac{x^4-1}{x^2-1}$
6. $\lim_{x\to \pi} \frac{\tan 2x}{x-\pi}$
7. $\lim_{x\to \pi} \frac{1+\cos x}{\sin 2x}$
8. $\lim_{t\to 0} \frac{\sin^2 t - \sin t^2}{t^2}$
9. $\lim_{x\to 1} \frac{x^3-3x+1}{x^4-x^2-2x}$
10. $\lim_{x\to 0} \frac{\mathrm{Arctan}\, x}{x^2}$
11. $\lim_{x\to 0} \frac{\mathrm{Arcsin}\, x}{\mathrm{Arctan}\, x}$
12. $\lim_{x\to 0} \frac{2\cdot\sin 3x}{\sin 5x}$
13. $\lim_{x\to 1} \frac{x^2-1}{x^6-1}$
14. $\lim_{x\to 0} \frac{2\cos x - 2 + x^2}{3x^4}$
15. $\lim_{x\to 0} \frac{x\sin(\sin x)}{1-\cos(\sin x)}$

Suggested Homework Set 16. *Do problems 3, 5, 8, 12, 14*

NOTES:

3.11 Chapter Exercises

Use any method to find the derivative of the indicated function. There is no need to simplify your answers.

1. $(x+1)^{27}$
2. $\cos^3(x)$
3. $\dfrac{x+1}{\sin 2x}$
4. $\sin\left((x+5)^2\right)$
5. $\dfrac{1}{\sin x + \cos x}$
6. $\sqrt{2x-5}$
7. $\sin 2x$
8. $\cos(\sin 4x)$
9. $\dfrac{1}{(\cos 2x)^3}$
10. $\dfrac{x^2+1}{\cos 2x}$
11. $\dfrac{1}{\sin 3x}$
12. $\dfrac{x+2}{\cos 2x}$
13. $\dfrac{x^2+1}{2x+3}$
14. $(\sin 3x)\cdot(x^{1/5}+1)$
15. $\sin(x^2+6x-2)$

Find the derivative of the following functions at the given point

16. $\dfrac{-1.4}{2x+1}$, *at* $x=0$
17. $(x+1)^{\frac{2}{3}}$, *at* $x=1$
18. $\dfrac{1}{x+\sqrt{x^2-1}}$, *at* $x=2$
19. $(2x+3)^{105}$, *at* $x=1$
20. $x\cdot\sin 2x$, *at* $x=0$
21. $\sin(\sin 4x)$, *at* $x=\pi$

Evaluate the following limits directly using any method

22. $\displaystyle\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$ where $f(x)=(x-1)^2$

23. $\displaystyle\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$ where $f(x)=|x+2|$

24. $\displaystyle\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$ where $f(x)=\sqrt{x+1}$,

    - Rationalize the numerator and simplify.

25. Find the second derivative $f''(x)$ given that $f(x)=(3x-2)^{99}$. Evaluate $f''(+1)$.

26. Let f be a differentiable function for every real number x. Show that $\frac{d}{dx}f(3x^2)=6x\cdot f'(3x^2)$. Verify this formula for the particular case where $f(x)=x^3$.

27. Find the equation of the tangent line to the curve $y=(x^2-3)^6$ at the point $(x,y)=(2,1)$.

28. Let $y=t^3+\cos t$ and $t=\sqrt{u}+6$. Find $\frac{dy}{du}$ when $u=9$.

29. Let $y=r^{1/2}-\frac{3}{r}$ and $r=3t-2\sqrt{t}$. Use the Chain Rule to find an expression for $\frac{dy}{dt}$.

30. Use the definition of the derivative to show that the function $y=x\cdot|x|$ has a derivative at $x=0$ but $y''(0)$ does not exist.

Use implicit differentiation to find the required derivative.

31. $x^3 + 2xy + y^2 = 1, \quad \dfrac{dy}{dx}$ at $(1, 0)$
32. $2xy^2 - y^4 = x^3, \quad \dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$
33. $\sqrt{x+y} + x^2y^2 = 4, \quad \dfrac{dy}{dx}$ at $(0, 16)$
34. $y^5 - 3y^2x - yx = 2, \quad \dfrac{dy}{dx}$
35. $x^2 + y^2 = 16, \quad \dfrac{dy}{dx}$ at $(4, 0)$

Find the equation of the tangent line to the given curve at the given point.

36. $2x^2 - y^2 = 1, \quad$ at $\ (-1, -1)$
37. $2x + xy + y^2 = 0, \quad$ at $\ (0, 0)$
38. $x^2 + 2x + y^2 - 4y - 24 = 0, \quad$ at $\ (4, 0)$
39. $(x - y)^3 - x^3 + y^3 = 0, \quad$ at $\ (1, 1)$
40. $\sin x + \sin y - 3y^2 = 0, \quad$ at $\ (\pi, 0)$

Evaluate the following limits at infinity.

41. $\displaystyle\lim_{x\to+\infty} \frac{x^2 - 1}{2x^2 - 1}$
42. $\displaystyle\lim_{x\to+\infty} \frac{x}{x - 1}$
43. $\displaystyle\lim_{x\to+\infty} \left(\frac{x^3}{x^2 + 1} - x\right)$
44. $\displaystyle\lim_{x\to-\infty} \frac{x^2}{x^3 - 1}$
45. $\displaystyle\lim_{x\to-\infty} x^2 \cdot (\frac{\pi}{2} + \operatorname{Arctan} x)$

Suggested Homework Set 17. *Work out problems* $14, 20, 21, 24, 28, 33, 37$

3.12 Challenge Questions

1. Use the methods of this section to show that Newton's First Law of motion in the form $F = ma$, where a is its acceleration, may be rewritten in *time independent* form as
$$F = p\frac{dv}{dx}.$$
Here $p = mv$ is the momentum of the body in question where v is the velocity. Conclude that the force acting on a body of mass m may be thought of as being

proportional to the product of the momentum and the rate of change of the momentum per unit distance, that is, show that

$$F = \frac{p}{m}\frac{dp}{dx}.$$

2. Conclude that if a force F is applied to a body of mass m moving in a straight line then its kinetic energy, E, satisfies the relation

$$\frac{dE}{dx} = F.$$

3.13 Using Computer Algebra Systems

Use your favorite Computer Algebra System (CAS), like Maple, MatLab, etc., or even a graphing calculator to answer the following questions:

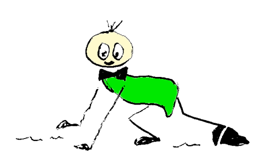
1. Two functions f, g have the property that $f'(9.657463) = -2.34197$, and $g(1.2) = 9.657463$. If $g'(1.2) = -6.549738$ calculate the value of the derivative of their composition, $D(f \circ g)(1.2)$.
2. Let $f(x) = x(x+1)(x-2)$ be defined on the interval $[-3, 3]$. Sketch the graph of $f(x)$ and then, on the same axes, sketch the graph of its derivative $f'(x)$. Can you tell them apart? If we hadn't told you what the function f was but only gave you their graphs, would you be able to distinguish f from its derivative, f'? Explain.
3. Find the equations of the tangent lines to the curve
$$y = \frac{x+1}{x^2+1}$$
through the points $x = 0$, $x = -1.2$, $x = 1.67$ and $x = 3.241$.
4. Remove the absolute value in the function f defined by $f(x) = |x^3 - x|$. Next, remove the absolute value in the function $g(x) = |x - 2|$. Now write down the values of the function h defined by the difference $h(x) = f(x) - 3g(x)$ where $-\infty < x < +\infty$. Finally, determine the points (if any) where the function h fails to be differentiable (*i.e.,* has no derivative). Is there a point x where $f'(x) = 0$?
5. Use implicit differentiation to find the first and second derivative of y with respect to x given that
$$x^2 + y^2 x + 3xy = 3.$$
6. Find a pattern for the first eight derivatives of the function f defined by $f(x) = (3x+2)^5 2$. Can you guess what the 25^{th} derivative of f looks like?
7. Use Newton's method to find the positive solution of the equation $x + \sin x = 2$ where $0 \le x \le \frac{\pi}{2}$. (Use Bolzano's theorem first to obtain an initial guess.)
8. Use repeated applications of the **Product Rule** to find a formula for the derivative of the product of *three* functions f, g, h. Can you find such a formula given *four* given functions? More generally, find a formula for the derivative of the product of n such functions, $f_1, f_2, \ldots, f_n$ where $n \ge 2$ is any given integer.
9. Use repeated applications of the **Chain Rule** to find a formula for the derivative of the composition of *three* functions f, g, h. Can you find such a formula given *four* given functions? More generally, find a formula for the derivative of the composition of n such functions, $f_1, f_2, \ldots, f_n$ where $n \ge 2$ is any given integer.

Chapter 4

Exponentials and Logarithms

The Big Picture

This Chapter is about exponential functions and their properties. Whenever you write down the expression $2^3 = 8$ you are really writing down the value (i.e., 8) of an exponential function at the point $x = 3$. Which function? The function f defined by $f(x) = 2^x$ has the property that $f(3) = 8$ as we claimed. So f is an example of such an exponential function. It's okay to think about 2^x when x is an integer, but what what happens if the '2' is replaced by an arbitrary number? Even worse, what happens if the *power, or, exponent*, x is an **irrational number**? (not an ordinary fraction). We will explore these definitions in this chapter. We will also study one very important function called **Euler's Exponential Function**, sometimes referred to by mathematicians as *The Exponential Function* a name which describes its importance in Calculus. **Leonhard Euler**, (pronounced 'oiler'), 1707-1783, is one of the great mathematicians. As a teenager he was tutored by Johann Bernoulli (the one who was L'Hospital's teacher) and quickly turned to mathematics instead of his anticipated study of Philosophy. His life work (much of which is lost) fills around 80 volumes and he is responsible for opening up many areas in mathematics and producing important trendsetting work in Physics in such areas as Optics, Mechanics, and Planetary Motion.



This exponential function of Euler will be denoted by e^x. It turns out that all other exponential functions (like $2^x, (0.5)^x, \ldots$) can be written in terms of it too! So, we really only need to study this one function, e^x. Part of the importance of this function of Euler lies in its applications to growth and decay problems in population biology or nuclear physics to mention only a few areas outside of mathematics *per se*. We will study these topics later when we begin solving differential equations. The most remarkable property of this function is that it is its own derivative! In other words, $D(e^x) = e^x$ where D is the derivative. Because of this property of its derivative we can solve equations which at one time seemed impossible to solve.

Review

You should review Chapter 1 and especially Exercise Set 3, Number 17 where the inequalities will serve to pin down Euler's number, "e", whose value is $e \approx 2.718\ldots$ obtained by letting $n \to \infty$ there.

4.1 Exponential Functions and Their Logarithms

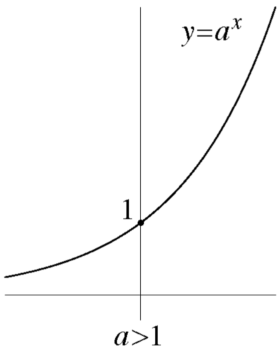


Figure 68.

Exponentials or **power functions** are functions defined by expressions which look like:

$$f(x) = a^x$$

where a is the base and x is the **power** or **exponent**. In this section we will be reviewing the basic properties of exponents first, leaving the formal definition of the exponential function until later. We will always assume that $a > 0$. It can be shown that such power functions have an **inverse** function. This inverse function will be called the **logarithm** (a quantity which must depend on the base) and the symbol used to denote the logarithm is:

$$F(x) = \boxed{\log_a(x)}$$

which is read as "the logarithm with base a of x ". Since this is an inverse function in its own right, it follows by definition of the inverse that:

$$\begin{aligned} x &= f(F(x)) \\ &= a^{F(x)} \\ &= a^{\log_a(x)} \end{aligned}$$

and, more generally:

$$\boxed{\square = a^{\log_a(\square)}}$$

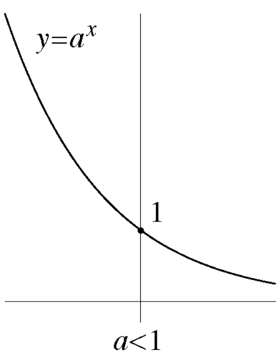


Figure 69.

for any '*symbol*' denoted by $\square$ for which $\square > 0$. Furthermore:

$$\begin{aligned} x &= F(f(x)) \\ &= \log_a(f(x)) \\ &= \log_a(a^x) \end{aligned}$$

and, once again,

$$\boxed{\square = \log_a(a^{\square})}$$

for any symbol $\square$ where now $-\infty < \square < \infty$.

Typical graphs of such functions are given in Figures 68, 69 in the adjoining margin along with their inverses (or logarithms) whose graphs appear in Figures 70, 71.

Example 161. Let, $f(x) = 3^x$. Then $F(x) = \log_3(x)$ is its inverse function and

$$\begin{aligned} 3^{\log_3(x)} &= x, \qquad x > 0 \\ \log_3(3^x) &= x, \qquad -\infty < x < \infty. \end{aligned}$$

Example 162. $3^2 = 9$ means the same as:

$$\log_3(9) = 2.$$

Now notice the following pattern: In words this is saying that,

$$\boxed{\log_{base}(result) = power}$$

or,

$$\boxed{base^{power} = result}$$

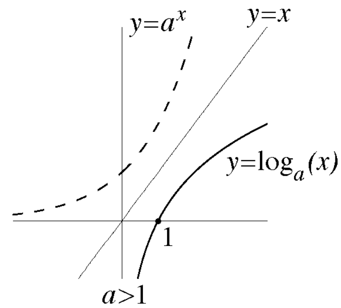


Figure 70.

Example 163. Write the following expression as a logarithm.

$$2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{\sqrt{2}}{2}$$

Solution In terms of logarithms, we can rewrite the equation

$$2^{-\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

as,

$$\log_2\left(\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$$

because here the $base = 2$, the $power$ is $-\frac{1}{2}$ and the $result$ is $\frac{\sqrt{2}}{2}$

Example 164. Write the following expression as a logarithm.

$$4^{\sqrt{2}} \approx 7.10299$$

means we set the $base = 4$, $power = \sqrt{2}$ and $result = 7.10299$. So,

$$\log_4(7.10299) = \sqrt{2}$$

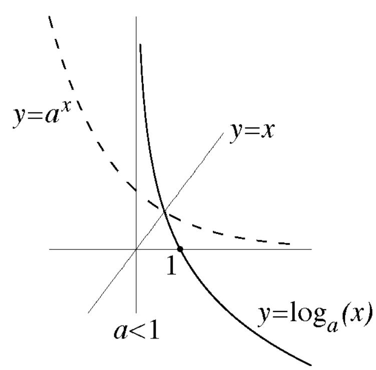


Figure 71.

Example 165. Write the following expression as a logarithm.

$$\sqrt[3]{2}^{\sqrt{3}} \approx 1.38646$$

Solution This means that we set the $base = \sqrt[3]{2}$, the $power = \sqrt{3}$ and the result:

$$\log_{\sqrt[3]{2}}(1.38646) = \sqrt{3}$$

Remember that the base does not have to be an integer, it can be any irrational or rational (positive) number.

Example 166. Sketch the graph of the following functions by using a calculator: use the same axes (compare your results with Figure 72).

a. $f(x) = 3^x$

b. $f(x) = \dfrac{1}{2^x}$

c. $f(x) = (\sqrt{2})^x$

Properties of the Logarithm

1. $\log_a(\square^{\triangle}) = \triangle \log_a(\square)$
2. $\log_a(\square\triangle) = \log_a(\square) + \log_a(\triangle)$
3. $\log_a(\frac{\triangle}{\square}) = \log_a(\triangle) - \log_a(\square)$

Table 4.1: Properties of the Logarithm

x	-2	-1	0	1	2
3^x	0.12	0.33	1	3	9

x	-2	-1	0	1	2
$\frac{1}{2^x}$	4	2	1	0.5	0.25

x	-2	-1	0	1	2
$(\sqrt{2})^x$	0.5	0.71	1	1.4	2

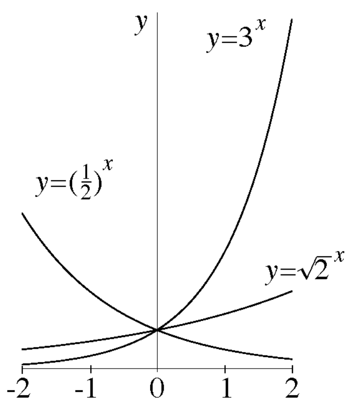


Figure 72.

Remarks: Note that as 'a' increases past 1 the graph of $y = a^x$ gets 'steeper' as you proceed from left to right (harder to climb).

If $0 < a < 1$ and 'a' is small but positive, the graph of $y = a^x$ also becomes steeper, but in proceeding from right to left.

From these graphs and the definitions of the exponential and logarithm functions we get the following important properties:

$$\boxed{a^{\triangle} = \square \text{ means } \log_a(\square) = \triangle}$$

and

$$\begin{aligned} a^0 &= 1 \\ \log_a(a) &= 1 \\ \log_a(1) &= 0 \end{aligned}$$

along with the very practical properties:

where, as usual, $\triangle, \square$ are any two '*symbols*' (usually involving x but not necessarily so).

Example 167. Show that for $\triangle > 0$ and $\square > 0$ we have the equality:

$$\log_a(\square\triangle) = \log_a(\square) + \log_a(\triangle)$$

Solution (**Hint:** Let $A = \log_a(\triangle), B = \log_a(\square)$. Show that $a^{A+B} = \triangle\square$, by using the definition of the logarithm. Conclude that $A + B = \log_a(\triangle\square)$.)

Example 168. Calculate the value of $\log_{\sqrt{2}}(4)$ exactly!

Solution Write the definition down...Here the *base* $= \sqrt{2}$, the *result* is 4 so the

Let's show that $\log_a(\square^{\triangle}) = \triangle \log_a(\square)$.

Write $y = \log_a(\square)$. Then, by definition of the logarithm, this really means,

$$a^y = \square$$

raising both sides to the power $\triangle$ we get,

$$(a^y)^{\triangle} = \square^{\triangle}$$

or, by the usual Power Laws,

$$a^{y\triangle} = \square^{\triangle}.$$

Using the definition of a logarithm once again we find,

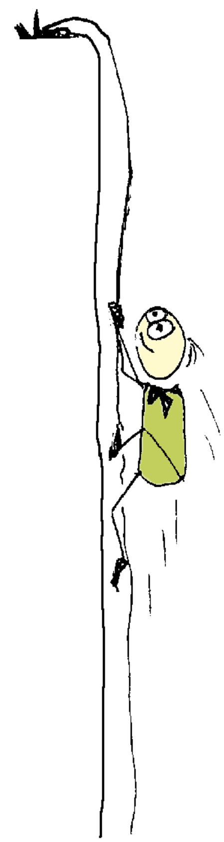
$$\log_a(\square^{\triangle}) = y\triangle$$

(as '$y\triangle$' and '$\square^{\triangle}$' are just two 'new' symbols). But $y\triangle = \triangle \log_a(\square)$, and so the result is true, namely that,

$$\log_a(\square^{\triangle}) = \triangle \log_a(\square)$$

The other properties may be shown using similar arguments.

Table 4.2: Why is $\log_a(\square^{\triangle}) = \triangle \log_a(\square)$?



power =? Well,

$$(\sqrt{2})^{power} = 4$$

means *power* $= 4$, by inspection (i.e.,$\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = 4$). So,

$$\log_{\sqrt{2}}(4) = 4$$

Example 169. Solve for x if $\log_2(x) = -3$

Solution By definition of this inverse function we know that $2^{-3} = x$ or $x = \frac{1}{8}$.

Example 170. If $\log_a(9) = 2$ find the *base* a.

Solution By definition, $a^2 = 9$, and so $a = +3$ or $a = -3$. Since $a > 0$ by definition, it follows that $a = 3$ is the *base*.

Example 171. Evaluate $\log_3(\sqrt{3})$ exactly.

Solution We use the rule $\log_a(\square^{\triangle}) = \triangle \log_a(\square)$ with $a = 3$. Note that

$$\log_3(\sqrt{3}) = \log_3(3^{\frac{1}{2}})$$

So we can set $\square = 3, \triangle = \frac{1}{2}$. Then

$$\begin{aligned} \log_3(\sqrt{3}) &= \log_3(3^{\frac{1}{2}}) \\ &= \frac{1}{2}\log_3(3), \quad \text{by Table 4.2,} \\ &= \frac{1}{2}(1) \quad (\text{because } \log_a(a) = 1) \\ &= \frac{1}{2} \end{aligned}$$

Example 172. Simplify the following expression for $f(x)$ using the properties above:

$$f(x) = \log_2(\sqrt{x}\sin x), \text{ if } 0 < x < \frac{\pi}{2}.$$

Solution:

$$\begin{aligned} f(x) &= \log_2(\sqrt{x}\sin x), \\ &= \log_2(\sqrt{x}) + \log_2(\sin x), \quad \text{by Table 4.1, (2)} \\ &= \log_2(x^{\frac{1}{2}}) + \log_2(\sin x) \\ &= \frac{1}{2}\log_2(x) + \log_2(\sin x), \quad \text{by Table 4.1, (1)} \end{aligned}$$

Example 173. Simplify the following expression for $f(x)$ using the properties above:

$$f(x) = \log_4(4^x 8^{x^2+1}), \ x \geq 0$$

Solution: All references refer to Table 4.1. OK, now

$$\begin{aligned} f(x) &= \log_4(4^x) + \log_4(8^{x^2+1}) \quad (\text{by Property 2}) \\ &= x\log_4(4) + (x^2+1)\log_4(8) \quad (\text{by Property 1}) \\ &= x(1) + (x^2+1)\log_4(2^3) \quad (\text{because } \log_a(a) = 1 \textit{ and } 8 = 2^3) \\ &= x + (x^2+1)\cdot 3\log_4(2) \quad (\text{by Property 1.}) \end{aligned}$$

Now if $\log_4(2) = z$, say, then, by definition, $4^z = 2$, but this means that $(2^2)^z = 2$ or $2^{2z} = 2$ or $2z = 1$, that is, $z = \frac{1}{2}$. Good, this means

$$\log_4(2) = \frac{1}{2}$$

and so,

$$f(x) = x + \frac{3}{2}(x^2+1),$$

hard to believe isn't it?

NOTES:

Exercise Set 20.

Simplify as much as you can:

1. $2^{\log_2(x^2+1)}$
2. $\left(\frac{1}{2}\right)^{\log_{\frac{1}{2}}(x)}$ $\left(\text{Hint}: \text{ Let } a = \frac{1}{2}\right)$
3. $\log_4(2^x 16^{-x^2})$
4. $\log_3\left(\frac{3}{4}\right)$ (Hint : Use Property 3)
5. $\log_a(2^x 2^{-x})$

Sketch the graphs of the following functions using your calculator:

6. $f(x) = 4^x$
7. $f(x) = \frac{1}{4^x}$
8. $(\sqrt{3})^x$

Write the following equations in logarithmic form (e.g. $2^3 = 8$ means $\log_2(8) = 3$):

9. $\sqrt{2}^{\sqrt{2}} \approx 1.6325$
10. $2^{-4} = \frac{1}{16}$
11. $3^{-2} = \frac{1}{9}$

Write the following equations as power functions (e.g. $\log_2(8) = 3$ means $2^3 = 8$).

12. $\log_2(f(x)) = x$
13. $\log_3(81) = 4$
14. $\log_{\frac{1}{2}}(4) = -2$
15. $\log_{\frac{1}{3}}(27) = -3$
16. $\log_a(1) = 0$
17. $\log_{\sqrt{2}}(1.6325) = \sqrt{2}$

Solve the following equations for x:

18. $\log_2(x) + \log_2(3) = 4$, (**Hint:** Use property 2)
19. $\log_3\left(\frac{x}{x+1}\right) = 1$
20. $\log_{\sqrt{2}}(x^2 - 1) = 0$
21. $\log_{\frac{1}{2}}(x) = -1$
22. Sketch the graph of the function y defined by $y(x) = \log_2(2^{x^2})$
    (**Hint:** Let $\square = x^2$)

4.2 Euler's Number, e = 2.718281828 ...

Now that we know how to handle various exponential functions and their logarithms we'll define one very special but really important exponential function. We'll introduce Euler's number, denoted by 'e', after Euler, whose value is approximately $\boldsymbol{e \approx 2.718281828459}$. Using this number as a 'base', we'll define the special exponential function, e^x, and its inverse function, called the **natural logarithm**. We'll then look at the various formulae for the derivative of exponential and logarithmic functions. The most striking property of e lies in that $y = e^x$ has the property that $y' = y$ (*a really neat property!*).

Before we move on to the definition of this number 'e' which is the cornerstone of differential and integral calculus, we need to look at **sequences**, or strings of numbers separated by commas. Well, more precisley, a sequence is the range of a function, denoted by 'a', whose domain is a subset of the integers. So, its values are given by $a(n) = a_n$, where a_n is just shorthand for this value and it can be thought of as representing the n^{th} term of the sequence, the term in the n^{th} position. For example, if $a(n) = 2n - 1$, then $a(6) = 2 \cdot 6 - 1 = 11$, and so we write $a_6 = 11$; $a(15) = 29$ and so we write $a_{15} = 29$, and so on. The whole sequence looks like

$$1, 3, 5, 7, 9, \underbrace{11}_{\text{the } 6^{th} \text{ term is } a_6}, 13, 15, 17, \ldots$$

Before we jump into this business of exponentials, we need to describe one central result. A sequence of numbers is said to be **monotone increasing** if consecutive terms get bigger as you go along the sequence. For example,

$$1, 2, 3, 4, 5, 6, \ldots$$

is an increasing sequence. So is

$$\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$$

or even,

$$1.6, 1.61, 1.612, 1.6123, 1.61234, \ldots$$

and so on. Mathematically we write this property of a sequence $\{a_n\}$ by the inequality,

$$\boxed{a_n < a_{n+1},} \quad n = 1, 2, 3, \ldots$$

which means that the n^{th} term, denoted by a_n, has the property that a_n *is smaller than the next one, namely,* a_{n+1}; makes sense right?!

Well, the next result makes sense if you think about it, and it is *believable* but we won't prove it here. At this point we should recall the main results on limits in Chapter 2. When we speak of the **convergence of an infinite sequence** we mean it in the sense of Chapter 2, that is, in the sense that

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} a(n)$$

or, if you prefer, replace all the symbols n by x above, so that we are really looking at the limit of a function as $x \to +\infty$ but x is always an integer, that's all.

Convergence of Increasing Sequences

Let $\{a_n\}$ be an increasing sequence. Then,

$$\lim_{n\to\infty} a_n$$

exists in the extended real numbers.

In other words; *either*

$$\lim_{n\to\infty} a_n = L < \infty$$

or,

$$\lim_{n\to\infty} a_n = +\infty$$

For *example*, the sequence $\{a_n\}$ where $a_n = n$ is increasing and $a_n \to \infty$ as $n \to \infty$.

On the other hand, the sequence $\{b_n\}$ where

$$b_n = \frac{n}{n+1}$$

is increasing and

$$\lim_{n\to\infty} b_n = 1$$

(Why? The simplest proof uses L'Hospital's Rule on the quotient $x/(x+1)$. Try it.)

The practical use of this result on the convergence of increasing sequences is shown in this section. We summarize this as:

The Increasing Sequence Theorem

Let $\{a_n\}$ be an increasing sequence.

1. If the a_n are smaller than some fixed number M, then

$$\lim_{n\to\infty} a_n = L$$

   where $L \le M$.

2. If there is no such number M, then

$$\lim_{n\to\infty} a_n = +\infty$$

Example 174. What is $\lim_{n\to\infty} \sqrt{\dfrac{n}{n+1}}$?

Solution Here $a_n = \sqrt{\dfrac{n}{n+1}}$ and $a_{n+1} = \sqrt{\dfrac{n+1}{n+2}}$. (Remember, just replace the

subscript/symbol 'n' by '$n+1$'.) Compare the first few terms using your calculator,

$$\begin{aligned} a_1 &= 0.7071 \\ a_2 &= 0.8165 \\ a_3 &= 0.8660 \\ a_4 &= 0.8944 \\ a_5 &= 0.9129 \end{aligned}$$

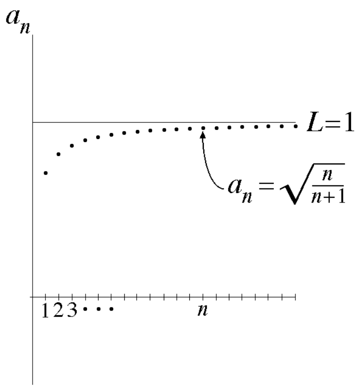


Figure 73.

and since $a_1 < a_2 < a_3 < a_4 < a_5 < \ldots$ it is *conceivable* that the sequence $\{a_n\}$ is increasing, right? Well, the proof of this is not too hard. You can get a proof as follows. For example, the statement

$$a_n < a_{n+1}$$

is equivalent to the statement

$$\sqrt{\frac{n}{n+1}} < \sqrt{\frac{n+1}{n+2}},$$

which, in turn is equivalent to the statement

$$\frac{n}{n+1} < \frac{n+1}{n+2},$$

and this is equivalent to the statement

$$n(n+2) < (n+1)^2,$$

Now this last statement is equivalent to the statement (after rearrangement)

$$n^2 + 2n < n^2 + 2n + 1$$

which is equivalent to the statement

$$0 < 1,$$

which is clearly true! Since all these statements are equivalent, you can 'go backwards' from the last statement that '$0 < 1$ to the first staement which is that '$a_n < a_{n+1}$' which is what we wanted to show, (see Figure 73 for a graph of the sequence $\{a_n\}$).

This is basically how these arguments go insofar as proving that a sequence is increasing: A whole bunch of inequalities which need to be solved and give another set of equivalent inequalities.

OK, so it has a limit (*by the Increasing Sequence Theorem*) and

$$\begin{aligned} \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} &= \lim_{n\to\infty} \sqrt{1 - \frac{1}{n+1}} \\ &= \sqrt{1-0} \\ &= 1 \end{aligned}$$

That's all. You could also have used L'Hospital's Rule to get this answer (replacing n by x and using this Rule on the corresponding function).

Now we proceed to define Euler's number, e.

Definition of Euler's number e.

We define e as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

It's value is approximately $e \approx 2.7182818284590\ldots$.

Why does this limit exist? Well, in Exercise 17 of Exercise Set 3, we proved that if n is any integer with $n \geq 2$, then

$$2 < \left(1+\frac{1}{n}\right)^n < 3.$$

OK, in that same Exercise we also found that the n^{th} term of the sequence $\{a_n\}$ defined by

$$a_n = \left(1+\frac{1}{n}\right)^n$$

can be rewritten as

$$a_n = \left(1+\frac{1}{n}\right)^n = 1+1+\left(1-\frac{1}{n}\right)\frac{1}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{3!} + \cdots + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\frac{1}{n!}$$

(where there are $(n+1)$ terms on the right side).

Now notice that $\{a_n\}$ is increasing because if we replace 'n' by 'n+1' in the expression for a_n above, we note that every factor of the form $\left(1-\frac{i}{n}\right)$, $i \geq 1$, 'increases' (because every such term is replaced by $\left(1-\frac{i}{n+1}\right)$ which is **larger** than the preceding one. So a_{n+1} is larger than a_n, in general for $n > 1$ and $\{a_n\}$ is increasing.

That's it! Using the Increasing Sequence Theorem we know that

$$\lim_{n\to\infty} a_n$$

exists and is not greater than 3. The first few terms of this sequence are given in Figure 75 in the margin.

The value of this limit, e, is given approximately by

$$\boxed{e \approx 2.7182818284590\ldots}$$

It was shown long ago that e is not a *rational number*, that is, it is not the quotient of two integers, (so it is said to be *irrational*) and so we know that its decimal expansion given above cannot repeat.

Remark Reasoning similar to the one above shows that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

so that x **does not have to converge** to infinity along integers only!

n	$\left(1+\frac{0.02}{n}\right)^n$
1	1.02
10	1.02018 ...
100	1.020199 ...
1,000	1.020201 ...
10,000	1.020201 ...
100,000	1.020201 ...

In monetary terms, the numbers on the right can be thought of as the bank balance at the end of 1 year for an initial deposit of 1.00 at an interest rate of 2% that is compounded n times per year. The limit of this expression as the number of compounding period approaches infinity, that is, when we continuously compound the interest, gives the number $e^{0.02}$, where e is Euler's number. Use your calculator to check that the value of $e^{0.02} \approx 1.020201340$ in very good agreement with the numbers above, on the right.

Figure 74.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
10	2.593742 ...
100	2.704813 ...
10000	2.718145 ...
100000	2.718268 ...

The first few terms of the sequence above that converges to Euler's number.

Figure 75.

Example 175. Evaluate $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$.

Solution There is no need to use the *Binomial Theorem* as in Example 17, but you can if you really want to! Observe that

$$\begin{aligned}\left(1+\frac{2}{n}\right)^n &= \left(1+\frac{1}{(\frac{n}{2})}\right)^{\frac{n}{2}\cdot 2}\\ &= \left(1+\frac{1}{m}\right)^{m\cdot 2} \quad \left(\text{if we set } m=\frac{n}{2}\right)\\ &= \left[\left(1+\frac{1}{m}\right)^m\right]^2.\end{aligned}$$

Now, as $n\to+\infty$ we also have $m\to+\infty$, right? Taking the limit we get that

$$\begin{aligned}\lim_{n\to+\infty}\left(1+\frac{2}{n}\right)^n &= \lim_{m\to+\infty}\left[\left(1+\frac{1}{m}\right)^m\right]^2\\ &= \left(\lim_{m\to+\infty}\left(1+\frac{1}{m}\right)^m\right)^2\\ &= e^2,\end{aligned}$$

where $e = 2.718...$ is Euler's number.

Exercise Set 21.

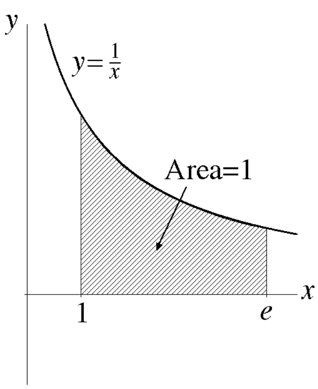


Figure 76.

1. Calculate the first few terms of the sequence $\{b_n\}$ whose n^{th} term, b_n, is given by
$$b_n = \left(1-\frac{1}{n}\right)^n, \qquad 1\le n$$
Find $b_1, b_2, b_3, \ldots, b_{10}$.
2. Can you guess $\lim_{n\to\infty} b_n$ where the sequence is as in Exercise 1? (**Hint:** It is a simple power (negative power) of e.)
3. Show that
$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$$
where a is any given number.

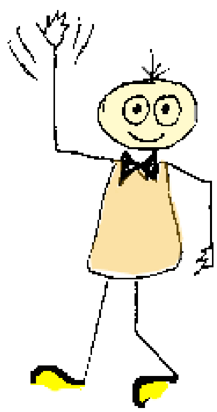
Known Facts About e.

1. $\boxed{e^{i\pi}+1=0}$ where $i=\sqrt{-1}$, $\pi = 3.14159\ldots$ [L. Euler (1707-1783)]
2. If you think (1) is nuts, what about $\boxed{e^{2\pi i}=1}$, (this is really nuts!) This follows from (1) by squaring both sides.
3. $\boxed{e^{\pi}=(-1)^{-i}}$, (what?), where $i=\sqrt{-1}$ [Benjamin Peirce (1809-1880)]
4. $\boxed{e=\lim_{n\to\infty}\left(1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)}$ [Leonard Euler (above)]
5. $\boxed{e=\lim_{x\to 0}(1+x)^{\frac{1}{x}}}$

6. $\boxed{i^i = e^{-\frac{\pi}{2}}}$, (this is completely nuts!) [Leonard Euler (1746)]
7. The number e is also that number such that the area under the graph of the curve $y = \frac{1}{x}$ (between the lines $x = 1, x = e$ and the x-axis) is equal to 1, (see Figure 76).

NOTES:

4.3 Euler's Exponential Function and the Natural Logarithm



One way of doing this is as follows. Let's recall that the rational numbers are *dense* in the set of real numbers. What does this mean? Well, *given any real number, we can find a rational number (a fraction) arbitrarily close to it!*

For example, the number 'e' is well-defined and

$$e \approx \frac{2718}{1000}$$

is a fair approximation correct to 3 decimal places (or 4 significant digits). The approximation

$$e \approx \frac{27182818284}{10000000000}$$

is better still and so on. Another example is

$$\pi \approx \frac{355}{113}$$

which gives the correct digits of π to 6 decimal places! It is much better than the classical $\pi \approx \frac{22}{7}$ which is only valid to 2 decimal places.

In this way we can *believe* that **every real number can be approximated by a rational number**. The actual proof of this result is beyond the scope of this book and the reader is encouraged to look at books on *Real Analysis* for a proof.

Approximations of e by rational numbers.

$$e = 2.71828182849504523536\ldots$$

Fraction	**Value**
$\frac{1957}{720}$	$2.7180555\ldots$
$\frac{685}{252}$	$2.7182539\ldots$
$\frac{9864101}{3628800}$	$2.718281801\ldots$
$\frac{47395032961}{17435658240}$	$2.71828182845822\ldots$ *etc.*

Better and better rational approximations to e may be obtained by adding up more and more terms of the **infinite series** for e, namely,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

The last term in this table was obtained by adding the first 15 terms of this series!

Exercise Find a rational number which approximates e to 9 decimal places.

Armed with this knowledge we can define the exponential function (or **Euler's exponential function**) as follows:

If $\boldsymbol{x = p}$ **is an integer** then

$$\begin{aligned} e^x &= e^p \\ &= \underbrace{e \cdot e \cdot e \cdots e}_{p \text{ times}} \end{aligned}$$

If $\boldsymbol{x = \frac{p}{q}}$ **is a rational number,** with $q > 0$, say, then

$$e^x = e^{\frac{p}{q}} = \sqrt[q]{e^p}.$$

If $\boldsymbol{x}$ **is irrational,** let $x_n = \frac{p_n}{q_n}$ be an infinite sequence of rational numbers converging to x. From what we said above, such a sequence always exists. Then

$$e^x = \lim_{n\to\infty} e^{x_n} = \lim_{n\to\infty} e^{\frac{p_n}{q_n}}.$$

That's all! This defines Euler's exponential function, or THE exponential function, e^x, which is often denoted by $\mathrm{Exp}(x)$ or $\exp(x)$.

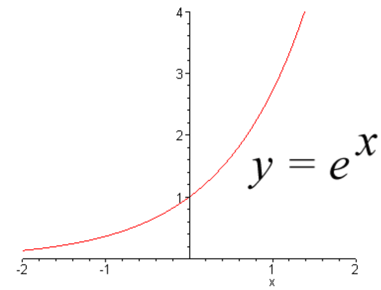


Figure 77. *Euler's exponential function*

Properties of $\boldsymbol{e^x}$

1. $e^{x+y} = e^x e^y$ for any numbers x, y and $e^0 = 1$
2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
3. $e^{x-y} = e^x e^{-y} = \dfrac{e^x}{e^y}$
4. $\lim\limits_{x\to+\infty} e^x = +\infty$
5. $\lim\limits_{x\to-\infty} e^x = 0$
6. One of the neatest properties of this exponential function is that

$$\boxed{\frac{d}{dx} e^x = e^x}.$$

Table 4.3: Properties of e^x

Let's look at that cool derivative property, Property (6) in Table 4.3. You'll need Exercise 3 from the previous Exercise Set, with $a = h$. OK, here's the idea: For $h \neq 0$,

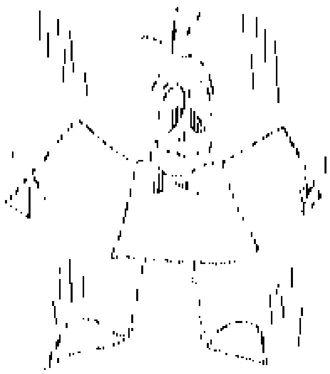
$$\begin{aligned} \frac{e^{x+h} - e^x}{h} &= \frac{e^x(e^h - 1)}{h} \\ &= e^x \cdot \frac{(e^h - 1)}{h} \\ &= e^x \cdot \frac{1}{h}\left\{\left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots\right] - 1\right\} \\ &= e^x \cdot \frac{1}{h}\left\{h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots\right\} \\ &= e^x \cdot \left\{1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots\right\} \qquad \text{so, passing to the limit,} \\ \frac{d}{dx} e^x &= e^x \lim_{h\to 0}\left\{1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots\right\} \\ &= e^x(1 + 0 + 0 + \ldots) \\ &= e^x(1) \\ &= e^x. \end{aligned}$$

That's the idea!

Now the inverse of the exponential function is called the **natural logarithm** and there are a few symbols in use for the natural logarithm:

$$\boxed{\text{Natural logarithm of } x\text{: } \ln(x),\ \log_e(x),\ \log(x)}$$

These two functions, e^x and $\ln(x)$ are just two *special transcendental functions* and they are the most important ones in Calculus. Their properties are exactly like those of general exponentials, a^x, and logarithms, $\log_a(x)$, which we'll see below



Properties of the Special Transcendental Functions

1. $e^{\ln \square} = \square$, if $\square > 0$
2. $\ln e^{\square} = \square$, for any '*symbol*', $\square$
3. $\ln(e) = 1$
4. $\ln(1) = 0$
5. $\ln \square^{\triangle} = \triangle \ln \square$, if $\square > 0$
6. $\ln(\triangle\square) = \ln\triangle + \ln\square$, $\triangle, \square > 0$
7. $\ln\left(\frac{\triangle}{\square}\right) = \ln(\triangle) - \ln(\square)$, $\triangle, \square > 0$

Table 4.4: Properties of the Special Transcendental Functions

Application

Find the derivative of the following function at the indicated point:

$$f(x) = \frac{e^x + e^{-x}}{2}, \text{ at } x = 2$$

This is called the **hyperbolic cosine of *x*** and is very important in applications to engineering. For example, the function defined by $g(x) = f(ax)$ where

$$g(x) = \frac{e^{ax} + e^{-ax}}{2}, \ a > 0$$

represents the shape of a hanging chain under the influence of gravity. The resulting curve is called a **catenary**. Another example of a curve that was inspired by a catenary is the *Gateway Arch* in St. Louis, Missouri, in a well known architectural monument by Finnish architect **Eero Saarinen** (1910-1961).

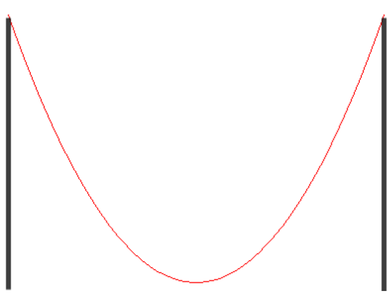


Figure 78. *A catenary curve: It is the result of gravitational effects upon a wire hanging between two telephone posts, for instance.*

Now that we've defined the Euler exponential function we can define the **general exponential function** $f(x) = a^x$, $a > 0$, by

$$\boxed{a^x = e^{x \ln a}, \text{ for any real x}}$$

Most pocket calculators actually work this way when evaluating expressions of the form a^x, if this key is present.

Remember: The term '$\ln a$' is a number which can be positive (or negative) depending on whether a is bigger than 1 or less than 1.

It makes sense to define a^x as we did above because we know that:

$$e^{\ln \square} = \square$$

so we can replace the symbol $\square$ by a^x and use Property (1), Table 4.1, of the logarithm, namely, $\ln(a^x) = x \ln a$ to get at the definition. Using the *change of base formula* (see the margin) for a logarithm we can then define the inverse function of the general exponential function by

$$\boxed{\log_a(\square) = \frac{\ln(\square)}{\ln a}}.$$

where $\square > 0$.

The Change of Base Formula for Logarithms is easy to show: Let $\square = a^x$. Then, by definition, $x = \log_a \square$. But $\ln \square = \ln(a^x) = x \ln a$. Solving for x in both expressions and equating we get the formula.

Example 176. Evaluate or rewrite the following quantities using the Euler exponential function.

1. $(2.3)^{1.2}$
   Solution: Here $a = 2.3$ and $x = 1.2$. Since $a^x = e^{x \cdot \ln a}$, it follows that $(2.3)^{1.2} = e^{(1.2)\ln(2.3)}$. We look up $\ln(2.3)$ on our (scientific) calculator which gives us 0.83291. Thus,
$$\begin{aligned}(2.3)^{1.2} &= e^{0.99949}\\ &= 2.71690.\end{aligned}$$
2. $f(x) = 2^{\sin x}$
   Solution: By Table 4.1, Property 1, $2^{\sin x} = e^{\ln 2 \cdot (\sin x)} = e^{0.693 \cdot (\sin x)}$ where $\ln 2 = 0.693\ldots$
3. $g(x) = x^x$, $(x > 0)$
   Solution: The right way of defining this, is by using Euler's exponential function. So,
$$x^x = e^{x \ln x}$$
   where now $\ln x$ is another function multiplying the x in the *exponent*. Thus we have reduced the problem of evaluating a complicated expression like x^x with a 'variable' *base* to one with a 'constant' *base*, that is,
$$\underset{\nearrow \atop \text{variable base}}{x^x} = \underset{\nwarrow \atop \text{constant base}}{e^{h(x)}}, \qquad \text{and } h(x) = x \ln x.$$

Historical Note: Leonard Euler, mentioned earlier, proved that if

$$f(x) = x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}$$

where the number of exponents tends to infinity then

$$\lim_{x \to a} f(x) = L < \infty$$

if $e^{-e} < a < e^{\frac{1}{e}}$ (or if $0.06599 < a < 1.44467$). What is L? *This is hard!*

4.4 Derivative of the Natural Logarithm

Let f, F be differentiable inverse functions of one another so that $x = f(F(x))$ and $x = F(f(x))$. Then

$$\frac{d}{dx}(x) = \frac{d}{dx} f(F(x))$$

or

$$1 = f'(F(x)) \cdot F'(x), \qquad \text{(by the Chain Rule).}$$

Solving for $F'(x)$ we get

$$F'(x) = \frac{1}{f'(F(x))}$$

Now we can set $f(x) = e^x$ and $F(x) = \ln x$. You can believe that F is differentiable for $x > 0$ since f is, right? The graph of $\ln x$ is simply a reflection of the graph of e^x about the line $y = x$ and this graph is 'smooth' so the graph of '$\ln x$' will also be 'smooth'. Actually, one proves that the natural logarithm function is differentiable by appealing to a result called the **Inverse Function Theorem** which we won't see here. It's basically a theorem which guarantees that an inverse function will have some nice properties (like being differentiable) if the original function is differentiable.

Exercise: Show that F defined by $F(x) = \ln x$ is differentiable at $x = a$ for $a > 0$ using the following steps:

a) The derivative of 'ln' at $x = a > 0$ is, by definition, given by

$$F'(x) = \lim_{x \to a} \frac{\ln(x) - \ln(a)}{x - a}$$

provided this limit exists.

b) Let $x = e^z$. Show that

$$F'(x) = \lim_{z \to \ln a} \left(\frac{z - \ln(a)}{e^z - a} \right)$$

provided this limit exists.

c) But note that $F'(a) = \dfrac{1}{\frac{d}{dz}(e^z)}$ evaluated at $z = \ln a$

d) Conclude that both limits in (b) and (a) exist.

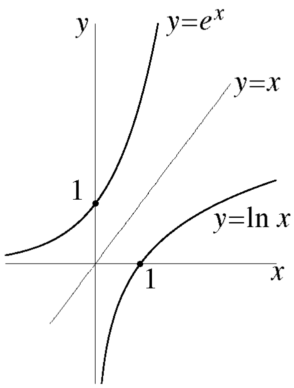


Figure 79.

If $\square < 0$, then $-\square > 0$ and so by the Chain Rule,

$$\frac{d}{dx} \ln(-\square) = \frac{1}{-\square} \frac{d}{dx}(-\square)$$

$$= -\frac{1}{\square}(-1)\frac{d}{dx}(\square) = \frac{1}{\square}\frac{d}{dx}(\square).$$

This means that so long as $\square \neq 0$ then

$$\frac{d}{dx} \ln|\square| = \frac{1}{\square}\frac{d}{dx}(\square)$$

a formula that will become very useful later on when dealing with the *antiderivative* of the exponential function.

OK, knowing that the natural logarithm function 'ln' is differentiable we can find it's derivative using the argument below:

$$\begin{aligned} F'(x) = \frac{d}{dx}\ln x &= \frac{1}{f'(F(x))} \\ &= \frac{1}{e^{F(x)}} \\ &= \frac{1}{e^{\ln x}} \\ &= \frac{1}{x} \end{aligned}$$

More generally (using Chain Rule) we can show that

$$\begin{aligned} \frac{d}{dx}\ln u(x) &= \frac{1}{u(x)}\left(\frac{du}{dx}\right) \\ &= \frac{u'(x)}{u(x)} \end{aligned}$$

and, in particular

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

In general it is best to remember that

$$\boxed{\frac{d}{dx}\ln\Box = \frac{1}{\Box}\frac{d}{dx}(\Box),\ \Box > 0}$$

where $\Box$ is any function of x which is positive and differentiable (but see the margin for something more general!)

Example 177. Find the derivatives of the following functions at the indicated point (if any).

a) $\ln(x^2+2)$, at $x=0$
b) $e^x\log(x)$ (remember $\log x = \ln x$)
c) $\ln(x^2+2x+1)$
d) $3\ln(x+1)$, at $x=1$
e) $e^{2x}\log(x^2+1)$

Solution:

a) Let $\Box = x^2+2$. Then

$$\begin{aligned}\frac{d}{dx}\ln(x^2+2) &= \frac{d}{dx}\ln(\Box)\\ &= \frac{1}{\Box}\frac{d}{dx}(\Box)\\ &= \frac{1}{x^2+2}\frac{d}{dx}(x^2+2)\\ &= \frac{1}{x^2+2}(2x)\end{aligned}$$

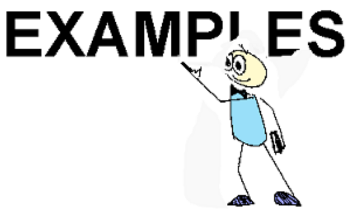
so that when evaluated at $x=0$ this derivative is equal to 0.

b) We have a product of two functions here so we can use the Product Rule.

$$\begin{aligned}\frac{d}{dx}(e^x\log x) &= \frac{d}{dx}(e^x)\log x + e^x\frac{d}{dx}\log x\\ &= e^x\log x + e^x\left(\frac{1}{x}\right)\\ &= e^x\left(\log x + \frac{1}{x}\right)\end{aligned}$$

c) Let $u(x) = x^2+2x+1$. We want the derivative of $\ln u(x)$. Now

$$\begin{aligned}\frac{d}{dx}\ln u(x) &= \frac{1}{u(x)}u'(x)\\ &= \frac{1}{x^2+2x+1}\frac{d}{dx}(x^2+2x+1)\\ &= \frac{1}{x^2+2x+1}(2x+2)\\ &= \frac{2x+2}{x^2+2x+1}\end{aligned}$$



d) Let $u(x) = x + 1$. Then $u'(x) = 1$. Furthermore,

$$\begin{aligned} \frac{d}{dx} 3\ln(x+1) &= 3\frac{d}{dx}\ln(x+1) \\ &= 3\frac{d}{dx}\ln u(x) \\ &= 3 \cdot \frac{1}{u(x)} \cdot u'(x) \\ &= 3\frac{1}{x+1} \cdot 1 \\ &= \frac{3}{x+1} \end{aligned}$$

So, at $x = 1$, we get that the derivative is equal to $\frac{3}{2}$.

e) Once again, this is a product of two functions, say, f and g, where

$$\begin{aligned} f(x) &= e^{2x} \\ g(x) &= \log(x^2+1) \end{aligned}$$

By the Product Rule we get $\frac{d}{dx}\left(e^{2x}\log(x^2+1)\right) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$. But

$$\frac{df}{dx} = \frac{d}{dx}(e^{2x}) = e^{2x}\frac{d}{dx}(2x) = 2e^{2x}$$

and

$$\begin{aligned} \frac{dg}{dx} &= \frac{d}{dx}\log(x^2+1) \\ &= \frac{d}{dx}\log(\Box) \text{ (where } \Box = x^2+1) \\ &= \frac{1}{\Box}\frac{d}{dx}(\Box) \\ &= \frac{1}{x^2+1}\frac{d}{dx}(x^2+1) \\ &= \frac{1}{x^2+1}(2x) \end{aligned}$$

Combining these results for $f'(x)$, $g'(x)$ we find

$$\frac{d}{dx}e^{2x}\log(x^2+1) = (2e^{2x})\log(x^2+1) + e^{2x}\left(\frac{2x}{x^2+1}\right)$$

Exercise Set 22.

Find the derivative of each of the following functions at the indicated point (if any).

1. $\ln(x^3+3)$, at $x = 1$
2. $e^{3x}\log x$
3. $\dfrac{e^x}{\log x}$, (**Hint:** Use the Quotient Rule)
4. $\ln(e^{2x}) + \ln(x+6)$, at $x = 0$
5. $\ln(x + \sqrt{x^2+3})$
(**Hint:** Let $\Box = x + \sqrt{x^2+3}$ and use the Chain Rule on the second term.)
6. $4\ln(x+2)$
7. $\ln(\sqrt{x^2+4})$ (**Hint:** Simplify the 'log' first using one of its properties.)

4.5 Differentiation Formulae for General Exponential Functions

In this section we derive formulae for the general exponential function f defined by

$$f(x) = a^x, \qquad \text{where } a > 0,$$

and its logarithm,

$$F(x) = \log_a x,$$

and then use the Chain Rule in order to find the derivative of more general functions like

$$\begin{aligned} g(x) &= a^{h(x)} \\ &\text{and} \\ G(x) &= \log_a h(x) \end{aligned}$$

where $h(x)$ is some given function. Applications of such formulae are widespread in scientific literature from the rate of decay of radioactive compounds to population biology and interest rates on loans.

OK, we have seen that, by definition of the general exponential function, if h is some function then

$$a^{h(x)} = e^{h(x)\cdot \ln a}.$$

Now the power '$h(x) \cdot \ln a$' is just another function, right? Let's write it as $k(x)$, so that

$$k(x) = h(x) \cdot \ln a.$$

Then

$$\begin{aligned} a^{h(x)} &= e^{k(x)} \\ &= f(k(x)) \end{aligned}$$

where $f(x) = e^x$ and $k(x) = \ln a \cdot h(x)$, right? Good. Now, since $f'(x) = f(x)$, $(f'(\square) = f(\square) = e^{\square})$,

$$\begin{aligned} \frac{d}{dx} a^{h(x)} &= \frac{d}{dx} f(k(x)) \\ &= f'(k(x)) \cdot k'(x) \qquad \text{(by the Chain Rule)} \\ &= e^{k(x)} \cdot k'(x) \qquad \left(\text{because } \frac{d}{dx} e^x = e^x\right) \\ &= e^{h(x)\cdot \ln a} \cdot \left(\frac{d}{dx}\, h(x) \cdot\ \ln a \right) \\ &= e^{\ln(a^{h(x)})} \left(\ln a\ \frac{d}{dx} h(x) \right) \\ &= a^{h(x)}\ (\ln a)\ h'(x) \end{aligned}$$

and we have discovered the general formula

$$\boxed{\frac{d}{dx} a^{h(x)} = a^{h(x)}\, \underbrace{(\ln a) h'(x)}_{\text{multiply original exponential by these two terms}}}$$

In the special case of $h(x) = x$, we have $h'(x) = 1$ so that

$$\frac{d}{dx}a^x = a^x \ln a$$

just a little more general than Euler's exponential function's derivative, because of the presence of the natural logarithm of a, on the right.

Example 178. Find the derivative of the exponential functions at the indicated points (if any).

1. $f(x) = e^{3x}$
2. $g(x) = e^{-(1.6)x}$, at $x = 0$
3. $f(x) = 2^{\sin x}$
4. $g(x) = (e^x)^{-2}$, at $x = 1$
5. $k(x) = (1.3)^{x^2} \cos x$

Solutions:

1. Set $a = e$, $h(x) = 3x$ in the boxed formula above. Then
$$\begin{aligned}\frac{d}{dx}e^{3x} &= \frac{d}{dx}a^{h(x)} \\ &= a^{h(x)}\ (\ln a)\ h'(x) \\ &= e^{3x}\ (\ln e)(3) \quad (\text{since } \frac{d}{dx}3x = 3) \\ &= e^{3x} \cdot 1 \cdot 3 \quad (\text{since } \ln e = 1) \\ &= 3e^{3x}.\end{aligned}$$

2. Set $a = e$, $\ln x = -(1.6)x$. Then
$$\begin{aligned}\frac{d}{dx}e^{-(1.6)x} &= \frac{d}{dx}a^{h(x)} \\ &= a^{h(x)}(\ln a)h'(x) \\ &= e^{-(1.6)x}(\ln e)\frac{d}{dx}((-1.6)x) \\ &= e^{-(1.6)x}(1) \cdot (-1.6) \\ &= -(1.6)e^{-(1.6)x}\end{aligned}$$
and at $x = 0$ its derivative is equal to -1.6.

3. Set $a = 2$ and $h(x) = \sin x$. Then
$$\begin{aligned}\frac{d}{dx}2^{\sin x} &= 2^{\sin x}(\ln 2)\frac{d}{dx}(\sin x) \\ &= 2^{\sin x}(\ln 2) \cdot (\cos x)\end{aligned}$$

4. Simplify first: $(e^x)^{-2} = (e^x)^{(-2)} = e^{(x)(-2)} = e^{-2x}$. So now we can set $a = e$, $h(x) = -2x$. Then
$$\begin{aligned}\frac{d}{dx}(e^x)^{-2} &= \frac{d}{dx}e^{-2x} \\ &= e^{-2x}(\ln e)(-2) \\ &= -2e^{-2x}\end{aligned}$$
So at $x = 1$ this derivative is equal to $-2e^{-2}$.

5. We have a **product of two functions** here so we have to use the Product Rule. Let $f(x) = (1.3)^{x^2}$ and $g(x) = \cos x$. We know that $k'(x) = f'(x)g(x)+f(x)g'(x)$ (by the Product Rule) and

$$\begin{aligned} f'(x) &= \frac{d}{dx}(1.3)^{x^2} \\ g'(x) &= (\cos x)' = -\sin x \end{aligned}$$

So, we only need to find $f'(x)$ as we know all the other quantities f, g, g'. Finally, we set $a = 1.3$ and $h(x) = x^2$. Then

$$\begin{aligned} \frac{d}{dx}(1.3)^{x^2} &= \frac{d}{dx}a^{h(x)} \\ &= a^{h(x)}\ (\ln a)\ h'(x) \\ &= (1.3)^{x^2} \cdot (\ln 1.3) \cdot (2x) \\ &= 2 \cdot (\ln 1.3)x \cdot (1.3)^{x^2} \end{aligned}$$

Historical Note

Long ago **Grégoire de Saint Vincent** (1584 - 1667) showed that the area (Fig. 80) under the curve

$$y = \frac{1}{x}$$

between the lines $x = 1$ and $x = t$ (where $t > 1$) and the x-axis is given by $\log t$ (and this was *before* Euler came on the scene!).

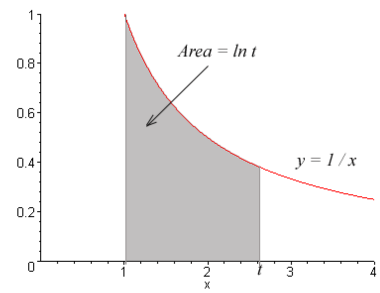


Figure 80.

Exercise Set 23.

Show that the following sequences are increasing and find their limits in the extended real numbers.

1. $a_n = n + 2,\ n \geq 1$
2. $a_n = \dfrac{n-1}{n},\ n \geq 1$
3. $a_n = \dfrac{n(n-2)}{n^2},\ n \geq 1$
4. $a_n = \dfrac{n}{n+3},\ n \geq 1$
5. $a_n = \dfrac{n-1}{n+1},\ n \geq 1$
6. Sketch the graph of the sequence $\{a_n\}$ given by
$$a_n = \sqrt{\frac{n-1}{n}}$$
for $n = 1, 2, \ldots, 15$, and find it's limit.
7. Evaluate
$$\lim_{n\to\infty}\left(1 - \frac{2}{n}\right)^n$$
(**Hint:** See Example 175.)

8. Show that

$$\lim_{x\to 0}(1+x^2)^{\frac{1}{x^2}} = e$$

(**Hint:** Assume you know Fact #5 about e.)

9. Find a rational approximation to e not given in the text.

10. Use your present knowledge of dilations and translations to sketch the graph of the function y defined by

$$y(x) = e^{2(x-1)}$$

using the graph of $y = e^x$.

11. Evaluate the following expressions and simplify as much as possible.
    a) $e^{3\ln x}$, at $x = 1$
    b) $\dfrac{e^{3\ln x} - e^{\ln(x^3)}}{2}$
    c) $\ln(e^{3x+2})$, at $x = 1$
    d) $\ln(e^{2x}) - 2x + 1$
    e) $\ln(\sin^2 x + \cos^2 x)$
    f) $\ln\left(\dfrac{1}{\sin\left(\frac{\pi}{2}\right)}\right)$
    g) $e^{(2x+1)\ln(2)} - 2^{2x}$
    h) $\ln\left(\dfrac{x^2-1}{x+1}\right)$, for $x > 1$
    i) $\ln\left(\dfrac{e^{3x}}{e^{2x+1}}\right)$
    j) $e^{x^2} - \ln\left(e^{e^{x^2}}\right)$

12. Evaluate

$$\lim_{x\to\infty}\left(\frac{x^2}{e^x}\right)$$

Hint Use L'Hospital's Rule.

13. Evaluate the following expressions using
    i) your calculator only and
    ii) writing them in terms Euler's exponential function and then using your calculator:

$$\begin{aligned} \text{Example: } \text{i) } (2.3)^{1.2} &= 2.71690 \\ \text{ii) } (2.3)^{1.2} &= e^{(1.2)\ln(2.3)} \\ &= e^{0.99949} \\ &= 2.71690 \end{aligned}$$

    a) $(1.2)^{2.1}$
    b) $\left(\dfrac{1}{2}\right)^2$
    c) $2^{6.25}$
    d) $3^{-2.61}$

e) $\left(\dfrac{1}{3}\right)^{-2.21}$

14. Write the function f defined by

$$f(x) = x^{\sin x}, \; x > 0$$

as an exponential function with base e.
(**Hint:** See Example 176, (3)).

15. Find the derivative of each of the following functions at the indicated points (if any).

a) $f(x) = 2e^{2x}$

b) $g(x) = e^{-(3.4)x+2}$, at $x = 0$

c) $f(x) = 3^{\cos x}$

d) $g(x) = (e^{3x})^{-2}$, at $x = 1$

e) $k(x) = e^{x^2} \sin x$, at $x = 0$

f) $f(x) = e^x \cos x$

g) $g(x) = -e^{-x}x^2$, at $x = 0$

h) $f(x) = x^2 e^{2x}$

i) $g(x) = \dfrac{e^{-2x}}{x^2}$

j) $f(x) = (1.2)^x$

k) $g(x) = x^{1.6} e^{-x}$

NOTES:

4.6 Differentiation Formulae for General Logarithmic Functions

Finally, the situation for the general logarithm is similar to the one for the natural logarithm except for an **additional factor** in the expression for its derivative.

OK. We know that for $a > 0$,

$$a^x = e^{x \ln a}$$

by definition, so that if we write $f(x) = a^x$ and let F denote its inverse function, then

$$F'(x) = \frac{1}{f'(F(x))}$$

as we saw earlier, but $f'(x) = a^x \ln a$ and so

$$\begin{aligned}\frac{d}{dx}\log_a(x) &= \frac{1}{f'(F(x))}\\ &= \frac{1}{a^{F(x)}\ln a}\\ &= \frac{1}{a^{\log_a(x)}\ln a}\\ &= \frac{1}{x\ln a}, \text{ (if } a>0,\ x>0)\end{aligned}$$

More generally we can see that (using the Chain Rule)

$$\boxed{\frac{d}{dx}\log_a(\Box) = \frac{1}{\ln a}\cdot\frac{1}{\Box}\cdot\frac{d}{dx}(\Box)}$$

Actually, more can be said: If $\Box \neq 0$, then

$$\boxed{\frac{d}{dx}\log_a(|\Box|) = \frac{1}{\ln a}\cdot\frac{1}{\Box}\cdot\frac{d}{dx}(\Box)}$$

so that you can replace the $\Box$ term by its *absolute value*. To show this just *remove the absolute value* and use the Chain Rule.

where $\Box > 0$, $a > 0$ (see the margin).

Exercise: Show the following **change of base** formula for logarithms:

$$\boxed{\log_a(\Box) = \frac{\ln(\Box)}{\ln a}}$$

This formula **allows one to convert from logarithms with *base a* to natural logarithms,** (those with base e).

Use the following steps.

1. Let $a^{\triangle} = \Box$ where $\triangle$, $\Box$ are symbols denoting numbers, functions, etc. Show that $\triangle = \log_a(\Box)$.
2. Show that $\ln(\Box) = \triangle \ln(a)$.
3. Show the formula by solving for $\triangle$.

Example 179. Find the derivatives of the following functions at the indicated point.

a) $\log_a(x^2+1)$, at $x=0$
b) $\log_2(3^x)$
c) $\log_4(2x+1)$, at $x=0$
d) $\log_{0.5}(e^{2x})$
e) $2^x\log_3(3x)$
f) $(\sin x)^x$ using any logarithm.

Solutions

a) Let $\square = x^2+1$. Then $\frac{d\square}{dx}=2x$, and

$$\begin{aligned}\frac{d}{dx}\log_a(\square) &= \frac{1}{\ln a}\cdot\frac{1}{\square}\cdot\frac{d\square}{dx}\\ &= \frac{1}{\ln a}\cdot\frac{1}{x^2+1}\cdot 2x\\ &= \frac{2x}{(x^2+1)\ln a}\end{aligned}$$

So, at $x=0$, its value is equal to 0.

b) Note that $\log_2(3^x)=x\log_2(3)$ by the property of logarithms. Thus,

$$\begin{aligned}\frac{d}{dx}\log_2(3^x) &= \frac{d}{dx}x\log_2(3)\\ &= \log_2(3)\frac{d}{dx}(x)\ (\text{since }\ \log_2(3)\text{ is a constant})\\ &= \log_2(3)\end{aligned}$$

You don't need to evaluate $\log_2(3)$.

c) Let $\square = 2x+1$. Then $\frac{d\square}{dx}=2$ and

$$\begin{aligned}\frac{d}{dx}\log_4(2x+1) &= \frac{d}{dx}\log_4(\square)\\ &= \frac{1}{\ln 4}\cdot\frac{1}{\square}\cdot\frac{d\square}{dx}\\ &= \frac{1}{\ln 4}\cdot\frac{1}{2x+1}\cdot 2\\ &= \frac{2}{(2x+1)\ln 4},\ \left(\text{for each } x>-\frac{1}{2}\right)\\ &= \frac{2}{\ln 4}\ (\text{at } x=0)\end{aligned}$$

d) Let $a=0.5$, $\square=e^{2x}$. Then $\frac{d\square}{dx}=2e^{2x}$ and

$$\begin{aligned}\frac{d}{dx}\log_a\square &= \frac{1}{\ln a}\cdot\frac{1}{\square}\cdot\frac{d\square}{dx}\\ &= \frac{1}{\ln(0.5)}\cdot\frac{1}{e^{2x}}\cdot 2e^{2x}\\ &= \frac{2}{\ln(0.5)}\\ &= \frac{2}{\ln(\frac{1}{2})}\\ &= -\frac{2}{\ln(2)}\ (\text{since }\ \ln(\frac{1}{2})=\ln 1-\ln 2)\end{aligned}$$

Thus, $\frac{d}{dx}\log_{0.5}(e^{2x}) = -\frac{2}{\ln 2}$ at $x = 0$.

NOTE: We could arrive at the answer more simply by noticing that

$$\begin{aligned} \log_{0.5} e^{2x} &= 2x \log_{0.5}(e) \\ &= x(\underbrace{2\log_{0.5}(e)}_{\text{constant}}) \end{aligned}$$

So its derivative is equal to $2\log_{0.5}(e) = -\frac{2}{\ln(2)}$ (by the change of base formula with $a = \frac{1}{2}$, $\square = e$).

e) Use the Product Rule here. Then

$$\begin{aligned} \frac{d}{dx}(2^x \log_3(3x)) &= \frac{d}{dx}(2^x)\log_3(3x) + 2^x \frac{d}{dx}(\log_3(3x)) \\ &= (2^x \ln 2)\log_3(3x) + 2^x \left(\frac{1}{\ln 3} \cdot \frac{1}{3x} \cdot 3\right) \\ &= 2^x(\ln 2)(\log_3(3x)) + \frac{3 \cdot 2^x}{3x(\ln 3)} \end{aligned}$$

f) Let $y = (\sin x)^x$. Then, $\ln y = \ln((\sin x)^x) = x\ \ln \sin x$. Now use *implicit differentiation* on the left, and the Product Rule on the right!

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}(x\ \ln \sin x), \\ &= x\frac{1}{\sin x}\cos x + \ln \sin x, \\ &= x\cot x + \ln \sin x, \qquad \text{and solving for the derivative we find,} \\ \frac{dy}{dx} &= y(x\cot x + \ln \sin x), \\ &= (\sin x)^x(x\cot x + \ln \sin x). \end{aligned}$$

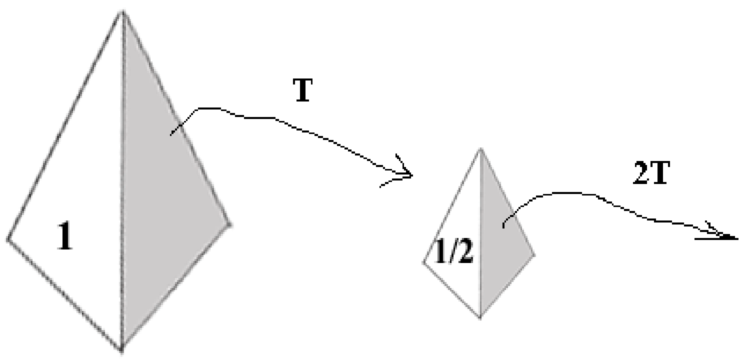
Exercise Set 24.

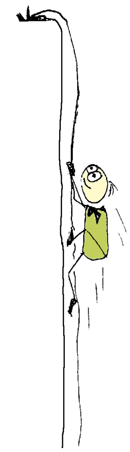
Find the derivative of each of the following functions.

a) $\log_a(x^3 + x + 1)$
b) $\log_3(x^x)$,
(**Hint:** Use a property of logarithms and the Product Rule.)
c) x^x, (**Hint:** Rewrite this as a function with a constant base.)
d) $\log_3(4x - 3)$
e) $\log_{1/3}(e^{4x})$
f) $3^x \log_2(x^2 + 1)$
g) $x\ln(x)$
h) $e^x \log_2(e^x)$
i) $\log_2(3x + 1)$
j) $\log_2(\sqrt{x+1})$, (**Hint:** Simplify first.)

4.7 Applications

The exponential function occurs naturally in physics and the use of nuclear reactors. Let $N(t)$ denote the amount of radioactive substance at time t (whose units may be seconds, minutes, hours or years depending on the substance involved). The **half-life** of a substance is, by definition, the time, $\mathbf{T}$, that it takes for one-half of the substance to remain (on account of radioactive decay).





After T units of time there is only $\frac{1}{2}$ the original amount left. Another T units of time results in only $\frac{1}{4}$. The original amount, and so on.

It is known that *the rate of decay* $\frac{dN}{dt}$ *is proportional to the amount of material present at time* t*, namely,* $N(t)$. This means that

$$\underbrace{\frac{dN}{dt}}_{\text{rate of decay}} = \underbrace{k}_{\text{proportionality constant}}\underbrace{N}_{\text{amount present at time } t}$$

This differential equation for $N(t)$ has solutions of the form (we'll see why later, in the chapter on Differential Equations)

$$\boxed{N(t) = Ce^{kt}}$$

where C and k are constants. The number $\tau = k$ is called the **decay constant** which is a measure of the rate at which the nuclide releases radioactive emissions. At $t = 0$ we have a quantity $N(0)$ of material present, so $C = N(0)$. Since $N(T) = \frac{N(0)}{2}$ if T is the half-life of a radionuclide, it follows that

$$\begin{aligned} \frac{N(0)}{2} &= N(T) \\ &= N(O)e^{kT} \text{ (since } C = N(0)) \\ \text{or } \frac{1}{2} &= e^{kT} \end{aligned}$$

which, when we solve for T gives

$$\begin{aligned} kT &= \ln(\frac{1}{2}) \\ &= -\ln 2 \end{aligned}$$

or

$$\boxed{T = -\frac{\ln 2}{k}}$$

So if we know the decay constant we can get the half-life and vice-versa. The formula for radioactive decay now becomes

$$\begin{aligned} N(t) &= N(0)e^{kt} \\ &= N(0)e^{-\frac{\ln 2}{T}t} \\ &= N(0)\left(e^{-\ln 2}\right)^{\frac{t}{T}} \\ &= N(0)\left(\frac{1}{2}\right)^{\frac{t}{T}} \end{aligned}$$

i.e.

$$\boxed{N(t) = \frac{N(0)}{2^{t/T}}}$$

Half-Life of Radioisotopes

Isotope	**Half − Life**
Kr87	1.27 hours
Sr89	50.5 days
Sr90	29.1 years
Pu240	6, 500 years
Pu239	24, 100 years

Table 4.5: Half-Life of Radioisotopes

Example 180. Plutonium 240 has a half-life of 6500 years. This radionuclide is extremely toxic and is a byproduct of nuclear activity. How long will it take for a 1 gram sample of Pu240 to decay to 1 microgram?

Solution We know that $N(t)$, the amount of material at time t satisfies the equation

$$N(t) = \frac{N(0)}{2^{t/T}}$$

where T is the half-life and $N(0)$ is the initial amount. In our case, $T = 6500$ (and all time units will be measured in years). Furthermore, $N(0) = 1$g. We want a time t where $N(t) = 1$ microgram $= 10^{-6}$g, right? So

$$\underbrace{10^{-6}}_{\text{amount left}} = \frac{\overbrace{1}^{\text{initial amount } =1 \text{ here}}}{2^{t/6500}}$$

or

$$2^{t/6500} = 10^6$$

or, by taking the natural logarithm of both sides we have

$$\frac{t}{6500}\ln(2) \quad = \quad \ln(10^6)$$

or, solving for t,

$$\begin{aligned} t &= \frac{6500\ln(10^6)}{\ln(2)} \text{ (years)} \\ &= \frac{6500 \cdot 6\ln 10}{\ln 2} \\ &= \frac{6500(6)(2.3026)}{(0.6931)} \\ &= 129564 \text{ years} \end{aligned}$$

approximately!

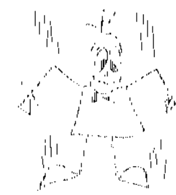


Figure 81.

Exercise Strontium 90 has a half-life of 29.1 years. How long will it take for a 5 gram sample of Sr90 to decay to 90% of its original amount?

The equation of motion of a body moving in free-fall through the air (Fig. 81) is given by

$$m\frac{dv}{dt} = mg - kv^2$$

where $v = v(t)$ is the velocity of the body in its descent, g is the acceleration due to gravity and m is its mass. Here k is a constant which reflects air resistance.

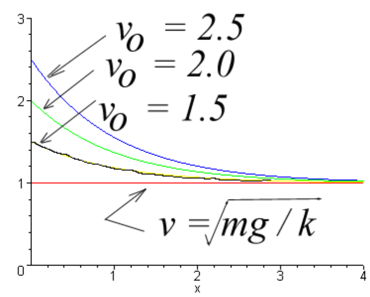


Figure 82.

We can learn to 'solve' this equation for the unknown velocity '$v(t)$' using methods from a later Chapter on Integration. At this point we can mention that this 'solution' is given by

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(t\sqrt{\frac{gk}{m}} + \operatorname{arctanh}\left(v_0\sqrt{\frac{k}{mg}}\right)\right)$$

where v_0 is its 'initial velocity'. For example, if one is dropping out of an airplane in a parachute we take it that $v_0 = 0$.

As $t \to \infty$ we see that $v(t) \to \sqrt{\dfrac{mg}{k}} = v_\infty$ (because the *hyperbolic tangent* term on the right approaches 1 as $t \to \infty$).

This quantity 'v_∞' called the limiting velocity is the 'final' or 'maximum' velocity of the body just before it reaches the ground. As you can see by taking the limit as $t \to \infty$, v_∞ *depends on the mass and the air resistance but does not depend upon the initial velocity!* See Figure 82.

NOTES:

Other Applications

1. The function f defined by $\boxed{f(x) = (2\pi)^{-1/2}e^{\frac{-x^2}{2}}}$ for $-\infty < x < \infty$, appears in probability theory, statistical mechanics, quantum physics, etc. It is referred to as a **normal distribution**, (see Fig. 83). It is also used by some teachers to 'curve' the grades of unsuccessful students!

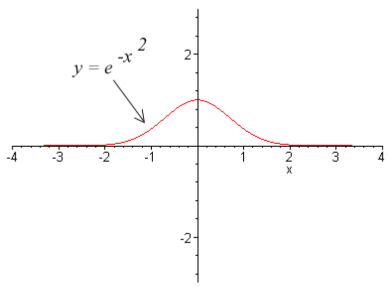


Figure 83.

2. The **entropy**, S, of a physical system is, by definition, an expression of the form

$$S = k \ln \Omega$$

where k is a physical constant and Ω is a measure of the number of states accessible to the system [Statistical Mechanics, Berkeley Physics (Vol 5). p. 143]. This notation is central to the study of Statistical Mechanics and Thermodynamics.

3. According to Newton, the temperature $T(t)$ of a cooling object drops at a rate proportional to the difference $T(t) - T_0$ where T_0 is the temperature of the surrounding space. This is represented analytically by a differential equation of the form

$$\frac{dT}{dt} = -k(T(t) - T_0)$$

where k is a constant.

It can be shown that the general solution if this equation looks like

$$T(t) = ae^{-kt} + T_0$$

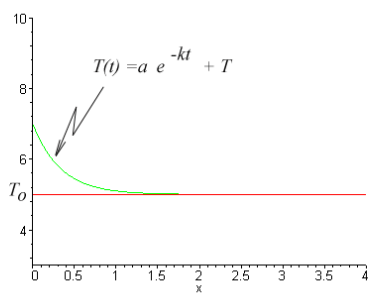


Figure 84.

where a is a constant. This law is called **Newton's Law of Cooling** as it represents the temperature of a heat radiating body (for example, coffee), as it cools in its surrounding space. Using this law, we can determine, for example, the temperature of a cup of coffee 10 minutes after it was poured, or determine the temperature of a hot pan, say, 5 minutes after it is removed from a heat source.

There are many other natural phenomena for which the rate of change of a quantity $y(t)$ at time t is proportional to the amount present at time t. That is, for which

$$\frac{dy}{dt} = ky \quad \text{with solution} \quad y(t) = Ce^{kt} \quad \text{where} \quad C = y(0).$$

Such a model is often called an **exponential decay** model if $k < 0$, and an **exponential growth** model if $k > 0$.

Example 181. If an amount of money P is deposited in an account at an annual interest rate, r, compounded continuously, then the balance $A(t)$ after t years is given by the exponential growth model

$$A = Pe^{rt} \qquad (\text{note that } P = A(0))$$

How long will it take for an investment of \$1000 to double if the interest rate is 10 % compounded continuously?

Solution Here $P = 1000$ and $r = .10$, so at any time t, $A = 1000e^{0.1t}$. We want to find the value of t for which $A = 2000$, so

$$2000 = 1000e^{0.1t}$$

so

$$2 = e^{0.1t}$$

and taking the natural logarithm of both sides

$$\ln 2 = \ln e^{0.1t} = 0.1t.$$

Thus

$$t = 10 \ln 2 \approx 6.9\text{years}.$$

Example 182.

A startup company that began in 1997 has found that its gross revenue follows an exponential growth model. The gross revenue was \$10,000 in 1997 and \$200,000 in 1999. If the exponential growth model continues to hold, what will be the gross revenue in 2000?

Solution Let $y(t)$ be the amount of the gross revenue in year t, so $y(t) = y(0)e^{kt}$. Taking 1997 as $t = 0$, $y(0) = 10,000$ so $y(t) = 10,000e^{kt}$. In 1999, $t = 2$, so

$$\begin{aligned} 200,000 &= 10,000e^{2k} \\ 20.5 &= e^{2k} \\ ln(20.5) &= 2k \\ k &= \frac{1}{2}\ln(20.5) = 1.51 \end{aligned}$$

and hence,

$$y(t) = 10,000e^{1.51t}.$$

Thus, in the year 2000, the gross revenue will be

$$y(3) = 10,000e^{1.51\times 3} \approx \$927586.$$

NOTES:

Summary of the Chapter

$$\boxed{\begin{aligned} e^x &= 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \dots \\ e &= \lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n \\ a^x &= e^{x\log a} = e^{x\ln a}, \quad \ln = \log \end{aligned}}$$

$$\frac{d}{dx}a^x = a^x \ln a, \ a > 0$$

$$\boxed{\begin{aligned} \frac{d}{dx}a^{\square} &= a^{\square}\ln a \ \tfrac{d\square}{dx} \\ \frac{d}{dx}\log_a(\square) &= \tfrac{1}{\square}\cdot\tfrac{1}{\ln a}\cdot\tfrac{d\square}{dx} \end{aligned}}$$

$$\frac{d}{dx}e^{\square} = e^{\square}\frac{d\square}{dx}$$

$$\frac{d}{dx}\ln\square = \frac{1}{\square}\frac{d\square}{dx}$$

($\square$ any '*symbol*' involving x, $\square > 0$, and differentiable)

The **exponential** and **logarithm** have the following properties:

(a) $a^0 = 1, \ a > 0$

(b) $\lim_{x\to+\infty} a^x = +\infty$

(c) $\lim_{x\to-\infty} a^x = 0$

(d) $a^{\triangle+\square} = a^{\triangle}a^{\square}$

(e) $a^{\triangle-\square} = \dfrac{a^{\triangle}}{a^{\square}}$

(f) $\log_a(1) = 0, \ a > 0$

(g) $\log_a(a) = 1$

(h) $\lim_{x\to+\infty}\log_a(x) = \begin{cases} +\infty & \text{if } a > 1 \\ -\infty & \text{if } 0 < a < 1 \end{cases}$

$$\boxed{\begin{aligned} &9. \quad \log_a(\triangle\square) &&= \log_a(\triangle) + \log_a(\square) \\ &10. \quad \log_a\left(\tfrac{\triangle}{\square}\right) &&= \log_a(\triangle) - \log_a(\square) \end{aligned}}$$

where $\triangle > 0$, $\square > 0$ are any '*symbols*' (numbers, functions, ...)

Table 4.6: Summary of the Chapter

4.8 Chapter Exercises

Show that the following sequences are increasing and find their limits in the extended real numbers.

1. $a_n = n + 3$, $n \geq 1$
2. $a_n = \dfrac{n-3}{n}$, $n \geq 1$
3. $a_n = \dfrac{n(n-1)}{n^2}$, $n \geq 1$
4. $a_n = \dfrac{n}{n+4}$, $n \geq 1$
5. Sketch the graph of the sequence $\{a_n\}$ given by
$$a_n = \sqrt{\frac{n-1}{2n}}$$
for $n = 1, 2, \ldots, 15$, and find it's limit.
6. Evaluate the following expressions and simplify as much as possible.
   a) $e^{x \ln x}$
   b) $\ln(e^{\sqrt{x}})$,
7. Evaluate the following expressions using
   i) your calculator only and
   ii) writing them in terms Euler's exponential function and then using your calculator:
$$\begin{aligned} \text{Example: } \text{i) } (2.3)^{1.2} &= 2.71690 \\ \text{ii) } (2.3)^{1.2} &= e^{(1.2)\ln(2.3)} \\ &= e^{0.99949} \\ &= 2.71690 \end{aligned}$$
   a) $(2.1)^{1.2}$
   b) $(0.465)^2$
   c) $(0.5)^{-0.25}$
8. Find the derivative of each of the following functions at the indicated points (if any).
   a) $f(x) = 3e^{5x}$
   b) $g(x) = 2e^{3x+2}$, at $x = 0$
   c) $f(x) = \cos(x\ e^x)$
   d) $g(x) = (e^{4x})^{-2}$, at $x = 1$
   e) $k(x) = e^{x^2} \sin(x^2)$, at $x = 0$
   f) $f(x) = e^x \ln(\sin x)$
   g) $g(x) = xe^{-x}$, at $x = 0$
   h) $f(x) = x^2 e^{-2x}$
   i) $g(x) = e^{-2x} \operatorname{Arctan} x$
   j) $f(x) = (x^2)^x$, at $x = 1$.
   k) $g(x) = \sqrt{x} \ln(\sqrt{x})$
   l) $f(x) = 2^x \log_{1.6}(x^3)$

m) $g(x) = 3^{-x} \log_{0.5}(\sec x)$

9. If \$500 is deposited in an account with an annual interest rate of 10 % , compounded continuously,

   (a) What amount will be in the account after 5 years?

   (b) How long will it be until the amount has tripled?

10. An annuity pays 12 % compounded continuously. What amount of money deposited today, will have grown to \$2400 in 8 years?

11. Four months after discontinuing advertising in Mcleans' Magazine, a manufacturer notices that sales have dropped from 10,000 units per month to 8,000 units per month. If the sales can be modelled by an exponential decay model, what will they be after another 2 months?

12. The revenue for a certain company was \$486.8 million in 1990 and \$1005.8 million in 1999.

   (a) Use an exponential growth model to estimate the revenue in 2001. (Hint: $t = 0$ in 1990.)

   (b) In what year will the revenue have reached \$1400.0 million?

13. The cumulative sales S (in thousands of units), of a new product after it has been on the market for t years is modelled by

$$S = Ce^{\frac{k}{t}}.$$

   During the first year 5000 units were sold. The saturation point for the market is 30,000 units. That is, the limit of S as $t \to \infty$ is 30,000.

   (a) Solve for C and k in the model.

   (b) How many units will be sold after 5 years?

Suggested Homework Set 18. *Work out problems* $6, 8c, 8d, 8f, 8j, 8l$

4.9 Using Computer Algebra Systems

Use your favorite Computer Algebra System (CAS), like Maple, MatLab, etc., or even a graphing calculator to answer the following questions:

1. Let $x > 0$. Calculate the quotient $\frac{\log x}{\log_3 x}$. What is the value of this quotient as a natural logarithm?

2. Find a formula for the first 10 derivatives of the function $f(x) = \log x$. What is the natural domain of each of these derivatives? Can you find a formula for ALL the derivatives of f?

3. Evaluate

$$\lim_{n\to\infty} n \log \left(1 + \frac{a}{n}\right)^n$$

   by starting with various values of a, say, $a = 0.1, 2.6, 5.2, 8.4, 10$ and then guessing the answer for any given value of a.Can you prove your guess using L'Hospital's Rule?

4. Let n be a given positive integer and $a_0, a_1, \ldots, a_n$ any given real numbers. Show that
$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{e^x} = 0.$$
Roughly, this says that the **exponential function grows to infinity faster than any polynomial regardless of its degree**. For example, plot the graphs of the functions $f(x) = e^x$ and $g(x) = x^10 + 3x^8 - 6$ on the same axes.
Even though this quotient is a very big number for $10 \le x \le 30$, it's easy to see that if $x = 40$ or above then the quotient is less than 1 (in fact, we know that it has to approach zero, so this inequality must be eventually true).

5. Show that
$$\lim_{x\to\infty} \frac{\log x}{x^a} = 0$$
regardless of the value of the exponent a so long as $a > 0$. Roughly, this says that the **logarithmic function grows to infinity more slowly than any polynomial regardless of its degree**.

6. Use a precise plot to show that
$$\left| \log_{10} x - 0.86304\frac{x-1}{x+1} - 0.36415\left(\frac{x-1}{x+1}\right)^3 \right| \le 0.0006$$
provided
$$\frac{1}{\sqrt{10}} \le x \le \sqrt{10}.$$

7. Using a graphical plotter prove the inequality
$$\frac{x}{1+x} < \log(1+x) < x$$
whenever $x > -1$ but $x \ne 0$. Can you prove this inequality using the Mean Value Theorem?

8. Calculate all the derivatives of the function $f(x) = e^{-x^2}$ at $x = 0$ and show that $f^(n)(0) = 0$ for any ODD integer n.

9. Compare the values $f(x) = e^x$ with the values of
$$g(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.$$
Can you guess what happens if we continue to add more terms of the same type to the polynomial on the right ?

NOTES:

Chapter 5

Curve Sketching

The Big Picture

In this Chapter much of what you will have learned so far in differential calculus will be used in helping you draw the graph of a given function. Curve sketching is one of the big applications of elementary calculus. You will see that the various types of limits and the methods used in finding them (e.g., L'Hospital's Rule) will show up again under the guise of **vertical asymptotes** or **horizontal asymptotes** to a graph. In addition, your knowledge of differentiation will help you determine whether a function is **increasing** or **decreasing** and whether or nor it is **concave up** or **concave down**. Furthermore, **Newton's method** for locating the roots of functions will come in handy in finding so-called **critical points** along with the various **intercepts**. All these ideas can be generalized to functions of two or even three variables, so a sound grasp of this chapter is needed to help you visualize the graphs of functions in the plane. We outline here the basic steps required in sketching a given planar curve defined by a function.

Review

Look over all the various **methods of differentiation**. A thorough review of Chapters 2 and 3 is needed here as all that material gets to be used in this chapter (at least do the Chapter Exercises at the end). The material in the first two sections of this chapter is also very important so don't skip over this unless you've already seen it before.

5.1 Solving Polynomial Inequalities

The subject of this section is the development of a technique used in solving inequalities involving polynomials or **rational functions** (quotients of polynomials) and some slightly more general functions which look like polynomials or rational functions. One of the main reasons for doing this in Calculus is so that we can use this idea to help us sketch the graph of a function. Recall that a polynomial in x is an expression involving x and multiples of its powers only. For example, $2x^2-3x+1$, $x-1$, -1.6, $0.5x^3+1.72x-5$, ... are all polynomials. Yes, even ordinary numbers are polynomials (of degree zero).

We'll learn how to solve a polynomial inequality of the form

$$(x-1)(x-\frac{1}{2})(x+2)<0$$

for all possible values of x, or a rational function inequality of the form

$$\frac{(x-2)(x+4)}{x^2-9}>0$$

for every possible value of x! For example, the inequality $x^2-1<0$ has the set of numbers which is the open interval $(-1,1)$ as the set of all of its solutions. Once we know how to solve such inequalities, we'll be able to find those intervals where the graph of a given (differentiable) function has certain properties. All this can be done without the help of a plotter or computer hardware of any kind, but a hand-held calculator would come in handy to speed up simple operations.

Why polynomials? It turns out that many, many functions can be approximated by polynomials, and so, if we know something about this polynomial approximation then we will know something about the original function (with some small errors!). So it is natural to study polynomials. Among the many approximations available, we find one very common one, the so-called **Taylor polynomial approximation** which is used widely in the sciences and engineering and in your pocket calculator, as well.

	Actual	Est.	Est.
x	$\sin(x)$	$p_5(x)$	$p_{13}(x)$
-2	-0.9093	-0.9333	-0.9093
-1	-0.8415	-0.8417	-0.8415
0	0	0	0
1	0.8415	0.8417	0.8415
2	0.9093	0.9333	0.9093
3	0.1411	0.5250	0.1411
4	-0.7568	1.8667	-0.7560

For example, the trigonometric function $y=\sin x$ can be approximated by this Taylor polynomial of degree $2n-1$, namely,

$$p_{2n-1}(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$$

The **larger the degree, the better the approximation** is a generally true statement in this business of Taylor polynomials and their related 'series', (see the margin for comparison). You don't have to worry about this now because we'll see all this in a forthcoming chapter on **Taylor series**.

The first step in solving polynomial inequalities is the factoring of the polynomial $p(x)$. Since all our polynomials have real coefficients it can be shown (but we won't do this here) that its factors are of exactly two types:

Either a polynomial, $p(x)$, has a factor that looks like

- A **Type I (or Linear) Factor**:

$$a_1x-a_2,$$

or it has a factor that looks like,

- A **Type II (Quadratic Irreducible) Factor**:

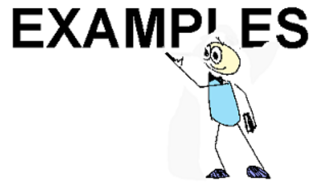
$$ax^2+bx+c, \text{ where } b^2-4ac<0$$

where a_1, a_2, a, b, c are all real numbers. All the factors of $p(x)$ must look like either Type I or Type II. This isn't obvious at all and it is an old and important result from Algebra. In other words, **every polynomial (with real coefficients) can be factored into a product of Type I and/or Type II factors and their powers**.

Example 183.

$$x^2 + 3x + 2 = \underbrace{(x+1)(x+2)}_{Type\ I\ factors}$$

Example 184.



$$\begin{aligned} x^2 + 2x + 1 &= (x+1)^2 \\ &= \underbrace{(x+1)(x+1)}_{Type\ I\ factors} \end{aligned}$$

Example 185.

$$2x^2 - 3x - 2 = \underbrace{(2x+1)(x-2)}_{Type\ I\ factors}$$

Example 186.

$$\begin{aligned} x^4 - 1 &= (x^2-1)(x^2+1) \\ &= \underbrace{(x-1)(x+1)}_{Type\ I}\underbrace{(x^2+1)}_{Type\ II} \end{aligned}$$

In this example, x^2+1 is a quadratic irreducible factor as $b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0$.

Example 187. $2x^4 + 19x^2 + 9 = (x^2+9)(2x^2+1)$. *Notice that there are* **no Type I** *factors at all in this example. Don't worry, this is OK, it can happen!*

Example 188. $x^2 + x + 1 = x^2 + x + 1$. *We cannot simplify this one further because $b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$. So the polynomial is identical with its Type II factor, and we leave it as it is!*

Example 189.

$$\begin{aligned} x^6 - 1 &= (x^3-1)(x^3+1) \\ &= (x-1)(x^2+x+1) \cdot (x+1)(x^2-x+1) \\ &= \underbrace{(x-1)(x+1)}_{Type\ I}\underbrace{(x^2-x+1)}_{Type\ II}\underbrace{(x^2+x+1)}_{TypeII} \end{aligned}$$

Example 190.

$$(x^2 - 2x + 1)(x^2 - 4x + 4) = \underbrace{(x-1)^2(x-2)^2}_{All\ Type\ I\ factors}$$

Don't worry about the powers which may appear in a linear factor (Type I factor), sometimes they show up, just use them.

Exercise Set 25.

Factor the following polynomials into Type I and Type II factors and identify each one as in the examples above.

1. $x^2 - 1$
2. $x^3 - x^2 + x - 1$ (Hint: x =1 is a root)
3. $x^2 + x - 6$
4. $x^3 - x^2 - x + 1$
5. $x^4 - 16$
6. $2x^2 + x - 1$
7. $x^4 - 2x^2 + 1$ (Hint: $x = 1$ and $x = -1$ are both roots.)
8. $x^3 + x^2 + x + 1$ (Hint: $x = -1$ is a root.)

For the purposes of solving inequalities we will call real points x where $p(x) = 0$, **break-points** (or real roots, or zeros, is more common). Thus,

$$x^2 - 1 = (x - 1)(x + 1)$$

has $x = \pm 1$ as break-points, while

$$x^4 - 16 = (x - 2)(x + 2)(x^2 + 4)$$

has $x = \pm 2$ as break-points, but no other such points (since $x^2 + 4 \neq 0$ for any x).

Remember: Quadratic irreducible factors (Type II factors) have no break-points. Break-points come from linear factors (Type I factors) **only**.

The Sign Decomposition Table of a Polynomial

The next step in our guide to solving polynomial inequalities is the creation of the so-called Sign Decomposition Table (SDT, for short) of a polynomial, $p(x)$. Once we have filled in this SDT with the correct '+' and '−' signs, we can **essentially read off the solution of our inequality**. In Table 5.1, we present an example of a SDT for the polynomial $p(x) = x^4 - 1$.

Look at the SDT, Table 5.1, of $x^4 - 1$. The solution of the inequality

$$x^4 - 1 < 0$$

can be "read off" the SDT by looking at the last column of its SDT and choosing the intervals with the '−' sign in its last column. In this case we see the row

$$|\ (-1, 1)\ \|\ +\ |\ -\ |\ +\ \|\ -\ \|$$

which translates into the statement

$$\text{“If } -1 < x < 1 \quad \text{then} \quad x^4 - 1 < 0.\text{”}$$

The Sign Decomposition Table of $x^4 - 1$

	$x+1$	$x-1$	x^2+1	sign of $p(x)$
$(-\infty,-1)$	$-$	$-$	$+$	$+$
$(-1,1)$	$+$	$-$	$+$	$-$
$(1,\infty)$	$+$	$+$	$+$	$+$

The SDT is made up of *rows containing intervals whose end-points are break-points* of $p(x)$ and *columns are the factors* of $p(x)$ and various $+/-$ signs. We'll explain all this below and show you how it works!

Table 5.1: The Sign Decomposition Table of $x^4 - 1$

Size of SDT $=$ $(r+1)$ **by** $(s+1)$

(rows by columns, excluding the margin and headers) where

r = the total number of **different** break-points of $p(x)$, and

s = (the total number of different break-points of $p(x)$) + (the total number of **different** Type II factors of $p(x)$).

Table 5.2: Size of SDT

The same kind of argument works if we are looking for all the solutions of $x^4 - 1 > 0$. In this case, there are **2** rows whose last column have a '+' sign namely,

$(-\infty,-1)$	$-$	$-$	$+$	$+$
$(1,\infty)$	$+$	$+$	$+$	$+$

The Size of a Sign Decomposition Table

This last piece of information tells us that,

"If $-\infty < x < -1$ **or** $1 < x < \infty$ then $x^4 - 1 > 0$"

So, all the information we need in order to solve the inequality $p(x) > 0$ can be found in its Sign Decomposition Table!

OK, so what is this SDT and how do we fill it in?

First, we need to decide on **the size of a SDT**. Let's say it has $r+1$ rows and $s+1$ columns (the ones containing the $+/-$ signs).

Example 191. What is the size of the STD of $p(x) = x^4 - 1$?

Solution The first step is to factor $p(x)$ into its linear (or Type I) and quadratic

irreducible (or Type II) factors. So,

$$\begin{aligned} x^4-1 &= (x^2-1)(x^2+1) \\ &= (x+1)(x-1)(x^2+1) \end{aligned}$$

The next step is to determine r and s. Now the break-points are ± 1 and so $r = 2$. There is only one quadratic irreducible factor , so $s = r+1 = 2+1 = 3$, by Table 5.2. So the SDT has size $(r+1)$ by $(s+1)$ which is $(2+1)$ by $(3+1)$ or 3 by 4. The SDT has 3 rows and 4 columns.

Example 192. Find the size of the SDT of the polynomial

$$p(x) = (x-1)(x-2)(x-3)(x^2+1)(x^2+4)$$

Solution The polynomial $p(x)$ is already in its desired factored form because it is a product of 3 Type I factors and 2 Type II factors! Its break-points are $x = 1, x = 2, x = 3$ and so $r = 3$, since there are 3 break-points. Next, there are only 2 distinct Type II factors, right? So, by Table 5.2, $s = r+2 = 3+2 = 5$. The SDT of $p(x)$ has size $(3+1)$ by $(5+1)$ or 4 by 6.

How to fill in a SDT?

OK, now that we know how big a SDT can be, what do we do with it?

Now **write down all the Type I and Type II factors and their powers** so that, for example,

$$p(x) = (x-a_1)^{p_1}(x-a_2)^{p_2}\dots(x-a_r)^{p_r}(A_1x^2+B_1x+C_1)^{q_1}\dots$$

Rearrange the break-points $a_1, a_2, \dots, a_r$ in increasing order, you may have to relabel them though, that is, let

$$(-\infty <) \quad a_1 < a_2 < a_3 < \dots < a_r \quad (< +\infty)$$

Form the following open intervals: $I_1, I_2, \dots, I_{r+1}$ where

$$\begin{array}{ll} I_1 : & (-\infty, a_1) = \{x : -\infty < x < a_1\} \\ I_2 : & (a_1, a_2) \\ I_3 : & (a_2, a_3) \\ I_4 : & (a_3, a_4) \\ & \dots \\ I_r : & (a_{r-1}, a_r) \\ I_{r+1} : & (a_r, +\infty) \end{array}$$

and put them in the **margin** of our SDT.

At the top of each column place every factor (Type I and Type II) along with their 'power':

	$(x-a_1)^{p_1}$	...	$(x-a_r)^{p_r}$	$(A_1x^2+B_1x+C_1)^{q_1}$	...
$(-\infty, a_1)$					
(a_1, a_2)					
(a_2, a_3)					
...					..
(a_{r-1}, a_r)					
$(a_r, +\infty)$					

Finally we "fill in" our SDT with the symbols '+' (for plus) and '−' (for minus).

So far, so good, but *how do we choose the sign?*

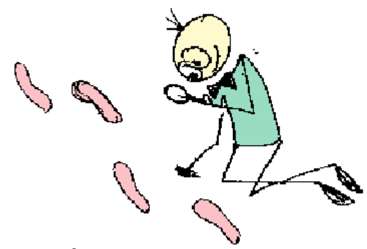
Actually, this is not hard to do. Let's say you want to know what sign (+/−) goes into the i^{th} row and j^{th} column.

How to find the signs in the SDT

Filling in a SDT

1. Choose **any** point in the interval $I_i = (a_{i-1}, a_i)$.
2. Evaluate the factor (at the very top of column j along with its power) at the point you chose in (1), above.
3. The sign of the number in (2) is the sign we put in this box at row i and column j.
4. The sign in the **last column** of row i is just the product of all the signs in that row.

Table 5.3: Filling in an Sign Decomposition Table



NOTE: For item (1) in Table 5.3, if the interval is *finite*, we can **choose the midpoint** of the interval $(a_{i-1}, a_i) = I_i$ or,

$$\text{midpoint } = \frac{a_i + a_{i-1}}{2}$$

Here's a few examples drawn from Table 5.1.

Example 193. In Table 5.1 we choose $x = -2$ inside the interval $(-\infty, -1)$, evaluate the factor $(x-1)$ at $x = -2$, look at its sign, (it is negative) and then place the plus or minus sign in the corresponding cell.

You have complete freedom in your choice of number in the given interval. The method is summarized in the diagram below:

$$\begin{array}{ccc} & & \boxed{(x-1)} \\ \text{Choose } x=-2 & \nearrow & \downarrow \text{ Sign of } (-2-1) \\ \boxed{(-\infty,-1)} & & \boxed{-} \end{array}$$

Example 194. In Table 5.1 we choose $x = 0$ inside the interval $(-1, 1)$, evaluate the factor $(x+1)$ at $x = 0$, look at its sign, and then place the plus or minus sign in the corresponding cell.

$$\begin{array}{ccc} & & \boxed{(x+1)} \\ \text{Choose } x=0 & \nearrow & \downarrow \text{ Sign of } (0+1) \\ \boxed{(-1,1)} & & \boxed{+} \end{array}$$

Example 195. In Table 5.1 we choose $x = 1.6$ inside the interval $(1, \infty)$,

evaluate the factor (x^2+1) at $x = 1.6$, look at its sign, and then place the plus or minus sign in the corresponding cell.

$$\begin{array}{ccc} & & \boxed{(x^2+1)} \\ & \text{Choose } x = 1.6 \nearrow & \downarrow \text{ Sign of } (2.56+1) \\ \boxed{(1,\infty)} & & \boxed{+} \end{array}$$

Example 196. In Table 5.1 we choose $x = -0.8$ inside the interval $(-1, 1)$, evaluate the factor $(x-1)$ at $x = -0.8$, look at its sign, and then place the plus or minus sign in the corresponding cell.

$$\begin{array}{ccc} & & \boxed{(x-1)} \\ & \text{Choose } x = -0.8 \nearrow & \downarrow \text{ Sign of } (-0.8-1) \\ \boxed{(-1,1)} & & \boxed{-} \end{array}$$

OK, now we are in a position to create the SDT of a given polynomial.

Example 197. Find the SDT of the polynomial

$$p(x) = (x-1)(x-2)^2(x^2+1)$$

Solution The first question is: What is the complete factorization of $p(x)$ into Type I and II factors? In this case we have nothing to do as $p(x)$ is already in this special form. Why?

Next, we must decide on the size of the SDT. Its size, according to our definition, is 3 by 4 (excluding the margin and headers).

We can produce the SDT: Note that its break-points are at $x = 1$ and $x = 2$.

	$(x-1)$	$(x-2)^2$	(x^2+1)	Sign of $p(x)$
$(-\infty,1)$				
$(1,2)$				
$(2,\infty)$				

Such SDT tables will be used later to help us find the properties of graphs of polynomials and rational functions!

We fill in the $3\times 4 = 12$ cells with $+/-$ signs according to the procedure in Table 5.3. We find the table,

	$(x-1)$	$(x-2)^2$	(x^2+1)	Sign of $p(x)$
$(-\infty,1)$	-	+	+	-
$(1,2)$	+	+	+	+
$(2,\infty)$	+	+	+	+

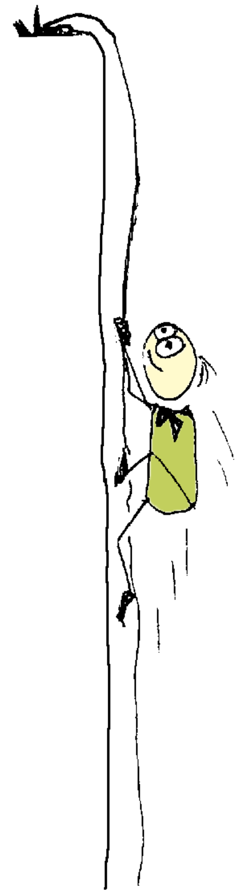
as its SDT, since the product of all the signs in the first row is negative (as $(-1)(+1)(+1) = -1$) while the product of the signs in each of the other rows is positive.

Example 198. Find the SDT of the polynomial

$$p(x) = (x+1)^2(x-1)(x+3)^3$$

Solution The break-points are given by $x = -3, -1, 1$. These give rise to some intervals, namely $(-\infty, -3), (-3, -1), (-1, 1), (1, \infty)$. The table now looks like

	$(x+3)^3$	$(x+1)^2$	$(x-1)$	Sign of $p(x)$
$(-\infty, -3)$				
$(-3, -1)$				
$(-1, 1)$				
$(1, \infty)$				



OK, now we have to fill in this SDT with $+/-$ signs, right? So choose some points in each one of the intervals in the left, find the sign of the corresponding number in the columns and continue this procedure. (See the previous Example). We will get the table,

	$(x+3)^3$	$(x+1)^2$	$(x-1)$	Sign of $p(x)$
$(-\infty, -3)$	$-$	$+$	$-$	$+$
$(-3, -1)$	$+$	$+$	$-$	$-$
$(-1, 1)$	$+$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$	$+$

That's all!

Example 199. Find the SDT of the polynomial

$$p(x) = \left(x - \frac{1}{2}\right)(x + 2.6)(x-1)^2(x^2 + x + 1)$$

Solution OK, first of all, do not worry about the type of numbers that show up here, namely, $\frac{1}{2}, 2.6$ etc. It doesn't matter what kind of numbers these are; they do not have to to be integers! The break-points are $-2.6, \frac{1}{2}, 1$ and $\boxed{?}$. Well, there is no other because x^2+x+1 is a quadratic irreducible (remember that such a polynomial has no real roots, or equivalently its discriminant is negative).

The SDT looks like (convince yourself):

SDT	$(x+2.6)$	$(x-\frac{1}{2})$	$(x-1)^2$	(x^2+x+1)	Sign of $p(x)$
$(-\infty, -2.6)$	$-$	$-$	$+$	$+$	$+$
$(-2.6, \frac{1}{2})$	$+$	$-$	$+$	$+$	$-$
$(\frac{1}{2}, 1)$	$+$	$+$	$+$	$+$	$+$
$(1, \infty)$	$+$	$+$	$+$	$+$	$+$

Example 200. Find the SDT of the polynomial

$$p(x) = 3(x^2 - 4)(9 - x^2)$$

Solution Let's factor this completely first. Do not worry about the number '3' appearing as the leading coefficient there, it doesn't affect the 'sign' of $p(x)$ as it is positive.

In this example the break-points are, $x = -3, -2, 2, 3$, in increasing order, because the factors of $p(x)$ are $(x-2)(x+2)(3-x)(3+x)$. (Notice that the 'x' does not come first in the third and fourth factors ... that's OK!). We produce the SDT as usual.

SDT	$(x+3)$	$(x+2)$	$(x-2)$	$(3-x)$	Sign of $p(x)$
$(-\infty,-3)$	$-$	$-$	$-$	$+$	$-$
$(-3,-2)$	$+$	$-$	$-$	$+$	$+$
$(-2,2)$	$+$	$+$	$-$	$+$	$-$
$(2,3)$	$+$	$+$	$+$	$+$	$+$
$(3,\infty)$	$+$	$+$	$+$	$-$	$-$

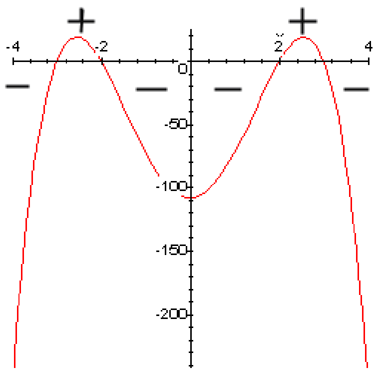


Figure 85.

The $+/-$ signs in the graph indicate the region(s) where the function is positive/negative, (see Figure 85).

How to solve polynomial inequalities?

Okay, now that you know how to find the SDT of a given polynomial it's going to be really easy to find the solution of a polynomial inequality involving that polynomial! All the information you need is in the SDT! Let's backtrack on a few examples to see how it's done.

Example 201. Solve the inequality $(x-1)(x-2)^2(x^2+1) < 0$

Solution The polynomial here is $p(x) = (x-1)(x-2)^2(x^2+1)$ and we need to solve the inequality $p(x) < 0$, right? Refer to Example 197 for its SDT. Just go to the last column of its SDT under the header 'Sign of $p(x)$' and look for *minus signs* only. There is only one of them, see it? It also happens to be in the row which corresponds to the interval $(-\infty, 1)$. There's your solution! That is, the solution of the inequality

$$(x-1)(x-2)^2(x^2+1) < 0,$$

is given by the set of all points x inside the interval $(-\infty, 1)$.

Example 202. Solve the inequality $p(x) = (x+1)^2(x-1)(x+3)^3 > 0$

Solution The polynomial is $p(x) = (x+1)^2(x-1)(x+3)^3$ and we need to solve the inequality $p(x) > 0$. Look at Example 198 for its SDT. Once again, go to the last column of its SDT under the header 'Sign of $p(x)$' and look for *plus signs* only. Now there are *two* of them, right? The rows they are in correspond to the two intervals $(-\infty, -3)$ and $(1, \infty)$. So the solution of the inequality is the *union* of these two intervals, that is, the solution of the inequality

$$(x+1)^2(x-1)(x+3)^3 > 0,$$

is given by the set of all points x where x is **either** in the interval $(-\infty, -3)$, **or**, in the interval $(1, \infty)$.

Example 203. Solve the inequality

$$p(x) = \left(x - \frac{1}{2}\right)(x+2.6)(x-1)^2(x^2+x+1) < 0.$$

Solution Now the polynomial is $p(x) = (x-(1/2))(x+2.6)(x-1)^2(x^2+x+1)$ and we need to solve the inequality $p(x) < 0$. Look at Example 199 for its SDT. Once again, go to the last column of its SDT under the header 'Sign of $p(x)$' and look for *minus signs* only. This time there is only one of them. The row it is in corresponds to the interval $(-2.6, 1/2)$. So the solution of the inequality

$$\left(x - \frac{1}{2}\right)(x+2.6)(x-1)^2(x^2+x+1) < 0$$

is given by the set of all points x in the interval $(-2.6, 0.5)$.

In case the polynomial inequality is of the form $p(x) \geq 0$ (or $p(x) \leq 0$), we simply

1. Solve the 'strict' inequality $p(x) > 0$ (or $p(x) < 0$) and
2. Add **all** the break-points to the solution set.

Let's look at an example.

Example 204. Solve the inequality $p(x) = 3(x^2 - 4)(9 - x^2) \leq 0$

Solution The polynomial is $p(x) = 3(x^2 - 4)(9 - x^2)$ and we need to solve the inequality $p(x) < 0$ and add all the break-points of p to the solution set, right? Look at Example 200 for its SDT. Once again, go to the last column of its SDT under the header 'Sign of $p(x)$' and look for *minus signs* only. This time there are three rows with minus signs in their last column. The rows correspond to the intervals $(-\infty, -3)$, $(-2, 2)$, and $(3, \infty)$. So the solution of the inequality

$$3(x^2 - 4)(9 - x^2) \leq 0,$$

is given by the union of all these intervals along with all the break-points of p. That is the solution set is given by the set of all points x where x is **either** in the interval $(-\infty, -3)$, **or**, in the interval $(-2, 2)$, **or**, in the interval $(3, \infty)$, along with the points {-2, 2, -3, 3}.

This can be also be written briefly as: $(-\infty, -3] \cup [-2, 2] \cup [3, \infty)$, where, as usual, the symbol '$\cup$' means the union.

Exercise Set 26.

1. Find all the break-points (or roots) of the following polynomials.
   a) $p(x) = (9x^2 - 1)(x + 1)$
   b) $q(x) = (x^4 - 1)(x + 3)$
   c) $r(x) = (x^2 + x - 2)(x^2 + x + 1)$
   d) $p(t) = t^3 - 1 \qquad (t - 1$ is a factor)
   e) $q(w) = w^6 - 1 \qquad (w - 1$ and $w + 1$ are factors)
2. Find the Sign Decomposition Table of each one of the polynomials in Exercise 1 above.
3. Find the Sign Decomposition Table of the function
$$p(x) = 2(x^2 - 9)(16 - x^2)$$
   (Hint: See Example 200).
4. What are the break-points of the function
$$p(x) = (2 + \sin(x))\ (x^2 + 1)(x - 2)?$$
   (Hint: $|\sin(x)| \leq 1$ for every value of x.)

5. Determine the SDT of the function

$$q(x) = (x^4 - 1)\ (3 + \cos(x))$$

(Hint: Use the ideas in the previous exercise and show that $3 + \cos(x) > 0$ for every value of x.)

6. Solve the polynomial inequality $x(x^2 + x + 1)(x^2 - 1) < 0$
7. Solve the inequality $(x^4 - 1)(x + 3) > 0$
8. Solve the polynomial inequality $(x + 1)(x - 2)(x - 3)(x + 4) \leq 0$
9. Solve the inequality $(x - 1)^3(x^2 + 1)(4 - x^2) \geq 0$
10. Solve the polynomial inequality $(9x^2 - 1)(x + 1) \geq 0$

NOTES:

5.2 Solving Rational Function Inequalities

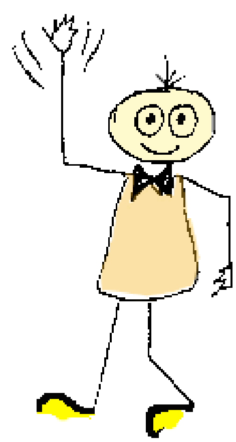
Recall that a rational function is, by definition, the quotient of two polynomials, so that, for example,

$$r(x) = \frac{x^3 - 3x^2 + 1}{x - 6}$$

is a rational function. When we study functions called **derivatives** we see that the way in which the graph of a rational function 'curves' around depends upon the need to solve inequalities of the form $r(x) > 0$ for certain values of x, or $r(x) < 0$, where r is some rational function. In this case we can extend the ideas of the previous sections and define an **SDT** for a given rational function. Let's see how this is done:

The idea is to extend the notion of a **break-point** for a polynomial to that for a rational function. Since a break-point is by definition a root of a polynomial, it is natural to define a break-point of a rational function to be a root of either its numerator or its denominator, and this is what we will do!

A **break-point** of a rational function r is any real root of either its numerator or its denominator but not a root of both.



This means that in the event that the numerator and denominator have a *common factor of the same multiplicity*, then we agree that there is *no break point there.* For instance, the rational function

$$r(x) = (x^2 - 9)/(x - 3)$$

has its only break-point at $x = -3$, because $x - 3$ is a factor in both the numerator and denominator! However, the slightly modified function

$$r(x) = (x^2 - 9)^2/(x - 3)$$

does have a break point at $x = 3$ since $x = 3$ is a double root of the numerator but only a simple root of the denominator. Since the multiplicities are different, we must include $x = 3$ as a break-point.

On the other hand, the break-points of the rational function

$$r(x) = \frac{x^2 - 4}{x^2 - 1}$$

are given by $x = \pm 2$ and $x = \pm 1$. Now we can build the SDT of a rational function by using the ideas in the polynomial case, which we just covered.

Example 205. Find the break-points of the following rational functions:

$$1)\ r(x) = \frac{x^2+1}{x} \quad 2)\ R(x) = \frac{x^3-1}{x+1} \quad 3)\ r(t) = \frac{3-t^2}{t^3+1}$$
$$4)\ R(t) = \frac{4t}{t^2+9} \quad 5)\ r(x) = x + 1 + \frac{2}{x-1}$$

Solution

1. Let's write

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x) = x^2+1$ and $q(x) = x$. The break-points of the $r(x)$ are by definition the same as the break-points of $p(x)$ and $q(x)$. But $p(x)$ is a quadratic irreducible (as its discriminant is negative) and so it has *no* break points, right? On the other hand, the break-point of the denominator $q(x)$ is given by $x = 0$ (it's the only root!). The collection of break points is given by $\{x = 0\}$.

2. Write $R(x) = \frac{p(x)}{q(x)}$, as before, where $p(x) = x^3 - 1$, $q(x) = x + 1$. We factor $R(x)$ completely to find
$$p(x) = x^3 - 1 = (x - 1)(x^2 + x + 1)$$
and
$$q(x) = x + 1$$
Now $x^2 + x + 1$ is an irreducible quadratic factor and so it has no break-points. The break-points of $p(x)$ are simply given by the single point $x = 1$ while $q(x)$ has $x = -1$ as its only break-point. The collection of break-points of $R(x)$ is now the set $\{x = 1, x = -1\}$.

3. Write $r(t) = \frac{p(t)}{q(t)}$ where $p(t) = 3 - t^2$ and $q(t) = t^3 + 1$. The factors of $p(t)$ are $(\sqrt{3} - t)(\sqrt{3} + t)$, right? The factors of $q(t)$ are $(t + 1)(t^2 - t + 1)$, so the break points are given by $t = -\sqrt{3}, -1, +\sqrt{3}$, since $t^2 - t + 1$ is an irreducible quadratic (no break-points).

4. In this example, the numerator $p(t) = 4t$ has only one break point, at $t = 0$. The denominator, $q(t) = t^2 + 9$ is an irreducible quadratic, right? Thus, the collection of break-points of $R(t)$ consists of only one point, $\{t = 0\}$.

5. This example looks mysterious, but all we need to do is find a common denominator, that is, we can rewrite $r(x)$ as
$$\begin{aligned} r(x) &= \frac{(x+1)(x-1)}{x-1} + \frac{2}{x-1} \\ &= \frac{(x+1)(x-1)+2}{x-1} \\ &= \frac{x^2+x-x-1+2}{x-1} \\ &= \frac{x^2+1}{x-1} \end{aligned}$$
From this equivalent representation we see that its break-points consist of the single point $\{x = 1\}$, since the numerator is irreducible.

Connections

Later on we'll see that the **break-points/roots of the denominator** of a rational function coincide with a vertical line that we call a **vertical asymptote**, a line around which the graph "peaks sharply" or "drops sharply", towards infinity.

For example, the two graphs in the adjoining margin indicate the presence of vertical asymptotes (v.a.) at $x = 0$ and $x = 1$.

The SDT of a rational function is found in exactly the same way as the SDT for a polynomial. The only difference is that we have to **include all the break-points** of the numerator and denominator which make it up. Remember that we never consider 'common roots'.

Example 206. Find the SDT of the rational function whose values are given by

$$\frac{(x-2)(x+4)}{x^2-9}$$

Solution From the SDT

	$(x+4)$	$(x+3)$	$(x-2)$	$(x-3)$	Sign of $r(x)$
$(-\infty,-4)$	$-$	$-$	$-$	$-$	$+$
$(-4,-3)$	$+$	$-$	$-$	$-$	$-$
$(-3,2)$	$+$	$+$	$-$	$-$	$+$
$(2,3)$	$+$	$+$	$+$	$-$	$-$
$(3,\infty)$	$+$	$+$	$+$	$+$	$+$

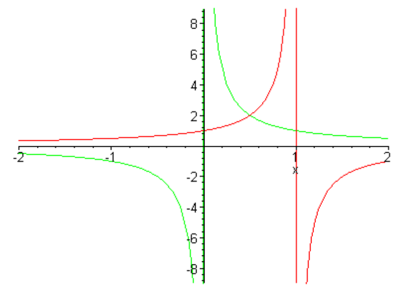
we see immediately that the solution of the inequality

$$\frac{(x-2)(x+4)}{x^2-9} < 0$$

is given by combining the intervals in rows 2 and 4 (as their last entry is negative). We get the set of points which is the union of the 2 intervals $(-4,-3)$ and $(2,3)$. Check it out with specific values, say, $x=-3.5$ or $x=2.5$ and see that it really works! On the other hand, if one wants the set of points for which

$$\frac{(x-2)(x+4)}{x^2-9} \leq 0$$

then one must add the break points $x=2$ and $x=-4$ to the two intervals already mentioned, (note that $x=\pm 3$ are not in the domain).



The graphs of Example 207 and $y = 1/x$ showing their vertical asymptotes at $x=1$ and $x=0$ respectively.

Example 207. Solve the inequality

$$\frac{x^2-3x+1}{x^3-1} \geq 0$$

Solution The break points of the values of this rational function, call them $r(x)$, are given by finding the roots of the quadratic in the numerator and the cubic in the denominator. Using the quadratic equation we get that the two roots of $x^2-3x+1=0$ are $x=(3+\sqrt{5})/2$ and $x=(3-\sqrt{5})/2$. Let's approximate these values by the numbers 2.618 and 0.382 in order to simplify the display of the SDT. The roots of the cubic $x^3-1=0$ are given by $x=1$ only, as its other factor, namely, the polynomial x^2+x+1 is irreducible, and so has no real roots, and consequently, no break-points.

Writing these break-points in increasing order we get: $0.382, 1, 2.618$. The SDT of this rational function is now

	$(x-0.382)$	$(x-1)$	$(x-2.618)$	Sign of $r(x)$
$(-\infty,0.382)$	$-$	$-$	$-$	$-$
$(0.382,1)$	$+$	$-$	$-$	$+$
$(1,2.618)$	$+$	$+$	$-$	$-$
$(2.618,\infty)$	$+$	$+$	$+$	$+$

The solution can be read off easily using the last column as the intervals corresponding to the '+' signs. This gives the union of the two intervals $(0.382,1),(2.618,\infty)$ *along* with the two break-points $x=0.382, 2.618$, (why?).

Example 208. Solve the inequality

$$\frac{x}{3x-6} + \frac{2x}{x-2} < 0$$

Solution Looks strange because it's not in the 'usual' form, right? No problem, just put it in the usual form (i.e., a polynomial divided by another polynomial) by finding a common denominator, in this case, $3(x-2)$. We see that

$$\frac{x}{3x-6} + \frac{2x}{x-2} = \frac{x}{3x-6} + \frac{(3)(2x)}{3(x-2)} = \frac{7x}{3x-6}$$

The break points of the values of this rational function are given by setting $7x = 0$ and $3x - 6 = 0$. Solving these two equations is easy and this gives us the two break-points $x = 0$ and $x = 2$. Writing these break-points in increasing order we find: $0, 2$. The SDT of this rational function is then

	$(x-0)$	$(x-2)$	Sign of $r(x)$
$(-\infty, 0)$	$-$	$-$	$+$
$(0, 2)$	$+$	$-$	$-$
$(2, \infty)$	$+$	$+$	$+$

The solution can be read off easily using the last column, by looking at the intervals corresponding to the '$-$' signs. This gives only *one* interval $(0, 2)$, and nothing else.

Example 209. Solve the inequality

$$\frac{x^2 - 9}{(x^2-4)^2} \le 0$$

Solution The 4 break-points are given in increasing order by $x = -3, -2, 2, 3$. The corresponding factors are then $(x+3)$, $(x+2)^2$, and $(x-2)^2$, $(x-3)$. (Don't forget the squares, because the denominator is squared !). Okay, now its SDT is then given by

	$(x+3)$	$(x+2)^2$	$(x-2)^2$	$(x-3)$	Sign of $r(x)$
$(-\infty, -3)$	$-$	$+$	$+$	$-$	$+$
$(-3, -2)$	$+$	$+$	$+$	$-$	$-$
$(-2, 2)$	$+$	$+$	$+$	$-$	$-$
$(2, 3)$	$+$	$+$	$+$	$-$	$-$
$(3, \infty)$	$+$	$+$	$+$	$+$	$+$

In this case, the solution set is the closed interval $[-3, 3]$ **without** the points $x = \pm 2$ (why?). This answer could have been arrived at more simply by noticing that, since the denominator is always positive (or zero) the negative sign on the right (or zero) can only occur if the numerator is negative (or zero), and the denominator is non-zero. This means that $x^2 - 9 \le 0$ and $x^2 - 4 \ne 0$ which together imply that $|x| \le 3$, and $x \ne \pm 2$ as before.

You could include the points $x = \pm 2$ in the solution if you allow *extended real numbers* as solutions. But, strictly speaking, we don't allow them as solutions.

SNAPSHOTS

Example 210. Find the solution of the inequality

$$\frac{4-x^2}{x-2} \le 0.$$

Solution The solution of the inequality

$$\frac{4-x^2}{x-2} \le 0$$

is given by the interval $[-2, \infty)$, excluding the point $x = 2$. This is because the given rational function is essentially a polynomial "in disguise". Factoring the difference of squares and simplifying what's left under the assumption that $x \neq 2$, we'll find the expression $-(x+2) \le 0$ from which the solution $x \ge -2$ follows. Note that we must include the break-point $x = -2$ to the solution set because we have a '$\le$' sign. On the other hand, we can't ever divide by '0' right? So, we have to forget about $x = 2$.

Example 211. Find the solution of the inequality

$$\frac{1}{x^2-4} \ge 0.$$

Solution The solution of the inequality

$$\frac{1}{x^2-4} \ge 0$$

is given by the set of points x with $|x| > 2$ (think about this). Notice that this rational function is never equal to zero, but that's OK. The points $x = \pm 2$ are not allowed since we are dividing by '0' if we decide to use them. No go! Its solution set is the set of points x where $|x| > 2$ and its SDT is

	$(x+2)$	$(x-2)$	Sign of $r(x)$
$(-\infty, -2)$	$-$	$-$	$+$
$(-2, 2)$	$+$	$-$	$-$
$(2, \infty)$	$+$	$+$	$+$

In some cases, another function may be multiplying a rational function. When this happens, it is still possible to use the methods in this section to solve an inequality associated with that function. Here's an example.

Example 212. Solve the inequality

$$\frac{x|\sin(x)|}{1+x} \ge 0.$$

Solution Now here, $|\sin(x)| \ge 0$ for **any** x so all we should think about is "what's left"? Well, what's left is the rational function $r(x) = x/(1+x)$. Its 2 break-points are $x = -1, 0$ and the corresponding factors are now $(x+1)$, x. Its SDT is then given by

	$(x+1)$	x	Sign of $r(x)$
$(-\infty, -1)$	$-$	$-$	$+$
$(-1, 0)$	$+$	$-$	$-$
$(0, \infty)$	$+$	$+$	$+$

and the solution set is the union of the intervals $(-\infty, -1)$ and $[0, \infty)$ with the break-point $x = 0$ included, but without $x = -1$ (division by zero, remember?), along with all the zeros of the sin function (namely $x = \pm\pi, \pm 2\pi, \ldots$). These extra points are already included in the union of the two intervals given above, so there's nothing more to do.

Exercise Set 27.

1. Find all the break-points of the following rational functions.

a) $\dfrac{1+x^2}{x-2}$ b) $\dfrac{3t-2}{t^3-1}$ c) $\dfrac{x+2}{1}-\dfrac{1}{x-1}$

d) $\dfrac{x^3+1}{x^3-1}$ e) $\dfrac{x^2-2x+1}{x^2-1}$ f) $\dfrac{4-x^2}{1-x^2}$

2. Find the Sign Decomposition Table (SDT) of each one of the following functions from above.

a) $\dfrac{1+t^2}{t-2}$ b) $\dfrac{3t-2}{t^3-1}$ c) $\dfrac{t+2}{1}-\dfrac{1}{t-1}$

d) $\dfrac{t^3+1}{t^3-1}$ e) $\dfrac{t^2-2t+1}{t^2-1}$ f) $\dfrac{4-t^2}{1-t^2}$

3. Use the results of the previous exercises to solve the following inequalities involving rational functions.

a) $\dfrac{1+t^2}{t-2}\le 0$ b) $\dfrac{3t-2}{t^3-1}\ge 0$ c) $\dfrac{t+2}{1}-\dfrac{1}{t-1}>0$

d) $\dfrac{t^3+1}{t^3-1}<0$ e) $\dfrac{t^2-2t+1}{t^2-1}\ge 0$ f) $\dfrac{4-t^2}{1-t^2}<0$

4. Find the break-points and SDT of the given rational functions, and solve the inequalities.

a) $\dfrac{x^2-16}{x-4}>0$ b) $3x+\dfrac{5}{x}<0$ c) $\dfrac{x^2-5}{x-5}>0$

d) $\dfrac{3x^2+4x+5}{x^2+8x-20}<0$ e) $\dfrac{x^3+x^2}{x^4-1}\ge 0$ f) $\dfrac{x^2\,|\cos x|}{x-2}<0$

5. Find the break-points and SDT of the given rational functions, and solve the inequalities.

a) $\dfrac{x^2-1}{x^2+4}>0$ b) $\dfrac{x^2+1}{x^4+1}>0$ c) $\dfrac{x^2-9}{x^2+x+1}<0$

d) $\dfrac{2x-3}{(x-4)^2}<0$ e) $\dfrac{x+1}{(x+2)(x+3)}>0$ f) $\dfrac{x^3-1}{x+1}<0$

Hint: 5b) $x^4+1=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$ is the product of two irreducible quadratic polynomials, or Type II factors. Don't worry, this isn't obvious!

Suggested Homework Set 19. *Work out problems* $4a, 4f, 5c, 5e, 5f$.

Group Project

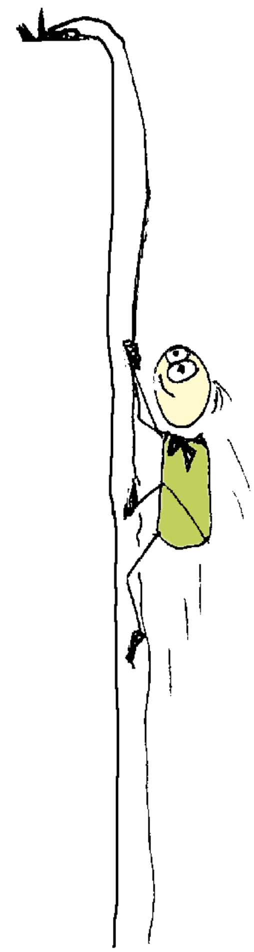
Get together with some of your classmates and try to extend the ideas in this section to a more general mathematical setting. By this we mean, for example, try to find a general method for making a SDT for a function of the form $f(x)r(x)$ where f is not necessarily either a polynomial or a rational function while $r(x)$ *is* a rational function. For instance, think of a way of making the SDT of the function

$$\frac{x}{x^2-1}\ \sin x$$

or more generally, a function whose values look like

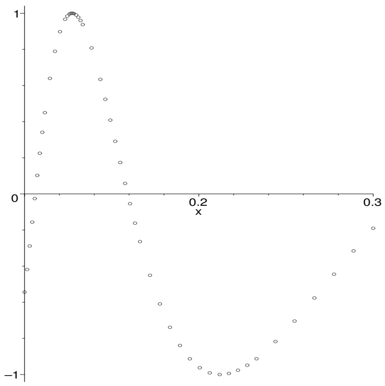
(rational function) (any single trig. function here) .

NOTES:



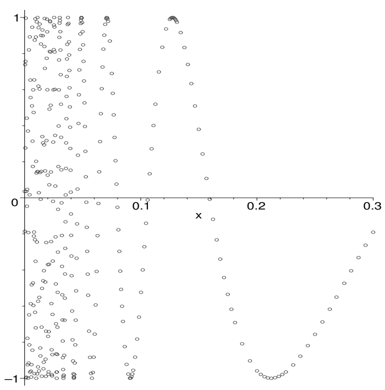
5.3 Graphing Techniques

The first thing you should do in this business of sketching the graph of a function is to just **plot a few points** to get a feel for what's happening to the graph, what you think it looks like. It's not going to be the best you can do, and there may be some missing data but still you'll get the basic idea. See Figure 86 for such an instance for the function f defined by $f(x) = sin(\frac{1}{x})$, where x is in radians. But **watch out**, this isn't enough as you'll see below!



A few points of $y = \sin \frac{1}{x}$

Figure 86.



More points of $y = \sin \frac{1}{x}$

Figure 87.

The next thing you may want to do is to find out where the function has a **zero**, that is, a point where the function is actually equal to zero. Another name for this is a **root**. Now, you've seen **Newton's Method** in action in the last chapter, and this is really all you need in case you get stuck and the roots are not obvious. For instance, in the case of Figure 86, there seems to be only a few roots in the interval $[0, 0.3]$, right? But actually there is an *infinite number* of them (see Figure 87); you just can't see them because they are really close to $x = 0$. This is one of the reasons you shouldn't rely on *just* plotting a few points. What this means is you'll need *additional* tools for analyzing the data you've plotted and this is where the derivative comes in handy.

One of the objects we're looking for has a name which describes it nicely. At these special points of the domain (or at its endpoints), a typical graph will have a 'peak' or a 'valley'. These last two words are only another way of describing the notions of a *maximum* and a *minimum*. These special points are called 'critical' because something really important happens there ... the function may 'blow up to $\pm\infty$' or the graph may reach its 'peak' or hit 'rockbottom'.

A point c in the domain (or one of its end-points) of a differentiable function f is called a **critical point** if either

1. $f'(c) = 0$ or
2. $f'(c)$ does not exist,

either as a finite number, or as a (two-sided) derivative. It follows from this definition that **if c is an end-point of the domain of f, then c is a critical point of** f (since there cannot exist a two-sided derivative there!). At a critical point $x = c$ defined by $f'(c) = 0$ the graph of the function looks like a 'rest area' for the graph of the function because its tangent line is horizontal at this special point.

Example 213. Show that the functions $f(x) = 1/x$, $g(x) = x^{\frac{2}{3}}$ have a critical point at $x = 0$.

Solution Here, $x = 0$ is an end-point of the domain of f (being $(0, \infty)$). So, by definition, f has a critical point there (see Figure 88; guess which graph corresponds to this function). On the other hand, $x = 0$ belongs to the natural domain of g but now $g'(0)$ is undefined, since $g'_+(0) = +\infty$. Thus, once again, $x = 0$ is a critical point but for a different reason.

Example 214. Show that $f(x) = x^2$ has a critical point at $x = 0$.

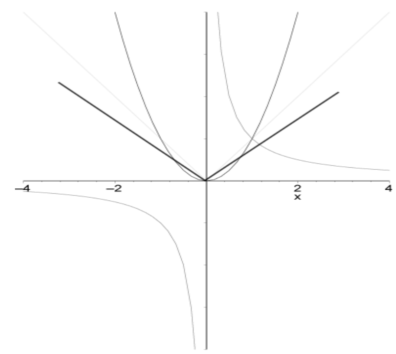
Solution In this case $f'(x) = 2x$ and this derivative is equal to zero when $x = 0$, i.e., $f'(0) = 0$. So, by definition, f has a critical point there (see Figure 88).

Example 215. Show that $f(x) = |x|$ has a critical point at $x = 0$.

Solution In this case we have to remove the absolute value. Now, by definition of the absolute value, if $x \geq 0$, then $f(x) = x$. On the other hand, if $x \leq 0$, then $f(x) = -x$. It follows that $f'_+(0) = 1$ while $f'_-(0) = -1$. So, f is not differentiable at $x = 0$ since the right-and left-derivatives are not equal there. So, again by definition, f has a critical point there (see Figure 88, the V-shaped curve).

The next tool that we can use has to do with how inclined (up or down) a graph can be. For this study of its inclination we use the ordinary derivative of the function and see when it's positive or negative. This motivates the next definition.

Zero is a Critical Point



Three curves $y = |x|, y = x^2, y = 1/x$

Figure 88.

A function f defined on an interval, I, is said to be **increasing** (resp. **decreasing**) if given any pair of points x_1, x_2 in I with $x_1 < x_2$, then we have $f(x_1) < f(x_2)$, (resp. $f(x_1) > f(x_2)$).

When does this happen? Let f be differentiable over (a, b).

(i) If $f'(x) > 0$ for all x in (a, b) then f is increasing on $[a, b]$

(ii) If $f'(x) < 0$ for all x in (a, b) then f is decreasing on $[a, b]$

In each of these cases, the slope of the tangent line is positive (resp. negative) at any point on the graph of $y = f(x)$.

Example 216. The function $f(x) = x$ is increasing on I (where I can be **any** interval that you wish to choose as its domain), because, obviously, the statement $x < y$ is equivalent to $f(x) < f(y)$, for x, y in I, on account of the definition of f. Alternately, since f is differentiable and $f'(x) = 1 > 0$, for any x the result in the table above shows that f is increasing.

Example 217. The function $g(s) = 3s + 9$, with domain $\boldsymbol{R}$, the set of all real numbers, is increasing there (use the definition) whereas the function $g(s) = -3s+9$ is decreasing on $\boldsymbol{R}$ (again, by definition).

Example 218. The function $f(x) = 4$ is *neither increasing nor decreasing* on a given domain, it is **constant** there. Of course, **every** constant function is neither increasing nor decreasing.

Example 219. The function $g(x) = x + \sin x$ is increasing on the *closed* interval $I = [3\pi/2, 5\pi/2]$ because its derivative, $g'(x) = 1 + \cos x > 0$ for these values of the independent variable x in I, by trigonometry (see Figure 89).

Example 220. Similarly, the function $h(z) = cos(z)$ is decreasing on $I = (2\pi, 3\pi)$ because its derivative $h'(z) = -\sin z < 0$ on this interval.

SNAPSHOTS

Example 221. The function $f(x) = x^4$ is increasing on $(0, \infty)$ and decreasing

on $(-\infty, 0)$. Just check its derivative.

Example 222. The function $g(z) = z^3$ is increasing on all of $\boldsymbol{R}$. If you sketch the graph of a typical polynomial (with real coefficients) you will notice that the graph is generally ondulatory (i.e., wave-like). This phenomenon is a particular case of the general property that the **graph of a non-constant polynomial consists of intervals on each of which the function is either increasing or decreasing**.

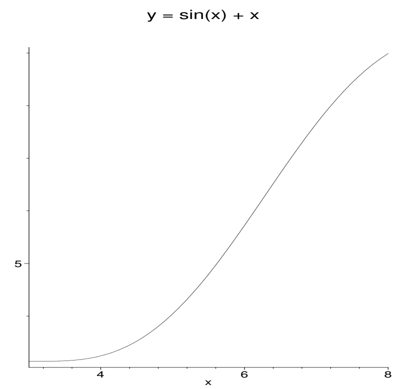


Figure 89.

Example 223. The polynomial $p(x) = (x-2)^2$ is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

Example 224. On the other hand, the polynomial function $q(x) = (x-1)(x-3)(x-4)$ with domain $\boldsymbol{R}$ is increasing on $(4, \infty)$ and its graph has one "bump" (or peak) and one "valley" (think of it as a point on the 'seafloor') between $x = 1$ and $x = 4$, (see Figure 90).

Here's a few more sophisticated examples:

Example 225. Determine the intervals on which the polynomial p defined by $p(x) = x(x^4 - 5)$ is increasing and decreasing.

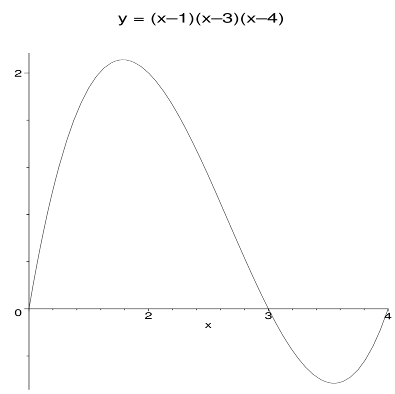


Figure 90.

Solution We use its Sign Decomposition Table, SDT, as defined in the previous sections. In this case, since every polynomial is differentiable, we have $p'(x) = D(x^5 - 5x) = 5x^4 - 5 = 5(x^4 - 1)$. This means that, aside from the constant factor of 5, the SDT for this p is identical to the SDT in Table 5.1! So, the SDT for $p'(x)$ looks like

	$x+1$	$x-1$	x^2+1	sign of $p'(x)$
$(-\infty, -1)$	$-$	$-$	$+$	$+$
$(-1, 1)$	$+$	$-$	$+$	$-$
$(1, \infty)$	$+$	$+$	$+$	$+$

It follows that $p'(x) > 0$ if x is in either $(-\infty, -1)$ or $(1, \infty)$, that is if $|x| > 1$. Similarly, p is decreasing when $p'(x) < 0$, which in this case means that x is in the interval $(-1, 1)$, or equivalently, $|x| < 1$.

Example 226. Determine the intervals on which the function f defined on its *natural domain* (see Chapter 1) by

$$f(x) = x + \frac{5}{6}\ln|x-3| - \frac{5}{6}\ln|x+3|$$

is increasing and decreasing.

Solution Now, this is pretty weird looking, right? Especially those absolute values! No, problem. Just remember that whenever $\square \neq 0$, (see Chapter 4),

$$\boxed{D \ln|\square| = \frac{1}{\square}\, D\square},$$

so that, ultimately, we really don't have to worry about these absolute values.

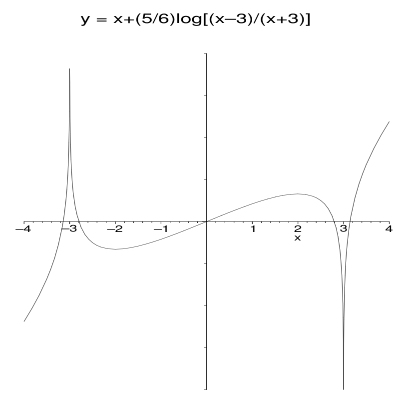
Let's find its derivative $f'(x)$ and then solve the inequalities. Now, the derivative of f is easily found using the methods of Chapters 3 and 4. We get

$$\begin{aligned} f'(x) &= 1 + \frac{5}{6}\frac{1}{x-3} - \frac{5}{6}\frac{1}{x+3} \\ &= \frac{(x^2-4)}{(x-3)(x+3)} \\ &= \frac{x^2-4}{x^2-9} \end{aligned}$$

which is a rational function ! Okay, so we can use the SDT method of the previous section to solve the inequalities $f'(x) > 0$ and $f'(x) < 0$. The break-points of f' are given by $-3, -2, 2, 3$ in increasing order. Its SDT is found to be

	$(x+3)$	$(x+2)$	$(x-2)$	$(x-3)$	Sign of $f'(x)$
$(-\infty, -3)$	$-$	$-$	$-$	$-$	$+$
$(-3, -2)$	$+$	$-$	$-$	$-$	$-$
$(-2, 2)$	$+$	$+$	$-$	$-$	$+$
$(2, 3)$	$+$	$+$	$+$	$-$	$-$
$(3, \infty)$	$+$	$+$	$+$	$+$	$+$

Now we can just read off the intervals where f is increasing and decreasing. We see that f is increasing when x is in either $(-\infty, -3)$, $(-2, 2)$, or $(3, \infty)$. It is decreasing when x is in either $(-3, -2)$, or $(2, 3)$, (see Figure 91).



$y = x + \frac{5}{6}\ln|x-3| - \frac{5}{6}\ln|x+3|$

Figure 91.

OK, now we know when a function is increasing or decreasing, what its zeros are and we know how to find its critical points. What else do these critical points tell us?

A function f is said to have a **local maximum at a point $x = a$** if there is a neighborhood of $x = a$ in which $f(x) \leq f(a)$, or there is a 'peak' or a 'big bump' in the graph of f at $x = a$. In this case, the value of $f(a)$ is called the **local maximum value**. It is said to have a **global maximum at $x = a$** if $f(x) \leq f(a)$ for every x in the domain of f. This means that the 'tallest peak' or the 'Everest' or the 'biggest bump' occurs when $x = a$. In this case, the value of $f(a)$ is called the **global maximum value of f**. We can define similar concepts for the notion of a *minimum* too.

So, a function f is said to have a **local minimum at a point $x = a$** if there is a neighborhood of $x = a$ in which $f(x) \geq f(a)$, or there is a 'valley' in the graph of f at $x = a$, or the graph 'hits rockbottom'. In this case, the value of $f(a)$ is called the **local minimum value**. It is said to have a **global minimum at $x = a$** if $f(x) \geq f(a)$ for every x in the domain of f. This means that the 'deepest valley' or the 'seafloor' or the 'biggest of all drops' occurs when $x = a$. In this case, the value of $f(a)$ is called the **global minimum value of f**. As you can gather, these are nice geometrical notions which are intuitively true.

First Derivative Test: Let $f'(c) = 0$ with f a differentiable function.

(i) If $f'(x) > 0$ in a left-neighbourhood of c **and** $f'(x) < 0$ in a right-neighbourhood of c then $f(c)$ is a **local maximum value** of f.

(ii) If $f'(x) < 0$ in a left-neighbourhood of c **and** $f'(x) > 0$ in a right-neighbourhood of c then $f(c)$ is a **local minimum value** of f.

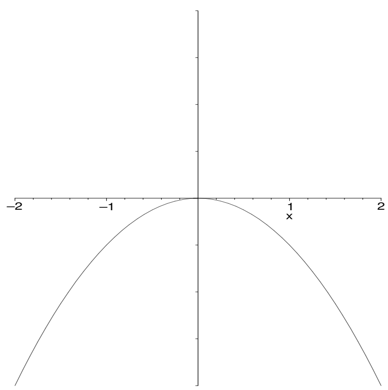
Example 227. Refer to Example 226. Show that $x = -2$ gives rise to a local

minimum value of f, while $x = +2$ gives rise to a local maximum value of f. Find the global maximum value and global minimum value of f. *Solution* Look at Example 226 above and Figure 91. From the SDT in that example, we gather that $f'(x) < 0$ just to the left of $x = -2$ and $f'(x) > 0$ just to the right of $x = -2$. So, according to the First Derivative Test, the value $f(-2)$ is a *local minimum value* of f, or we say that **a local minimum occurs at** $x = -2$**.** The value of this minimum is $f(-2) \approx -0.6588$.

A point is called an **extreme point** of f if it is either a local maximum or a local minimum for the given function, f

On the other hand, from the same SDT we see that $f'(x) > 0$ just to the left of $x = 2$ and $f'(x) < 0$ just to the right of $x = 2$. So, according to the First Derivative Test again, the number, $f(2)$, is a *local maximum value* of f, or we say that *a local maximum occurs at* $x = 2$. The value of this maximum is $f(2) \approx +0.6588$.

Notice from the graph, Figure 91, that there is NO global maximum value of f, since the point where it should be, namely, $x = -3$, is NOT a point in the domain of f. Similarly, there is no global minimum value of f, (because $x = 3$ in not in the domain of f).



The graph of $y = -x^2$

Figure 92.

QUICKIES

If you look at Figure 90 you'll see that the function f defined there has a local maximum somewhere near $x = 2$ and a local minimum in between $x = 3$ and $x = 4$. If the domain of f is defined to be the interval $[0, 4]$ only, then these two special points would represent **global extrema**, just a fancy word for saying that there is either a global maximum or global minimum. On the other hand, if you refer to Figure 88, then you see that the function with $f(x) = x^2$ has a local (actually global) minimum at $x = 0$ and so does the function with $f(x) = |x|$. Finally, you would probably believe that, in the case of Figure 86, there is an *infinite* number of local and global extrema.

Now, so far we have seen that a study of the points where a given function is increasing and decreasing, the position of its critical points, its zeros, and a few points on its graph tells us a lot about the curve it defines. But still, the picture is not complete. We need to know how the graph 'bends' this way and that, or 'how it curves around'. For this we need one more derivative, the **second derivative** of f which, we assume, exists. Then we need to define a new notion called **concavity**, which some authors prefer to call *convexity.* This has something to do with whether a graph is 'concave up' or 'concave down'. The graph of a twice differentiable function is said to be **concave up** on an interval I, or 'U - shaped' if $f''(x) > 0$ for every x in I. Similarly, the graph of a twice differentiable function is said to be **concave down** on an interval I, or 'hill-shaped' if $f''(x) < 0$ for every x in I.

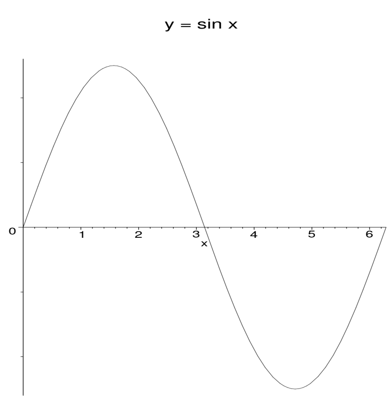


Figure 93.

Example 228. A simple example of these concepts is seen in Figure 88 for the function f defined by $f(x) = x^2$. It's easy to see that $f'(x) = 2x$, so $f''(x) = 2 > 0$ which, by definition means that the graph of this f is *concave up.* As you can see, this means that the graph is sort of U-shaped (or bowl-shaped).

On the other hand, if f is defined by $f(x) = -x^2$. then $f''(x) = -2 < 0$ which, by definition, means that the graph of this f is *concave down.* This means that the graph is similar to a smooth mountain or hill, see Figure 92.

Example 229. Show that the sine function, sin, is concave up on the interval

$(\pi, 2\pi)$ and concave down on $(0, \pi)$.

Solution Let $f(x) = \sin x$. Then $f'(x) = \cos x$, and $f''(x) = -\sin x > 0$, since when $\pi < x < 2\pi$, we know from trigonometry that $\sin x < 0$. So, by definition, the graph is concave up. On the other hand, if $0 < x < \pi$, then $\sin x > 0$ which means that $f''(x) = -\sin x < 0$ and the graph is concave down, (see Figure 93).

We mimic what we did earlier in the case of the 'first derivative' and so we define a new kind of point, a point around which the function changes its **concavity**. Basically, this means that the graph changes from concave up (or down) to concave down (or up) around that point. Such a point is called a *point of inflection* and we define it formally below.

A point P with coordinates $(c, f(c))$ on the graph of a *twice differentiable* function f is called a **point of inflection** or *inflection point* if either:

(i) $f''(x) > 0$ in a left-neighborhood of c **and** $f''(x) < 0$ in a right-neighborhood of c, OR,

(ii) $f''(x) < 0$ in a left-neighborhood of c **and** $f''(x) > 0$ in a right-neighborhood of c

Example 230. Show that the polynomial function of Example 225 has an inflection point at $x = 0$.

Solution We need to find the SDT of its *second derivative*, right? We have only found the SDT of its (first) derivative so we have a little more work to do. We know that $p(x) = x^5 - 5x$ and $p'(x) = 5x^4 - 5$. So, $p''(x) = 20x^3$. Well, this is really easy to handle since p'' has only one root, namely, $x = 0$. So, we don't really have to display this SDT since we can just read off any information we need from the expression for p''. We see from the definitions that when $x < 0$ then $p''(x) = 20x^3 < 0$ and so the graph is concave down, while for $x > 0$, $p''(x) = 20x^3 > 0$ and so the graph is concave up. So, by definition, $x = 0$ is an inflection point (since the graph changes its concavity around that point).

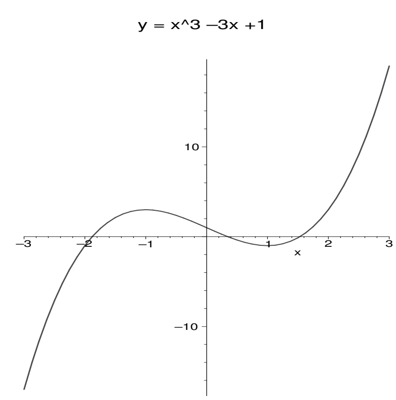


Figure 94.

In practice, when you're looking for points of inflection you look for the roots of $f''(x) = 0$. Check these first.

Example 231. Show that the polynomial function of Figure 90 above has an inflection point at $x = \frac{8}{3}$.

Solution We do a few calculations. First, $q(x) = (x - 1)(x - 3)(x - 4)$ means that $q'(x) = (x - 3)(x - 4) + (x - 1)(x - 4) + (x - 1)(x - 3)$ and that $q''(x) = 6x - 16$. All the candidates for 'point of inflection' solve the equation $q''(x) = 0$. This means that $6x - 16 = 0$ or $x = 8/3$. This is the only candidate! Note that to the just to the left of $x = 8/3$, we have $q''(x) = 6x - 16 < 0$ while just to the right of $x = 8/3$ we have $q''(x) = 6x - 16 > 0$. By definition, this means that $x = 8/3$ is a point of inflection. You can see from the graph of q, Figure 90, that there is indeed a change in the concavity of q around $x = 8/3$, although it appears to be closer to 3, because of plotting errors.

LOOK OUT! The statement "$f''(c) = 0$ implies c is a point of inflection" is generally FALSE. Look at $f(x) = x^4$ which has the property that $f''(c) = 0$ when $c = 0$ only, but yet there is no point of inflection at $x = 0$ (because its graph is *always* concave up), so there is no change in concavity. Its graph is similar to Fig. 88, for $f(x) = x^2$ but it is flatter.

Some General Terminology

The graph of a twice differentiable function f is said to be **concave up (down) on an interval** if it is concave up (down) at every point of the interval, *i.e.* $f''(x) \geq 0$ (or $f''(x) \leq 0$) for all x in the interval.

Now, we're lucky that there is *another* test which can handle the question of whether or not a function has a maximum or a minimum value at a point $x = c$. It is easier to apply than the First Derivative Test but the disadvantage is that you have to compute one more derivative ... This is how it sounds: Let c be a critical point of a twice differentiable function f, then there is the

The Second Derivative Test: Let $f'(c) = 0$ (c is a critical point)

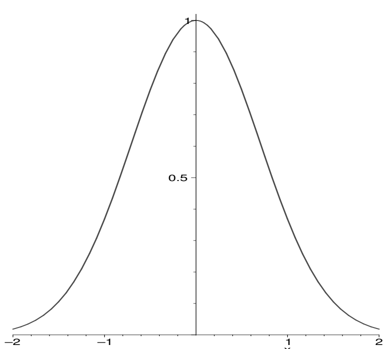
(i) If $f''(c) < 0$ then c is a local maximum of f.

(ii) If $f''(c) > 0$ then c is a local minimum of f.

(iii) If $f''(c) = 0$ more information is needed. We don't know.

Example 232. Find all the critical points of the function defined by $f(x) = x^3 - 3x + 1$ on the interval $[-3, 3]$ and determine the **nature of these critical points** (i.e., local maximum/local minimum). Find its points of inflection, if any.

A Bell curve



The curve $y = e^{-x^2}$

Figure 95.

Solution We know that since f is a polynomial, it is differentiable and $f'(x) = 3x^2 - 3$. If c is a critical point, then, by definition, $f'(c) = 0$ which is equivalent to saying that $3c^2 - 3 = 0$ which is equivalent to $c = \pm 1$ and there cannot be any other critical points (as the derivative of f always exists and is finite).

Now, we use the Second Derivative Test to determine their nature. Here $f''(x) = 6x$. Now, when $x = +1$, $f''(+1) = 6 > 0$ which means that there must be a local minimum at $x = 1$. On the other hand, when $x = -1$, then $f''(-1) = -6 < 0$ which means that there must be a local maximum at $x = -1$. The values of f at these extrema are $f(1) = -1$, (the local minimum value) and $f(-1) = 1$, (the local maximum value). See Figure 94 for its graph.

The only *candidate* for a point of inflection is when $f''(c) = 6c = 0$, that is, when $c = 0$, right? Now, just to the left of 0, (i.e, $x < 0$) we have $f''(x) = 6x < 0$ while just to the right of $x = 0$ (i.e, $x > 0$) we have $f''(x) = 6x > 0$. So there is a change in concavity (from concave down to concave up) as you move from $x < 0$ to $x > 0$ through the point $x = 0$. So, by definition, this is a point of inflection. It is the only one since there are no other roots of $f''(x) = 0$.

NOTE: Notice that the global maximum and the global minimum values of this polynomial on the given interval, $[-3, 3]$ occur at the endpoints, and not, as you might think, at the critical points. This is why we call these things 'local', meaning that this property only holds just around the point but not everywhere in the interval.

Example 233. Find all the critical points of the function defined by $f(x) = e^{-x^2}$ on the interval $(-\infty, \infty)$ and determine the nature of these critical points (i.e., local maximum/local minimum). Determine any points of inflection, if any.

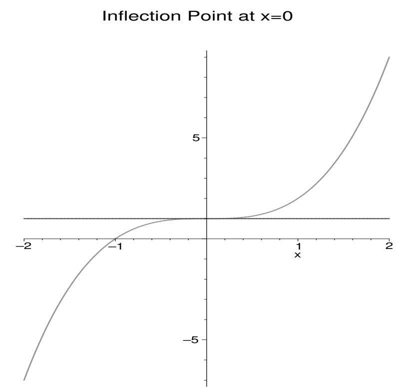
Solution We see that f is a nice exponential, and by the Chain Rule we see that it is differentiable as well and $f'(x) = -2xe^{-x^2}$. If c is a critical point, then, by definition, $f'(x) = 0$ and this is equivalent to saying that $-2xe^{-x^2} = 0$. This, in turn, is equivalent to $x = 0$ and there cannot be any other critical points (because an exponential function is never zero).

We apply the Second Derivative Test to this f. Since $f''(x) = (4x^2 - 2) \cdot e^{-x^2}$, at the critical point $x = 0$ we have $f''(0) = -2 < 0$ which means that $x = 0$ is a local maximum of f.

For (candidates for) points of inflection we set $f''(x) = 0$. This occurs only when $(4x^2 - 2) \cdot e^{-x^2} = 0$, and this is equivalent to saying that $4x^2 - 2 = 0$, that is,

$$x = \pm\frac{1}{\sqrt{2}} \approx \pm 0.7071.$$

We observe that around these points there is a change of concavity since, for example, $f''(x) < 0$ just to the left of $x = 1/\sqrt{2}$ while $f''(x) > 0$ just to the right of $x = 1/\sqrt{2}$. The same argument works for the other candidate, $x = -1/\sqrt{2}$. See Figure 95 for its graph.



The curve $y = x^3 + 1$

Figure 96.

But, geometrically, what is a point of inflection?

Look at the graph of $y = x^3 + 1$, Figure 96. The tangent line at the inflection point, $x = 0$ here, *divides the curve into two parts*, one below the line and one above the line. More generally the tangent line at an inflection point divides a small portion of the curve into two parts, one part of the curve is on one side of the tangent line, while the other part of the curve is on the other side.

Horizontal and Vertical Asymptotes:

Another calculation which is helpful in drawing the graph of a given function involves the finding of *special straight lines called asymptotes.* We already referred to these in Chapter 2, in our study of limits, so it will be an easy matter for us to review what we saw then and apply it here.

The horizontal lines $y = L$, $y = M$ on the xy-plane are called **horizontal asymptotes for the graph of f** if

$$\lim_{x \to +\infty} f(x) = L$$

or

$$\lim_{x \to -\infty} f(x) = M,$$

or both these limits exist (and may or may not be equal).

Remark: In other words, a horizontal asymptote is a horizontal line, $y = L$, with the property that the graph of f gets closer and closer to it, and may even cross this line, looking more and more like this line at infinity. Remember that the value of L is a number which is equal to a limiting value of f at either $\pm\infty$.

The vertical line $x = a$ is called a **vertical asymptote** for the graph of f provided either

$$\lim_{x\to a} f(x) = \infty, \quad (\text{or } -\infty)$$

or

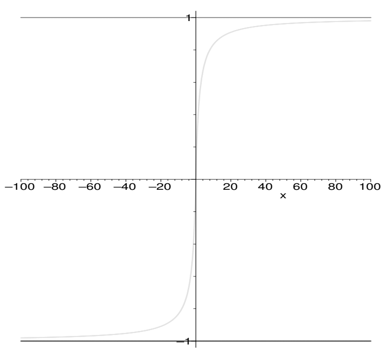
$$\lim_{x\to a^+} f(x) = \infty, \quad (\text{or } -\infty)$$

or

$$\lim_{x\to a^-} f(x) = \infty, \quad (\text{or } -\infty)$$

Example 234. We refer to Figure 97. In this graph we see that the lines $y = \pm 1$ are both horizontal asymptotes for the graph of the function f defined on its natural domain, by

Some horizontal asymptotes



Two asymptotes in $y = \frac{x}{|x|+2}$

Figure 97.

$$f(x) = \frac{x}{|x|+2}.$$

Why? We simply calculate the required limits ... Since $|x| = x$ when $x > 0$ we see that,

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{x}{|x|+2}, \\
&= \lim_{x\to+\infty} \frac{x}{x+2} \\
&= \lim_{x\to+\infty} \frac{D(x)}{D(x+2)}, \quad \text{by L'Hospital's Rule,} \\
&= \lim_{x\to+\infty} \frac{1}{1} \\
&= 1.
\end{aligned}$$

So, $L = 1$ here. In the same way we can show that

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x}{|x|+2}, \\
&= \lim_{x\to-\infty} \frac{x}{-x+2}, \quad \text{since } |x| = -x \text{ when } x < 0, \\
&= \lim_{x\to+\infty} \frac{1}{-1} \\
&= -1,
\end{aligned}$$

once again, by L'Hospital's Rule. In this case, $M = -1$ and these two limits are different.

NOTE: These two limiting values, when they exist as finite numbers, may also be the same number. For example, the function $f(x) = x^{-1}$ has $L = M = 0$, (see Figure 88).

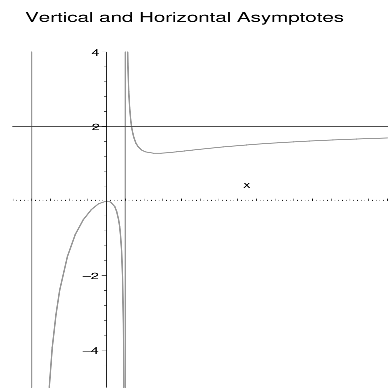
Example 235. Determine the vertical and horizontal asymptotes of the function f defined by

$$f(x) = \frac{2x^2}{x^2+3x-4}.$$

Solution The horizontal asymptotes are easily found using L'Hospital's Rule (because the values of f at infinity are 'indeterminate', of the form ∞/∞). In this case,

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{2x^2}{x^2+3x-4}, \\
&= \lim_{x\to+\infty} \frac{4x}{2x+3}, \quad \text{by L'Hospital's Rule,} \\
&= \lim_{x\to+\infty} \frac{4}{2}, \quad \text{once again, by L'Hospital's Rule ,} \\
&= 2.
\end{aligned}$$

So, $L = 2$ and the line $y = 2$ is a horizontal asymptote, (see Figure 98).



$y = \frac{2x^2}{x^2+3x-4}$

Figure 98.

In order to find a vertical asymptote, we simply have to look for points where the graph 'takes off' and the best place for this is the zeros of the denominator of our function. When we factor the denominator $x^2 + 3x - 4$ completely, we get $x^2 + 3x - 4 = (x + 4)(x - 1)$. So, its roots are $x = -4$ and $x = 1$. These are merely candidates for vertical asymptotes, right? Let's check the values of the limits as required by the definition. In case of doubt, always use one-sided limits, first.

$$\begin{aligned}
\lim_{x\to-4^+} f(x) &= \lim_{x\to-4^+} \frac{2x^2}{x^2+3x-4}, \\
&= \lim_{x\to-4^+} \frac{2x^2}{(x+4)(x-1)}, \\
&= -\infty,
\end{aligned}$$

since $x \to -4^+$ implies that $x > -4$, or $x + 4 > 0$, so $(x + 4)(x - 1) < 0$. A very similar calculation shows that

$$\begin{aligned}
\lim_{x\to-4^-} f(x) &= \lim_{x\to-4^-} \frac{2x^2}{x^2+3x-4}, \\
&= \lim_{x\to-4^-} \frac{2x^2}{(x+4)(x-1)}, \\
&= +\infty,
\end{aligned}$$

since $x \to -4^-$ implies that $x < -4$, or $x + 4 < 0$, so $(x + 4)(x - 1) > 0$. So, it follows by the definition that the vertical line $x = -4$ is a vertical asymptote. In the same way we can show that

$$\begin{aligned}
\lim_{x\to1^+} f(x) &= \lim_{x\to1^+} \frac{2x^2}{x^2+3x-4}, \\
&= \lim_{x\to1^+} \frac{2x^2}{(x+4)(x-1)}, \\
&= +\infty,
\end{aligned}$$

since $x \to 1^+$ implies that $x > 1$, or $x + 1 > 0$, so $(x + 4)(x - 1) > 0$. Furthermore,

The concept of **Left and right continuity** is defined like Continuity except that we take the limit from the left or the limit from the right in the definition of continuity (instead of a two-sided limit); see Chapter 2.

$$\begin{aligned}
\lim_{x\to1^-} f(x) &= \lim_{x\to1^-} \frac{2x^2}{x^2+3x-4}, \\
&= \lim_{x\to1^-} \frac{2x^2}{(x+4)(x-1)}, \\
&= -\infty,
\end{aligned}$$

since $x \to 1^-$ implies that $x < 1$, or $x + 1 < 0$, so $(x + 4)(x - 1) < 0$. Part of the graph of this function showing both vertical asymptotes and the one horizontal asymptote can be seen in Figure 98.

Monotonicity	Horizontal Asymptotes	Zeros: All intercepts
↖	↑	↗

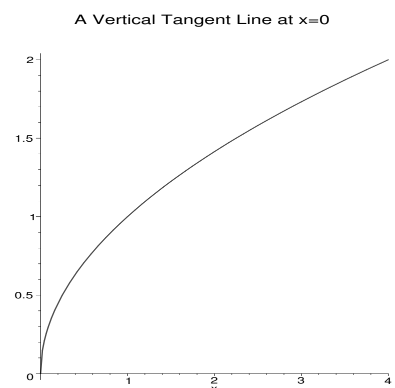
C M E HA VA PI ZZ A

↙	↙	↓	↘
Concavity	Extrema	Vertical Asymptotes	Points of Inflection

Table 5.4: The 'C Me Hava Pizza' Rule

> The graph of a function f is said to have a **vertical tangent line** at a point $P(a, f(a))$ if f is (left- or right-) continuous at $x = a$ and if either its left- or right- derivative is infinite there, that is, if
>
> $$\lim_{x \to a^{\pm}} |f'(x)| = \infty.$$

NOTE: This business of a 'vertical tangent line' is not the same as a vertical asymptote. Why? You see, a function cannot be continuous at a point where an asymptote occurs because, by definition, we are looking at the values of the function at infinity, and the function can't be continuous there. But, in the case of a vertical tangent line, the function *is* continuous (from the left or right) at the point on its graph where this line is tangent, so the graph actually touches that value, $f(a)$. So, even though it *looks* like a vertical asymptote, it certainly isn't! Let's look at an example.



$y = \sqrt{x}$

Figure 99.

Example 236. Show that $y = \sqrt{x}$ has a vertical tangent line at $x = 0$.

Solution Its graph appears in Figure 99. We calculate the derivative $y'(x)$ using the Power Rule and find that,

$$D(\sqrt{x}) = y'(x) = \frac{1}{2\sqrt{x}}.$$

For the tangent line to be vertical the derivative there must be 'infinite' and so this leaves only $x = 0$ as a candidate. We easily see that

$$\lim_{x \to 0^+} \left|\frac{1}{2\sqrt{x}}\right| = \lim_{x \to 0^+} \frac{1}{2\sqrt{x}} = +\infty,$$

so, by definition, this graph has a vertical tangent at $x = 0$, namely, the $y-$axis (or the line $x = 0$). Note that the (two-sided) limit cannot exist in this case because the square root function is only defined for $x \geq 0$. So, we had to take a one-sided limit from the right.

Phew! That's about it! But how are you going to remember all this stuff? Well, let's look at the following jingle or *mnemonic device*, that is, something to help you remember what you have to do. It goes like this: We call it the *C me Hava Pizza* Rule... See Table 5.4 for an explanation.

The boxed words in the 'C Me Hava Pizza' Rule describe the following instructions:

Let the graph of the function f be represented on the $xy-$plane, where, as usual, we assume that $y = f(x)$.

1. $\boxed{\text{Monotonicity}}$ $\Longleftrightarrow$ Look for intervals where the graph is **Increasing** and **Decreasing**,

2. $\boxed{\text{Horizontal Asymptotes}}$ $\Longleftrightarrow$ Look for all the **horizontal asymptotes**, if any.

3. $\boxed{\text{Zeros: All intercepts}}$ $\Longleftrightarrow$ Look for all the points where the **graph crosses the $x-$axis** (set $y = 0$ and solve), AND look for all the points where the **graph crosses the $y-$axis** (set $x = 0$ and solve).

4. $\boxed{\text{Concavity}}$ $\Longleftrightarrow$ Look for all the intervals where the graph is **concave up** or **concave down**.

5. $\boxed{\text{Extrema}}$ $\Longleftrightarrow$ Look for all the **critical points**, and classify them as local maximum, and local minimum.

6. $\boxed{\text{Vertical Asymptotes}}$ $\Longleftrightarrow$ Look for all the **vertical asymptotes**, if any.

7. $\boxed{\text{Points of Inflection}}$ $\Longleftrightarrow$ Look for all the **points of inflection** to see how the graph bends this way and that.

OK, good, all we have to do now is put this Rule into action. Let's try a few examples.

Example 237. Sketch the graph of $f(x) = \dfrac{4}{9 + x^2}$.

Solution These are the questions you should be thinking about ...

Questions to Ask:

1. What are the zeros of f and the $y-$intercepts? (*i.e.* When is $f(x) = 0$?)
2. What are the critical points of f, and what is their nature?
3. Where is f increasing/decreasing?
4. Where is the graph of f concave up/concave down?
5. What are the points of inflection?
6. Where are the asymptotes of f, if any, and identify them (horizontal asymptotes, vertical asymptotes)?
7. Take a deep breath, assimilate all this, and finally, sketch the graph of f.

Solution (continued)

1. $\boxed{\text{Zeros: All intercepts}}$ Well, we see that $y = 0$ is impossible since
$$f(x) = \frac{4}{9 + x^2} \neq 0,$$
for any x. It just can't be zero, so there are NO $x-$intercepts. So, $f(x) > 0$ and its graph lies in the first two quadrants (think trig. here). Setting $x = 0$ we get

the one and only $y-$intercept value, namely, $y = 4/9$. We note that the natural domain of f is $-\infty < x < \infty$, (*i.e.* the real line).

2. $\boxed{\textbf{E}\text{xtrema}}$ Next, let's look for the critical points of f ...

$$f'(x) = \frac{(9+x^2)\cdot 0 - 4(2x)}{(9+x^2)^2} = -\frac{8x}{(9+x^2)^2}.$$

Since the denominator is never 0, the derivative always exists, and so the critical points are given by solving the equation $f'(c) = 0$. So,

$$f'(c) = -\frac{8c}{(9+c^2)^2} = 0 \Rightarrow c = 0.$$

Therefore, $\boldsymbol{c = 0}$ **is the only critical point.** Let's classify this point as to whether a local maximum or local minimum occurs here. We know that $f'(0) = 0$ so we can apply the Second Derivative Test. Now,

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left\{\frac{-8x}{(9+x^2)^2}\right\} = \frac{(9+x^2)^2(-8) - (-8x)(4x(9+x^2))}{(9+x^2)^4} \\ &= \frac{(-8)(9+x^2)^2 + 32x^2(9+x^2)}{(9+x^2)^4} = \frac{(9+x^2)[-8(9+x^2)+32x^2]}{(9+x^2)^4} \\ &= \frac{24x^2-72}{(9+x^2)^3} = 24\frac{x^2-3}{(9+x^2)^3}. \end{aligned}$$

So,

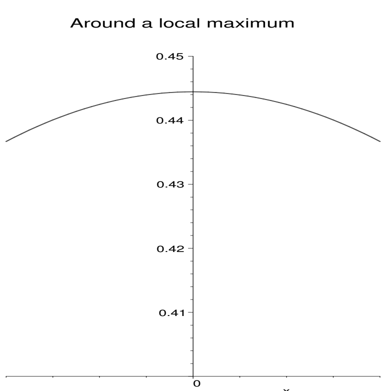
$$f''(0) = \frac{-72}{9^3} = -\frac{8}{81} < 0,$$

which means that there is a **local maximum** at $x = 0$ and so the *graph bends down and away from its local maximum value* there. Since there are no other critical points it follows that there is NO local minimum value. So, $\boldsymbol{f(0) = \frac{4}{9} \approx 0.444}$ **is a local maximum value of $\boldsymbol{f}$.**

At this point the only information we have about this graph is that it looks like Figure 100. Don't worry, this is already quite a lot.

3. $\boxed{\textbf{M}\text{onotonicity}}$ To find where f is increasing (decreasing) we look for those x such that $f'(x) > 0$ ($f'(x) < 0$), respectively. Note that we have already calculated the derivative in item (2). Now,

$$\begin{aligned} f'(x) &= -\frac{8x}{(9+x^2)^2} > 0 \quad \text{only when } x < 0 \text{ and,} \\ f'(x) &< 0 \text{ only when } x > 0. \end{aligned}$$



A first guess at $y = \frac{4}{9+x^2}$

Figure 100.

Therefore, $\boldsymbol{f}$ **is increasing on** $\boldsymbol{(-\infty, 0)}$ and $\boldsymbol{f}$ **is decreasing on** $\boldsymbol{(0, \infty)}$. This says something about the values of the function, right? In other words, its values, $f(x)$, are getting 'smaller' as $x \to -\infty$ and the same is true as $x \to +\infty$. But still, $f(x) > 0$, so you think "they must be getting closer and closer to the $x-$axis".

4. $\boxed{\textbf{C}\text{oncavity}}$ We know that f is concave up if $f''(x) > 0$, f is concave down if $f''(x) < 0$. But

$$f''(x) > 0 \quad \text{only when} \quad 24\frac{x^2-3}{(9+x^2)^3} > 0 \;\; i.e. \text{ when } \;\; x^2 - 3 > 0,$$

and

$$f''(x) < 0 \quad \text{only when} \quad 24\frac{x^2-3}{(9+x^2)^3} < 0 \;\; i.e. \text{ when } \;\; x^2 - 3 < 0.$$

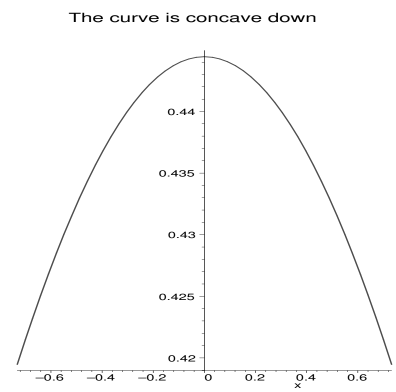
Now, $x^2 - 3 > 0$ when $x^2 > 3$, which is equivalent to $|x| > \sqrt{3}$. Therefore, $x^2 - 3 < 0$ whenever $|x| < \sqrt{3}$. Hence $\boldsymbol{f}$ **is concave up** on the interval $|x| > \sqrt{3}$, *i.e.* on each of the intervals

$$(-\infty, -\sqrt{3}) \quad \text{and} \quad (\sqrt{3}, \infty),$$

and **concave down** on the interval $|x| < \sqrt{3}$, *i.e.* on the interval

$$(-\sqrt{3}, \sqrt{3})$$

Now, let's re-evaluate the information we have about this graph. It should look like Figure 101, because the graph is concave down on the interval $(-\sqrt{3}, \sqrt{3}) = (-0.732, +0.732)$. It must be *bell-shaped* here. On the other hand, we know that it becomes concave up after we pass the two points $x = \pm\sqrt{3}$, which we must suspect are points of inflection! Furthermore, the function is decreasing whenever $x > 0$, from item (3), which means that it's always getting smaller, too! So, we already have a pretty good idea about what will happen here. Let's see.



A better guess at $y = \frac{4}{9+x^2}$

Figure 101.

5. $\boxed{\textbf{P}\text{oints of }\textbf{I}\text{nflection}}$ This is almost obvious. We know, from item (4) above, and by definition that

$$f''(x) = 0 \quad \text{only when} \quad 24\frac{x^2-3}{(9+x^2)^3} = 0 \;\; i.e. \text{ when } \;\; x^2 - 3 = 0,$$

and this forces

$$|x| = \sqrt{3}.$$

These two points, $x = \pm\sqrt{3}$, are the candidates for being such points of inflection. But we already showed in passing that there *is* a change in concavity around these points (from item (4)). So, they are, in fact, the two points of inflection. There can be no other.

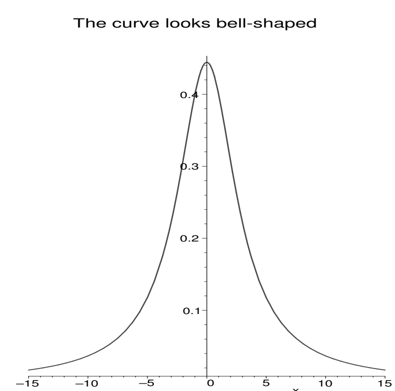
6. $\boxed{\textbf{H}\text{orizontal }\textbf{A}\text{symptotes}}$ The horizontal asymptotes are given by evaluating the $\lim_{x\to\infty} f(x)$, if it exists. Now,

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{4}{9+x^2} = 4 \lim_{x\to\infty} \frac{1}{9+x^2} \\ &= 4 \cdot 0 = 0.\end{aligned}$$

This one was really easy, because it was not an indeterminate form, right? Therefore $\lim_{x\to\infty} f(x) = 0$ means that the horizontal line $\boldsymbol{y = 0}$ **is a horizontal asymptote**. Note also that

$$\lim_{x\to-\infty} f(x) = 0$$

therefore the graph of f tends to the x-axis (*i.e.* $y = 0$) at both ends of the real line. Since $f(x) > 0$, this information along with the above data indicates that the graph of f at the "extremities" of the real line will look like Figure 102, in the sense that the curve is approaching the $x-$axis *asymptotically.*



A final analysis of $y = \frac{4}{9+x^2}$

Figure 102.

$\boxed{\textbf{V}\text{ertical }\textbf{A}\text{symptotes}}$ The graph has **no vertical asymptotes** since $\lim_{x\to a} f(x)$ is always finite, for any value of the number a. In fact,

$$\lim_{x\to a} f(x) = \frac{4}{9+a^2}, \qquad \text{for any real } a.$$

7. Note that f has no absolute minimum on $(-\infty, \infty)$, since it is always positive and approaching the value 0 at infinity. Because of the shape of the graph of f, we see that $f(0) = 4/9$ is the **absolute maximum value** of f, and this means the values of f are never larger than this number, $4/9$. The final graph of f appears in Figure 102.

Example 238. Sketch the graph of the function $f(x) = x \ln x$ on its natural domain, the interval $I = (0, \infty)$.

Solution Now we can start accelerating the process initiated in the previous example. The zeros of this function are given by setting either $x = 0$, which is not allowed since this number is not in the natural domain, or by setting $\ln x = 0$.

But $\ln x = 0$ only when $x = 1$. So, the only zero is at $x = 1$. The critical points are found by solving $f'(c) = 0$ for c or by finding any points where the derivative does not exist. Let's see.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\ x \ln x, \\ &= 1 \cdot \ln x + x \cdot \frac{1}{x}, \\ &= 1 + \ln x. \end{aligned}$$

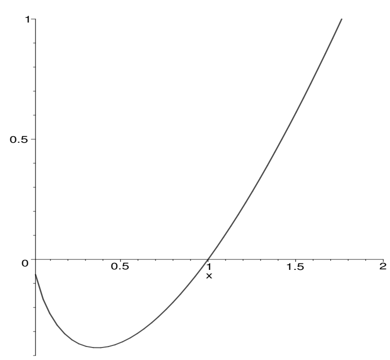


Figure 103.

The derivative fails to exist only when $x = 0$, right? Note that this point is NOT in the domain of our function, though. No problem, it is on the 'edge', at the left end-point of our interval, and strange things can happen there, you'll see. Anyways, the only critical point inside our interval is obtained by setting $f'(x) = 0$. This means that $1 + \ln x = 0$, which is equivalent to $\ln x = -1$. But, by definition of the logarithm, this means that $x = e^{-1}$, where 'e' is Euler's number, right? OK, so the only critical point inside I is $x = e^{-1} \approx 0.3679$. What is its nature? Let's use the Second Derivative Test to determine this. Here $c = 1/e$ and $f'(c) = 0$. Next,

$$\begin{aligned} f''(x) &= \frac{d}{dx}\ (1 + \ln x), \\ &= 0 + \frac{1}{x}, \\ &= \frac{1}{x}. \end{aligned}$$

So, $f''(c) = 1/c = e > 0$ which indicates that $x = 1/e$ is local minimum of f. The value of f at our critical point is $f(1/e) = (1/e)\ln(1/e) = (1/e)(\ln 1 - \ln e) = (1/e)(0-1) = -1/e < 0$. So, up to now we're sure that the graph looks like a small U-shaped bowl around the value $-1/e = -0.3679$ at its bottom. We also see that the requirement that $f'(x) = 1 + \ln x > 0$ is equivalent to $\ln x > -1$, or solving this inequality by taking the exponential of both sides, we get, $x > 1/e$. This means that f is increasing for $x > 1/e$. Similarly, we can show that f is decreasing when $x < 1/e$.

Next, since we have already found the second derivative of f, we see that $f''(x) = 1/x > 0$ only when $x > 0$. This means that the graph of f is concave up when $x > 0$. Since x is never less than 0 (because these negative points are not in the domain of f), it follows that the graph of f is *always* concave up on its domain. Furthermore, there can't be any points of inflection, because $f''(x) \neq 0$, never, for any x!

OK, now you *assimilate all this information.* You know that there is a zero at $x = 1$, the graph is always concave up and the graph is increasing to the right of the local

minimum at $x = 1/e$. It is also 'decreasing' to the left of the local minimum. So, the graph should look like Figure 103 ... the only problem is, we don't know what's happening near $x = 0$. We said earlier that strange things may be going on near this point because the derivative does not exist there. So let's take the limit

$$\begin{aligned}
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x\ln x \qquad \text{(Indeterminate Form: } (0)\cdot(-\infty))\\
&= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \qquad \text{(Indeterminate Form: } (\infty)/(\infty))\\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \qquad \text{(by L'Hospital's Rule)}\\
&= \lim_{x\to 0^+} -x \qquad \text{(after simplifying the fraction)}\\
&= 0.
\end{aligned}$$

So, amazingly enough, the graph is trying to reach the point $(0, 0)$ as $x \to 0^+$, but never quite makes it! But the definition of a vertical asymptote requires us to have an infinite limit as we approach zero, right? So, what's happening here? Well, we know that f is not defined at $x = 0$. But we CAN define $f(0)$ to be this limit, that is, let's agree to define $f(0)$ by

$$f(0) = \lim_{x\to 0^+} f(x) \quad = \quad \lim_{x\to 0^+} x\ln x = 0.$$

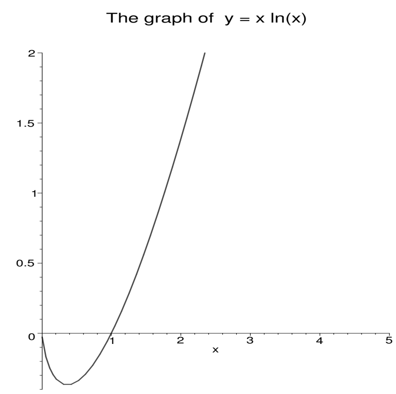


Figure 104.

In this case we can prove, using the methods of Chapter 2, that f is actually right-continuous at $x = 0$ (because we can never define f for negative values so we can't expect more). So, $x = 0$ is actually a *vertical tangent line* because

$$\begin{aligned}
\lim_{x\to 0^+} |f'(x)| &= \lim_{x\to 0^+} |1+\ln x|,\\
&= |-\infty|, \quad (\text{since } \lim_{x\to 0^+} \ln x = -\infty),\\
&= \infty.
\end{aligned}$$

What about horizontal asymptotes? In this case,

$$\begin{aligned}
\lim_{x\to +\infty} f(x) &= \lim_{x\to \infty} x\ln x \qquad \text{(Form: } (+\infty)\cdot(\infty))\\
&= +\infty
\end{aligned}$$

since the form of the product of x with $\ln x$ is NOT indeterminate at $+\infty$ (so you can't use L'Hospital's Rule!). Since this limit is infinite there are no horizontal asymptotes. Our guess that Figure 103 is the graph of this f is a good one, except that it should look more like Figure 104, because of the additional information we gathered from the vertical tangent, above.

Let's work out one more example, but this time of a trigonometric function-polynomial combo. This long example uses *a lot* of what we have studied.

Example 239. Sketch the graph of the function f defined by

$$f(x) = \frac{\sin x}{x}$$

on the open interval $I = (0, 6)$, and determine what happens near $x = 0$ by taking a limit.

Solution In this example we will be using many of the ideas we learned so far about Differential Calculus. This is the kind of example that shows up in real world applications, and where you need to use Newton's method to find roots and make other estimates. You'll see that it's no too bad. Ok, so let's start off by looking for the zeros of f.

In this case the zeros of f are the same as the zeros of the sin function right? Now, the positive zeros of the sin function are given by the roots of $\sin x = 0$. These occur at $x = 0, \pi, 2\pi, 3\pi, \ldots$. But notice that $x = 0$ is excluded because $x = 0$ isn't in I. Next, $x = \pi$ is in I, because $\pi \approx 3.14 < 6$. So that's one zero. The next zero is at $2\pi \approx 6.28 > 6$. So this zero is NOT in I. So, there's only ONE zero in I, and it is at $x = \pi$. So, the only $x-$intercept is $x = \pi$. Since $x = 0$ is not in I, we're not allowed to plug-in the value $x = 0$ into the expression for $f(x)$. So, there are no $y-$intercepts.

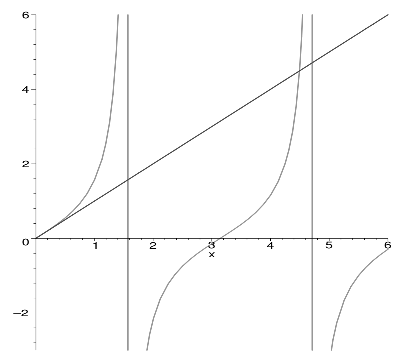


Figure 105.

Next, we'll look for the critical points of f in I. Note that f is differentiable for every value of x in I (because 0 is excluded). So, the critical points are given by the solution(s) of $f'(c) = 0$, for c in I. Now, its derivative is given by (use the Quotient Rule),

$$f'(x) = \frac{x \ \cos x - \sin x}{x^2}.$$

The critical points are given by solving for c in the equation

$$f'(c) = \frac{c \ \cos c - \sin c}{c^2} = 0.$$

or, equivalently, we need to find the solution(s) of the equation

$$c \ \cos c - \sin c = 0, \qquad \text{since } c \neq 0.$$

So, we're really looking for the zero(s) in I of the function, let's call it g, defined by

$$g(x) = x \ \cos x - \sin x.$$

This is where Newton's Method comes in (see Chapter 3). Before we use it though, it would be nice to know *how many zeros there are*, right? Otherwise, we don't know if we'll 'get them all' using the Newton iterations. How do we do this? We simplify the expression for the zero of g. For example, let c be any one of its zeros. Note that $\cos c \neq 0$ (think about this). So, dividing $g(c)$ by $\cos c$, we get, since $\cos c \neq 0$,

$$c - \tan c = 0.$$

Newton iterates for
$g(x) = x \cos x - \sin x$ and $x_0 = 4$

m	$g(x_m)$	x_{m+1}
0	−1.857772	4.613691
1	0.540511	4.495964
2	0.011213	4.493411
3	0.000063	4.493409
4	0.000000	$\boxed{4.493409}$
...	...	...

Figure 106.

This means that these zeros of g (i.e., the critical points of f) coincide with the *points of intersection of the graph of the function* $y = \tan x$ *with the graph of* $y = x$. Furthermore, we only want those points of intersection whose $x-$coordinates are in I. But these two graphs are easy to draw, see Figure 105. You see from this Figure that there are only *two points of intersection*, one at $x = 0$ (which is not in I, so we forget about it) and another one near $x = 4$. That's the one! It looks like there's only ONE ROOT of $x - \tan x = 0$ in I, and so only one root of $g(x) = 0$ in I and so only one root of $f'(x) = 0$ in I. The conclusion is that there is **only one critical point** of f in I.

Now, recall that the approximations to this root of $g(x) = 0$ are given by x_m where the **larger the subscript m the better the approximation** and,

$$x_{m+1} = x_m - \frac{g(x_m)}{g'(x_m)} = x_m + \frac{x_m \ \cos x_m - \sin x_m}{x_m \sin x_m}, \quad \textit{for } m \geq 0.$$

Let's use $x_0 = 4$ as our initial guess from Figure 105. Then $x_1 \approx 4.6137$ (check this using your calculator). The remaining better estimates are given in Figure 106. So, the one and only critical point of f in I is given by the boxed entry in Figure 106, that is,

$$f'(c) = 0 \quad \Rightarrow \quad c \approx 4.4934.$$

What is the nature of this critical point? Is it a $\boxed{\text{local maximum, local minimum?}}$. The best thing to do here is to use the Second Derivative Test. Now,

$$f''(x) \quad = \quad \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3},$$

so, evaluating this expression at our critical point, $x = 4.4934$, we see that

$$f''(4.4934) \approx 0.2172 > 0,$$

which means that our critical point $c = 4.4934$, is a *local minimum* with a local minimum value equal to $f(c) = f(4.4934) \approx -0.2172$. Actually, it can be shown that $f(c) = -f''(c)$, and not only to five significant digits. Can you show why?

Now, let's $\boxed{\text{look for the intervals where } f \text{ is increasing or decreasing.}}$ We can rewrite $f'(x)$ for x in I as

$$f'(x) = \begin{cases} (x - \tan x) \cdot \frac{\cos x}{x^2}, & \text{if} \quad x \neq \frac{\pi}{2}, \\ \frac{-4}{\pi^2}, & \text{if} \quad x = \frac{\pi}{2}, \end{cases}$$

because $x = \pi/2$ is the only place in I where $\tan x$ is undefined. Now, look at Figure 105. Note that the graph of $y = x$ lies *above* the graph of $y = \tan x$ if $\pi/2 < x < 4.4934$. This means that $x - \tan x > 0$ for such values of x, right? Furthermore, for these same values of x, $\cos x < 0$, so,

$$f \text{ is decreasing if } \pi/2 < x < 4.4934 \quad \textit{i.e.,} \ f'(x) < 0.$$

Similarly, if $0 < x < \pi/2$, $\cos x > 0$ but now the graph of $y = x$ is *under* the graph of $y = \tan x$. This means that $x - \tan x < 0$ and so,

$$f \text{ is decreasing if } 0 < x < \pi/2 \quad \textit{i.e.,} \ f'(x) < 0.$$

We already know that $f'(\pi/2) = -4/\pi^2 < 0$, so all this information tells us that f is decreasing on the interval $(0, 4.4934)$. A similar argument shows that f is increasing on the interval $(4.4934, 6)$, (you should convince yourself of this) .

Let's recap. Up to now we have a graph that looks like a shallow bowl, because the minimum value is so small (≈ -0.2172). This is because we know that f has a local minimum at $x \approx 4.4934$ and that f is increasing to the right of this point and decreasing to the left of this point. We also know that $f(\pi) = 0$. So, the picture so far is similar to Figure 107.

Now, our intuition tells us that the graph is concave up on the interval $(\pi, 6)$, based on the rough figure we sketched in the margin. What happens near $x = 0$? Is there going to be a change in concavity? Is it going to be concave down anywhere near $x = 0$? Before we go on finding out the concavity of our graph let's look at what's happening near $x = 0$. First, we take the limit of $f(x)$ as $x \to 0$ and then we'll look at its right-derivative there. So,

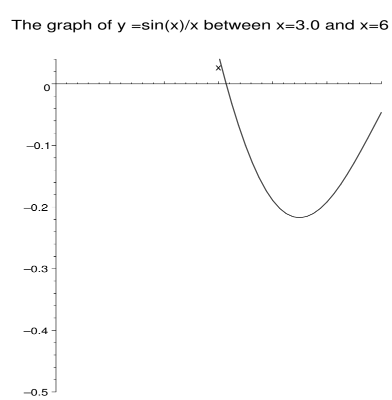


Figure 107.

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{\sin x}{x},\\ &= \lim_{x\to 0^+} \frac{\cos x}{1}, \qquad \text{(by L'Hospital's Rule; Form is 0/0)}\\ &= 1.\end{aligned}$$

Next,

$$\begin{aligned}\lim_{x\to 0^+} f'(x) &= \lim_{x\to 0^+} \frac{x\ \cos x - \sin x}{x^2},\\ &= \lim_{x\to 0^+} \frac{-x\sin x}{2x}, \qquad \text{(by L'Hospital's Rule; Form is 0/0)}\\ &= \lim_{x\to 0^+} \frac{-\sin x}{2},\\ &= 0.\end{aligned}$$

So the derivative is 'leveling off' at $x = 0$ (the tangent line becomes horizontal) and the function itself tends to a finite value of 1 at $x = 0$. But then, we reason like this: The derivative is positive near $x = 3$ and it's 0 at $x = 0$ which means, we think, that maybe it's always getting *smaller* as $x \to 0$. But this is the same as saying that f' is decreasing, i.e., the graph of f is concave down (or, $f''(x) < 0$) in that inteval! But wait! The graph is known to be concave up in $(3, 6)$. So, there should be a point of inflection, c, with $0 < c < 3$. Let's see if our intuition is right...

Newton iterates for
$h(x) = -x^2 \sin(x) - 2x\cos(x) + 2\sin(x)$
and $x_0 = 2$

m	$h(x_m)$	x_{m+1}
0	−0.154007	2.092520
1	0.023531	2.081737
2	0.000340	2.081576
3	0.000000	$\boxed{2.081576}$
...	...	...

Figure 108.

Well, let's spare us the details in using Newton's Method one more time to find the roots of $f''(c) = 0$, whose solution(s) define the inflection point, c. We use the starting value, $x_0 = 2$, and define the function h to be the numerator of $f''(x)$. So,

$$h(x) = -x^2 \sin x - 2x\cos x + 2\sin x,$$

and the iterations converge to the root whose value is $c \approx 2.081576$, see Figure 108. This value of c is indeed a point of inflection since, just to the left of c you can calculate that $f''(x) < 0$ while just to the right of c we have $f''(x) > 0$.

Finally, we can now infer almost without a doubt that the graph of f is concave up in $(2.0816, 6)$ and concave down in $(0, 2.0816)$.

The graph of y =sin(x)/x between x=0 and x=6

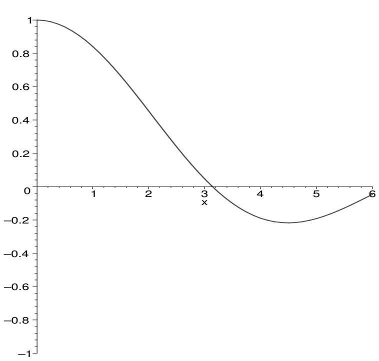


Figure 109.

As for asymptotes, there are no horizontal asymptotes since the interval is finite, and there cannot be any vertical asymptotes since the only candidate, $x = 0$, is *neither* a vertical asymptote or a vertical tangent line. The complete graph of this function appears in Figure 109.

NOTE: What is really nice here is the graph of the function $f(x) = \sin x/x$, we just described on the interval $(0, \infty)$. Unfortunately, this would likely take us years (really!) to do with just a calculator, and it could never be done entirely (since the interval is infinite), but its general form could be described using arguments which are very similar to the one just presented. You'll then get the graph in Table 5.5. In this case the graph *does* have a horizontal asymptote at $y = 0$, since

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0,$$

by the Sandwich Theorem for Limits (Chapter 2), and NOT by L'Hospital's Rule, because $\sin(\infty)$ is meaningless in the limiting sense.

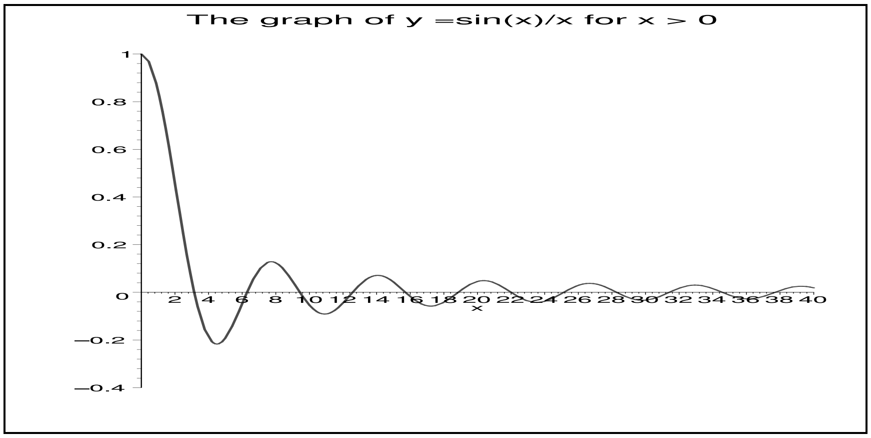


Table 5.5: The Graph of $\frac{\sin x}{x}$ on the Interval $(0, \infty)$.

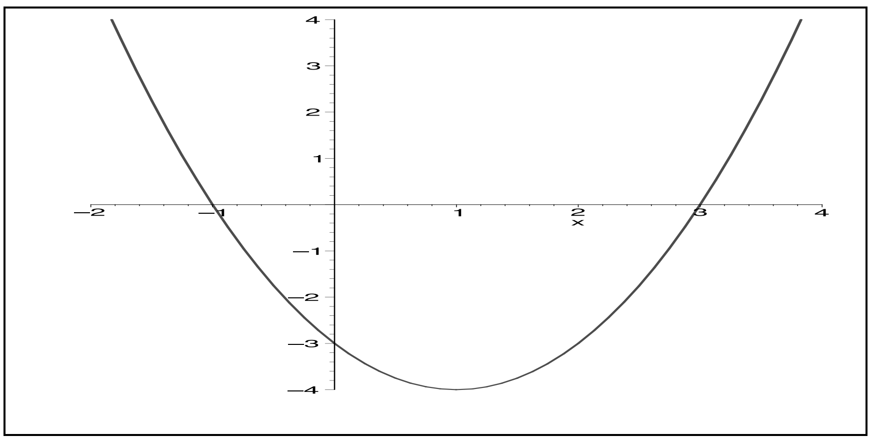


Table 5.6: The Graph of $x^2 - 2x - 3$ on the Interval $(-\infty, \infty)$.

SNAPSHOTS

Example 240. Sketch the graph of the function f defined by $f(x) = x^2 - 2x - 3$.

Solution The function is a polynomial which factors easily as $f(x) = (x-3)(x+1)$, so its zeros are at $x = 3$ and $x = -1$. So, its $x-$intercepts are at these points $x = 3, -1$. Its $y-$intercept is $y = -3$. Furthermore, since $f'(x) = 2x - 2$ and $f''(x) = 2 > 0$ the graph is a function which is *always concave up*, (so has no inflection points) and has one critical point c where $f'(c) = 2c - 2 = 0$, i.e., when $c = 1$. This point $x = 1$ is a local minimum (by the Second Derivative Test). A few extra points plotted for convenience generate the graph, Table 5.6.

Example 241. Sketch the graph of the function f defined by

$$f(x) = \frac{1}{x^2 - 1}.$$

Solution This function is a rational function whose denominator factors easily as $x^2 - 1 = (x-1)(x+1)$, so its zeros are at $x = 1$ and $x = -1$. This means that f is undefined at these points, $x = \pm 1$, so the lines $x = \pm 1$ qualify as vertical asymptotes. Since $f(x) \neq 0$, the graph has no $x-$intercepts. Its $y-$intercept (set $x = 0$) is $y = -1$. Next,

$$\begin{aligned} f'(x) &= \frac{-2x}{(x^2-1)^2}, \quad \textit{and} \\ f''(x) &= \frac{2(3x^2+1)}{(x^2-1)^3}. \end{aligned}$$

The only critical point in the domain of f is when $x = 0$. There cannot be *any* inflection points since, for any x, $f''(x) \neq 0$. The point $x = 0$ is a local maximum (by the Second Derivative Test) and $f(0) = -1$. In order to find the intervals of increase and decrease and the intervals where the graph of f is concave up or down, we use the SDT's of each rational function $f'(x), f''(x)$.

Intervals	$(x+1)^2$	$-x$	$(x-1)^2$	sign of $f'(x)$
$(-\infty,-1)$	$+$	$+$	$+$	$+$
$(-1,0)$	$+$	$+$	$+$	$+$
$(0,1)$	$+$	$-$	$+$	$-$
$(1,\infty)$	$+$	$-$	$+$	$-$

The graph of y= 1/(x^2 - 1)

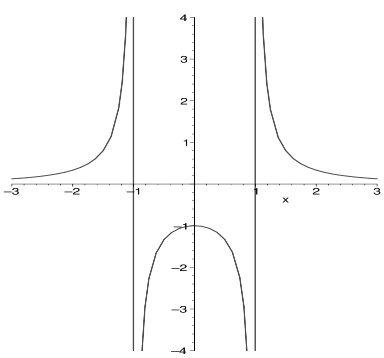


Figure 110.

NOTE: We've included the '$-$ sign' in front of the function f', into the 'x column' in our Table, just so we don't forget it. The SDT for $f''(x)$ is (since its numerator is irreducible),

Intervals	$(x+1)^3$	$(x-1)^3$	sign of $f''(x)$
$(-\infty,-1)$	$-$	$-$	$+$
$(-1,1)$	$+$	$-$	$-$
$(1,\infty)$	$+$	$+$	$+$

Assimilating all this we see that f is increasing on $(-\infty,0)$, decreasing on $(0,\infty)$, concave up on $(-\infty,-1),(1,\infty)$, and concave down on $(-1,1)$. Lastly, there is a horizontal asymptote at both 'ends' of the real line given by $y = 0$, since

$$\lim_{x\to\pm\infty} \frac{1}{x^2-1} = 0,$$

and two vertical asymptotes given by the lines $x = \pm 1$ (as mentioned above). Combining all this data and adding a few extra points using our calculator, we get the graph in Figure 110.

Example 242. Sketch the graph of the function f defined by $f(x) = x^3 - 3x + 2$.

Solution By inspection we note that $x = 1$ is a zero of f $(f(1) = 0)$. To find the remaining factors we need to divide $(x-1)$ into $f(x)$. We do this using **long division**, (a quick review of this procedure is given elsewhere in this book). So, we see that

$$\begin{aligned} \frac{x^3-3x+2}{(x-1)} &= x^2 + x - 2, \\ &= (x+2)(x-1). \end{aligned}$$

We see that the factors of f are given by $f(x) = (x-1)^2(x+2)$. From this we see that f has a **double root** at $x = +1$ (i.e., $f(1) = 0$ and $f'(1) = 0$). Its $x-$intercepts are at these points $x = 1, -2$. Its $y-$intercept is $y = 2$.

Next, $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$ and $f''(x) = 6x$. The only critical points are at $x = \pm 1$ and the Second Derivative Test shows that $x = +1$ is a local minimum while $x = -1$ is a local maximum. The graph is a function which is concave up when $x > 0$ and concave down when $x < 0$. Since $f''(x) = 0$, it follows that $x = 0$ is a point of inflection. Using the SDT of f', or, more simply this time, by solving the inequalities $x^2 - 1 > 0, x^2 - 1 < 0$, we see that if $|x| > 1$, then f is increasing , while if $|x| < 1$, then f is decreasing. Since f is a polynomial, there are **no vertical or horizontal asymptotes**, (can you show this?). A few extra points added to the graph for convenience generate Figure 111.

NOTES:

POSSIBLE SHORTCUTS

The graph of a cubic with a double root

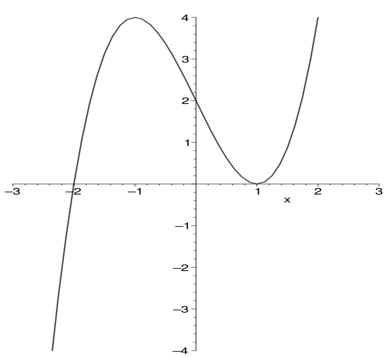


Figure 111.

1. **EVEN SYMMETRY**. There is an additonal tool that one can use to help draw the graph of a function. This tool is not always useful in practice, but it *can* be useful in some cases. The tool involves some *symmetry* and depending on whether or not the given f has this or that property we can cut down the amount of time needed to draw the graph. It works like this.

   If the given function is an **even function**, then the graph of f is symmetric with respect to the $y-$axis. This means that you can fold the graph of f for $x > 0$ along the $y-$axis and obtain the same graph on the other side, $x < 0$. In other words, if a mirror is placed along the $y-$axis, then the graph is its own mirror image. OK, so what's an even function? A function is said to be *even* if it satisfies

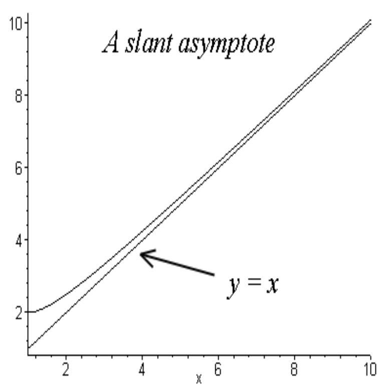
$$f(-x) = f(x), \quad \text{for every value of } x \text{ in its domain.}$$

   You can use the Box principle (Chapter 1) to check this. For example, the function $f(x) = x^2$ is even, since $f(-x) = (-x)^2 = x^2 = f(x)$. Another example is $f(x) = x^2 \cos(x)$, since the cosine function is, itself an even function, that is, $\cos(-x) = \cos x$. Other examples include the functions f defined in Examples 233, 237, and 241. In each of these Examples, the graph is 'symmetric' with respect to the $y-$axis.

2. **ODD SYMMETRY**. On the other hand, if the given function is an **odd function**, then the graph of f is symmetric with respect to the origin, $(0, 0)$, (the fancy word for this is a **central reflection**). This means that the graph of f can be found by folding the graph of f for $x > 0$ about the $y-$axis, and then folding *that graph* along the $x-$axis.

   But what's an odd function? A function is said to be *odd* if it satisfies

$$f(-x) = -f(x), \quad \text{for every value of } x \text{ in its domain.}$$



The curve $y = \frac{x^2+1}{x}$

Figure 112.

   For example, the function $f(x) = x^3$ is odd, since $f(-x) = (-x)^3 = -x^3 = -f(x)$. Another example is $f(x) = \sin(x)$, which declares that the sine function is, itself an odd function. Another such example includes the functions f defined in Example 234. In each of these Examples, the graph is 'symmetric' with respect to the origin, $(0, 0)$. However, in the remaining examples the functions are **neither even nor odd.**

   Why these odd names, even and odd? Well, because someone, somewhere, ages ago figured that if you associate an even function with a *plus sign* and an odd function with a *minus sign*, then the 'product of two such functions' behaves much like the product of the corresponding *plus and minus signs*. So, one can show without much difficulty that

$$\begin{aligned}
(Even\ function) \cdot (Even\ function) &\implies Even\ function,\\
(Even\ function) \cdot (Odd\ function) &\implies Odd\ function,\\
(Odd\ function) \cdot (Even\ function) &\implies Odd\ function,\\
(Odd\ function) \cdot (Odd\ function) &\implies Even\ function,
\end{aligned}$$

   For example, let's show that if f is an odd function and g is an even function, their product is an odd function. let f be odd and g be even.Writing $h = fg$, their product, (not their *composition*), we know that $h(x) = f(x)g(x)$, by

definition, and so,

$$\begin{aligned} h(-x) &= f(-x)g(-x), \\ &= (-f(x))(g(x)), \quad \text{(since } f \text{ is odd and } g \text{ is even)}, \\ &= -f(x)g(x), \\ &= -h(x), \quad \text{(which means that } f \text{ is odd, by definition.)} \end{aligned}$$

3. **SLANT ASYMPTOTES** These are lines which are neither vertical nor horizontal asymptotes. They are 'slanted' or 'tilted' at an arbitrary angle. If the function f has the property that there are two real numbers, m and b with

$$\lim_{x\to\infty} (f(x) - (mx + b)) = 0,$$

then we say that the line $y = mx + b$ is a **slant asymptote** of the graph of f. For example, the function f defined by

$$f(x) = \frac{x^2+1}{x},$$

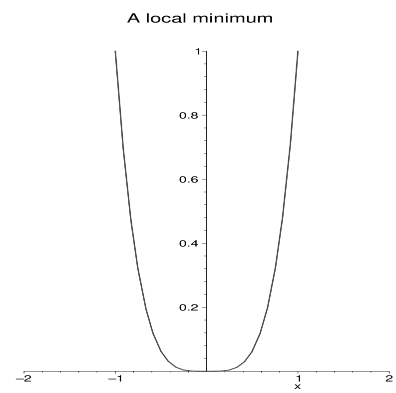
has the property that

$$\lim_{x\to\infty} \left(\frac{x^2+1}{x} - x\right) = 0,$$

because $1/x \to 0$ as $x \to \infty$. Its graph can be seen in Figure 112.

4. **WHAT IF YOU CAN'T USE THE SECOND DERIVATIVE TEST** because $f'(c) = 0$ and $f''(c) = 0$? What do you do?

We use the following neat result which is more general than the Second Derivative Test. Its proof requires something called **Taylor's Formula** which we will see in a later Chapter. So here is



The curve $y = x^4$

Figure 113.

THE GENERAL EXTREMUM TEST

Let f be $n-$times differentiable over an interval I. Assume that for $x = c$ we have,

$$f'(c) = 0, \quad f''(c) = 0, \quad f'''(c) = 0, \ldots, \quad f^{(n-1)}(c) = 0, \quad f^{(n)}(c) \neq 0.$$

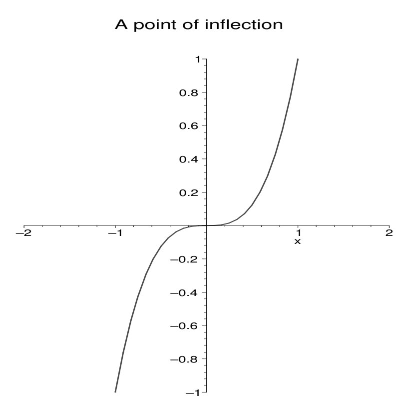
(a) If n is even and $f^{(n)}(c) > 0$, $\Longrightarrow$ c is a local minimum

(b) If n is even and $f^{(n)}(c) < 0$, $\Longrightarrow$ c is a local maximum

(c) If n is odd, $\Longrightarrow$ c is a point of inflection

The classic examples here are these: Let f be defined by $f(x) = x^4$. Then $f'(0) = 0, f''(0) = 0, f'''(0) = 0, f^{(4)}(0) = 24 > 0$. So, $n = 4$, and the General Extremum Test tells us that $x = 0$ is a local minimum (see Figure 113).

On the other hand, if we let $f(x) = x^3$, then $f'(0) = 0, f''(0) = 0, f'''(0) = 6$. Now, $n = 3$ is an odd number, so regardless of the sign of the third derivative we know that $x = 0$ is a point of inflection (see Figure 114).



The graph of $y = x^3$

Figure 114.

NOTES:

5.4 Chapter Exercises

Sketch the graphs of the following functions using your CAS or a graphing calculator or failing this, your hand calculator and a few points. Then sketch the graphs using the methods in this Chapter. Compare your results.

1. $y = (x+1)^2$
2. $y = 4 - x^2$
3. $y - 3x^2 - 3x = 0$
4. $y = x^2 - 2x + 1$
5. $y = x\sqrt{x}$
6. $y = x^3 - 27$
7. $y = \sin 2x$
8. $y = x^2 - 5x + 6$
9. $y = \cos 2x$
10. $y(x-2) = 1$
11. $y = (x-1)(x+1)^3$
12. $y + x^2y = 4x$
13. $y = \dfrac{x^2+1}{2x+3}$
14. $y = x^3 + x$
15. $y = \sin(2x+1)$

Use the Sign Decomposition Table (SDT) to help you sketch the following graphs.

16. $y = \dfrac{x+1}{x^2+1}$
17. $y = \dfrac{x-2}{x^2-2x+1}$
18. $y = \dfrac{x+1}{x^3-1}$
19. $y = \dfrac{x^2-1}{(x-1)(x+2)^2}$
    - Simplify this expression first!
20. $y = \dfrac{x^3}{1+x^2}$

Sketch the graphs of the following functions

21. $y = xe^{-x}$
22. $y = |x-4|$
23. $y = x^2 \ln x,\ x > 0$
24. $y = \ln(x^2)$
25. $y = e^{2\ln|x|},\ x \neq 0$
26. $y = x \sin x$
27. $y = (x-1)^2(x+1)$
28. $y + x^2y = 4$
29. $y = x^2e^{-x}, x \geq 0$

30. **Challenge Problem:** Sketch the graph of the function f defined by

$$f(x) = \frac{1-\cos(2x)}{x}$$

for $0 < x < 4\pi$.

31. A company that makes and sells stereos has found that its daily cost function and revenue function, respectively, are given by

$$C(x) = 5 + 35x - 1.65x^2 + 0.1x^3 \qquad R(x) = 32x.$$

If the company can produce at most 20 stereos per day, what production level x will yield the maximum profit and what is the maximum profit?

32. A company has determined that it can sell x units per day of a particular small item if the price per item is set at $p = 4 - 0.002x$ dollars, with $0 \leq x \leq 1200$.

(a) What production level x and price p will maximize daily revenue?

(b) If the corresponding cost function is $C(x) = 200 + 1.5x$ dollars, what production level will maximize daily profit?

(c) What is the maximum daily profit?

(d) Find the marginal cost and marginal revenue functions.

(e) Equate the marginal cost and marginal revenue functions and solve for x. Notice that this is the same value of x that maximized the profit. Can you explain why this is true?

33. A manufacturer of cabinets estimates that the cost of producing x cabinets can be modelled by

$$C(x) = 800 + 0.04x + 0.0002x^2.$$

How many cabinets should be produced to minimize the **average cost** per cabinet, $C(x)/x$?

34. In economics, a **demand equation** is an equation which gives the relationship between the price of a product, p, and the number of units sold, x. A small company thinks that the demand equation for its new product can be modelled by

$$p = Ce^{kx}.$$

It is known that when $p = \$45$, $x = 1000$ units, and when $p = \$40$, $x = 1200$ units.

(a) Solve for C and k.

(b) Find the values of x and p that will miximize the revenue for this new product.

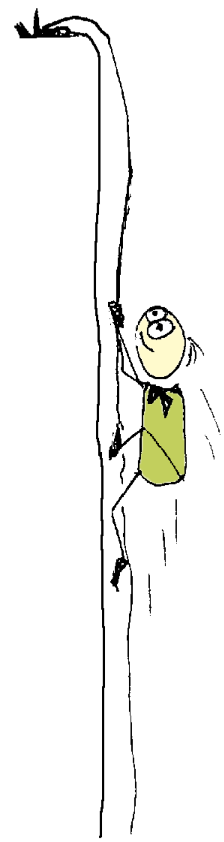
Suggested Homework Set 20. *Do problems 2, 6, 11, 18, 23*

NOTES:

Chapter 6

Integration

The Big Picture



This Chapter is about the subject of *Integration.* One can argue that the reason for the existence of this subject of integration is intimately tied with the existence of the derivative. With the notion of the *derivative* we have seen that we can model the world around us. The integral allows to go 'backwards', so to speak. In other words, let's say that you know the speed of a particle moving in a straight line in some fixed direction and you want to know where it is... Mmm, not enough information, right? You only know its speed, so you can't know where it *is* unless you ask a question like: "Where was it, say, 5 seconds ago?". Then we can tell you where it is now and where it will be later (assuming its speed remains the same). Well, this whole business of finding out where a body is given its speed is part of the subject of integration. You can imagine that Newton and Leibniz and Bernoulli and all the early discoverers of Calculus had to wrestle with this concept so that they could actually make calculations about where planetary bodies were and where they would be, given their present position and Newton's own Laws. They also solved other problems of practical interest and needed to find the solution of a *differential equation* using the *integral.*

Sometimes this mathematical subject called *integration* is not classified correctly by librarians and one may find it in some libraries under the heading of *Social Studies*, or *Sociology.* Anyways, this topic of integration together with the topic of *Differentiation* which we saw in Chapters 3 and 4, comprise the main tools used in the study of Calculus of functions of one variable. These two topics really go hand-in-hand through an important result called the **Fundamental Theorem of Calculus**, called by some authors, the *Fundamental Theorem of Integral Calculus.* The statement of this Fundamental Theorem says something like 'The operation of taking the derivative of a function and the operation of *taking the integral of a function*' are basically "inverse operations" to one another. The word 'inverse' here is meant in a sense analogous to *inverse functions*, a topic which we saw in Chapter 3.

So, one thinks that this last statement about inverses should mean: "The derivative of the integral of a function is the function itself" while "the integral of the derivative of a function is *almost* the function, itself, because there is a constant missing". This explains why we said " ... *basically* inverses of one another", above. Among the applications of the integral we'll see that it is used to find the area under a curve, or the **area between two curves**, or even for finding the volume of a region called a **solid of revolution**. It is also used to calculate **centers of mass** in

engineering, or to find the probability that an electron is in such and such a place (in quantum mechanics). Since most physical dynamical phenomena are modeled using differential equations it becomes reasonable that their *solutions* will be given by the integral of something or other. There are many, many more applications of the integral that we could give but these will suffice for our purposes. This chapter is part of the heart of Calculus, its importance cannot be underestimated.

Review

Review all the Tables involving **derivatives of exponential and trigonometric functions and their inverses**. Soon enough you'll see that you don't really have to memorize any more formulae so long as you've committed to memory those Tables from Chapters 3 and 4.

6.1 Antiderivatives and the Indefinite Integral

The procedure of finding the **antiderivative** of a given function is the **inverse operation** to that of differentiation or the "taking of a derivative". Basically, given a function f continuous over an interval $[a, b]$, its *antiderivative* is a differentiable function, $\mathcal{F}$, (read this as 'script F') whose derivative, $\mathcal{F}'$, satisfies

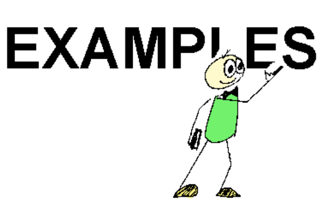
$$\mathcal{F}'(x) = f(x), \quad a < x < b,$$

i.e. the *derivative of an antiderivative of a function is the function itself.* We will use the symbol, script F, or $\mathcal{F}(x)$ for an antiderivative of a given function f, script G, or $\mathcal{G}(x)$ for an antiderivative of g, etc.

Example 243. Find an antiderivative, $\mathcal{F}(x)$, of the function whose values are $f(x) = 2x$.

Solution We want to find a function $\mathcal{F}$ such that $\mathcal{F}'(x) = 2x$, right? Let's try a function like $\mathcal{F}(x) = x^r$, where we have to find r. Why this type? Because we know that the derivative of such a power function is another such power function (by the Power Rule for Derivatives in Chapter 3). Now,

$$\begin{aligned} \mathcal{F}'(x) &= rx^{r-1}, \quad \textit{in general}, \\ &= 2x, \quad \textit{in particular}, \end{aligned}$$



when we choose (by inspection) $r = 2$, right? This means that if we choose $\mathcal{F}(x) = x^2$, then $\mathcal{F}'(x) = 2x$, which, by definition, means that $\mathcal{F}$ is an antiderivative of f. On the other hand, the function whose values are $\mathcal{F}(x) = x^2 + 4.123$ is **also** an antiderivative of f (because the derivative of a constant, like 4.123, is zero). The moral of all this is in Table 6.1, below.

The final result in Table 6.1 is not hard to show. We need to show that the difference between any two antiderivatives of a given function is always a constant. This is because if $\mathcal{F}_1, \mathcal{F}_2$ are each antiderivatives of f, then

$$\begin{aligned} D(\mathcal{F}_1 - \mathcal{F}_2) &= D(\mathcal{F}_1) - D(\mathcal{F}_2), \\ &= f - f, \\ &= 0, \end{aligned}$$

When it exists, the derivative f' of a given function f is always unique, in the sense that it is the only function with the name "derivative of f". On the other hand, the *antiderivative*, $\mathcal{F}$, of a given function f is *not unique.* This means that if $\mathcal{F}$ is an antiderivative of f then the function $\mathcal{F} + C$, where C is a constant, is a NEW antiderivative of f.

Mathematicians summarize this property of an antiderivative by saying that: **"an antiderivative of a function is defined up to the addition of an arbitrary constant"**.

Mathematically, this means that if $\mathcal{F}_1, \mathcal{F}_2$ are each antiderivatives of f then, their difference, $\mathcal{F}_1 - \mathcal{F}_2 = C$ where C is some constant (number).

Table 6.1: The Basic Property of an Antiderivative

since $D(\mathcal{F}_i) = f$, by assumption, for $i = 1, 2$. This means that the function $\mathcal{F} = \mathcal{F}_1 - \mathcal{F}_2$ has a derivative which is equal to zero everywhere and so, by Example 97, $\mathcal{F}(x) = constant$, that is, $\mathcal{F}_1 - \mathcal{F}_2 = constant$, which is what we wanted to show.

Another notation (symbol) for an antiderivative of a given function f is given by Leibniz's own 'elongated S' notation, which is called the **indefinite integral** of f, that is,

$$\boxed{\mathcal{F}(x) = \int^x f(t)\ dt, \qquad (6.1)}$$

where the variable t appearing on the right is sometimes called a *free variable*, the older term being a *dummy variable.* The shape of the symbol defining a free variable is not important, so, for instance, (6.1) can be rewritten as

$$\mathcal{F}(x) = \int^x f(u)\ du$$

also defines an antiderivative of f, or even $\int^x f(\square)\ d\square$. Combining the above results and notation we can write

$$\boxed{\mathcal{F}'(x) = \frac{d}{dx}\int^x f(t)\ dt = f(x), \qquad (6.2)}$$

for $a < x < b$, whenever $\mathcal{F}$ is an antiderivative of f over $[a, b]$.

Example 244. Show that every antiderivative of the function f defined by $f(x) = cx^r$, where c is a given constant and $r \neq -1$, looks like

$$\int^x ct^r\ dt = \frac{cx^{r+1}}{r+1} + C \qquad (6.3)$$

where C is a constant, called a **constant of integration**.

Solution Here, $f(x) = cx^r$, or $f(t) = ct^r$. Notice that the function $\mathcal{F}$ whose values are $\mathcal{F}(x)$ where

$$\mathcal{F}(x) = \frac{cx^{r+1}}{r+1} + C$$

has the property that $\mathcal{F}'(x) = cx^r = f(x)$, by the Power Rule for derivatives. It follows that this function is an antiderivative, and every other antiderivative of f differs from this one by a constant, by Table 6.1. So,

$$\begin{aligned} \mathcal{F}(x) &= \int^x ct^r \, dt, \\ &= \frac{cx^{r+1}}{r+1} + C. \end{aligned}$$

Equation (6.3) is very useful when combined with the next result, because it will allow us to find the most general antiderivative (or indefinite integral) of a general polynomial of degree n.

Example 245. If $\mathcal{F}, \mathcal{G}$ are antiderivatives of f, g, respectively, and c is any constant (positive or negative), then the function $\mathcal{F} + c \cdot \mathcal{G}$ is an antiderivative of $f + c \cdot g$, that is,

$$\int^x (f(t) + c \cdot g(t)) \, dt = \int^x f(t) \, dt + c \int^x g(t) \, dt$$

This result is sometimes stated as: **an antiderivative of a sum of two or more functions is the sum of the antiderivatives of each of the functions**, and the antiderivative of a scalar multiplication of a function is the antiderivative of this function multiplied by the same scalar.

Solution We use the definition of an antiderivative. All we have to do is to find the derivative of the function $\mathcal{F} + c \cdot \mathcal{G}$ and show that it is the same as $f + c \cdot g$. So,

$$\begin{aligned} \frac{d}{dx}(\mathcal{F} + c \cdot \mathcal{G}) &= \mathcal{F}'(x) + c \cdot \mathcal{G}'(x), \\ &= f(x) + c \cdot g(x), \end{aligned}$$

by the usual properties of the derivative.

Let's use the result in Example 245 in the next few Examples.

Example 246. Find $\displaystyle\int (2x+3) \; dx$.

Solution In order to find an antiderivative of $2x + 3$ we find an antiderivative of $2x$ and 3 and use the results of Example 245 and Equation (6.3). So, we let $f(x) = 2x,\; g(x) = 3$. Then, $\mathcal{F}(x) = x^2 + C_1$ and $\mathcal{G}(x) = 3x + C_2$ (by Equation (6.3) with $r = 0, c = 3$). So,

$$\begin{aligned} \int^x (2t+3) \, dt &= \int^x 2t \, dt + \int^x 3 \, dt, \\ &= (x^2 + C_1) + (3x + C_2), \\ &= x^2 + 3x + \underbrace{C_1 + C_2}, \\ &= x^2 + 3x + \quad C, \end{aligned}$$

where $C = C_1 + C_2$ is a new constant of integration.

Example 247. Find $\displaystyle\int (x^3 - 3x^2)\ dx$.

Solution We use the results of Example 245 and Equation (6.3). In this case, we let $f(x) = x^3$, $g(x) = -3x^2$. Then, $\mathcal{F}(x) = \frac{x^4}{4} + C_1$ and $\mathcal{G}(x) = -x^3 + C_2$ (by Equation (6.3) with $r = 2, c = -3$). So,

$$\begin{aligned}
\int^x (t^3 - 3t^2)\ dt &= \int^x t^3\ dt + \int^x -3t^2\ dt, \\
&= (\frac{x^4}{4} + C_1) + (-x^3 + C_2), \\
&= \frac{x^4}{4} - x^3 + \underbrace{C_1 + C_2}, \\
&= \frac{x^4}{4} - x^3 + \qquad C,
\end{aligned}$$

where $C = C_1 + C_2$ is, once again, a new constant of integration.

Many authors will use the following notation for an indefinite integral, that is,

$$\int f(x)\ dx = \int^x f(t)\ dt.$$

Both symbols mean the same thing: They represent an antiderivative of f.

Now, repeated applications of Example 245 shows that we can find an antiderivative of the sum of three, or four, or more functions by finding an antiderivative of each one in turn and then adding them. So, for example,

Example 248. Evaluate $\displaystyle\int (3x^2 - 2x + (1.3)x^5)\ dx$.

Solution In this case,

$$\begin{aligned}
\int (3x^2 - 2x + (1.3)x^5)\ dx &= \int 3x^2\ dx + \int (-2x)\ dx + \int (1.3)x^5\ dx, \\
&= (x^3 + C_1) + (-x^2 + C_2) + ((1.3)\frac{x^6}{6} + C_3), \\
&= x^3 - x^2 + (1.3)\frac{x^6}{6} + \underbrace{C_1 + C_2 + C_3}, \\
&= x^3 - x^2 + \frac{13x^6}{60} + \qquad C,
\end{aligned}$$

where now $C = C_1 + C_2 + C_3$ is, once again, a NEW constant of integration. All the other constants add up to make another constant, and we just don't know what it is, so we label it by this generic symbol, C.

So, **why are these 'constants of integration' always floating around?** The reason for this is that these 'constants' allow you to distinguish one antiderivative from another. Let's see ...

Example 249. Find that antiderivative $\mathcal{F}$ of the function f defined by $f(x) = 4x^3 - 2$ whose value at $x = 0$ is given by $\mathcal{F}(0) = -1$.

Solution First, we find the most general antiderivative of f. This is given by

$$\mathcal{F}(x) = \int (4x^3 - 2)\ dx = x^4 - 2x + C,$$

where C is just a constant of integration. Now, all we have to do is use the given information on $\mathcal{F}$ in order to evaluate C. Since we want $\mathcal{F}(0) = -1$, this means that $\mathcal{F}(0) = 0 - 0 + C = -1$, which gives $C = -1$. So, the antiderivative that we want is given by $\mathcal{F}(x) = x^4 - 2x + (-1) = x^4 - 2x - 1$.

Now, let's recall the **Generalized Power Rule**, from Chapter 3. To this end, let u denote any differentiable function and let r be any real number. We know that

$$\frac{d}{dx}\underbrace{u(x)^r}_{\mathcal{F}(x)} \quad = \quad \underbrace{r\ u(x)^{r-1}\ \frac{du}{dx}}_{f(x)},$$

One of the reasons why Leibniz's notation was, and still is, superior over Newton's is this boxed formula. You almost want to 'cancel' the dx's out of the symbol on the left leaving one with a true formula, namely that,

$$\int u^r\ du = \frac{u^{r+1}}{r+1} + C,$$

if we forget about the fact that u is really a function of x ... Well, Leibniz 'got away with this' back then, and the form of this formula makes it easy for students to remember it because one 'thinks' about canceling out the dx's, on the left, but you really don't do this!

so, if we denote all the stuff on the right by $f(x)$, and the function being differentiated on the left by $\mathcal{F}(x)$, then we just wrote that $\mathcal{F}'(x) = f(x)$, or, by definition, that $\mathcal{F}(x)$ is an antiderivative of $f(x)$. In other words, the Generalized Power Rule for derivatives tells us that

$$\int r\ u(x)^{r-1}\ \frac{du}{dx}\ dx \quad = \quad u(x)^r + C,$$

or, rewriting this in another way by replacing r by $r+1$, dividing both sides by $r+1$ and simplifying, we see that,

$$\boxed{\int u(x)^r\ \frac{du}{dx}\ dx = \frac{u(x)^{r+1}}{r+1} + C, \qquad r \neq -1,} \tag{6.4}$$

where now, the 'new' C is the 'old' C divided by $r+1$. Still, we replace this 'old' symbol by a generic symbol, C, always understood to denote an arbitrary constant. As is usual in this book, we may simplify the 'look' of (6.4) by replacing every occurrence of the symbols '$u(x)$' by our generic symbol, $\square$. In this case, (6.4) becomes

$$\int \square^r\ \frac{d\square}{dx}\ dx = \frac{\square^{r+1}}{r+1} + C, \qquad r \neq -1, \tag{6.5}$$

Let's see how this works in practice:

Example 250. Evaluate the (most general) antiderivative of f where $f(x) = \sqrt{5x+1}$, that is, find

$$\int \sqrt{5x+1}\ dx$$

Solution The first thing to do is to bring this into a more recognizable form. We want to try and use either one of (6.4) or (6.5). The square root symbol makes us think of a *power*, so we let $\square = 5x+1$. The whole thing now looks like

$$\int \sqrt{\square}\ dx = \int \square^{\frac{1}{2}}\ dx.$$

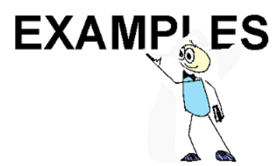
but, don't forget, in order to use (6.5) there should be the symbol $\frac{d\square}{dx}$ on the left. It's not there, right? So, what do we do? We'll have to **put it in there and also divide by it!** OK, so we need to calculate the derivative of $\square$, right? (because it has to be part of the left-hand side of the formula). We find $\square'(x) = \frac{d\square}{dx} = 5$. Now,

let's see what happens...

$$\begin{aligned}
\int \sqrt{5x+1}\, dx &= \int \square^{\frac{1}{2}}\, dx \\
&= \int \square^{\frac{1}{2}} \cdot 5 \cdot \frac{1}{5}\, dx, \quad \text{(Multiply and divide by } \square'(x) \text{ here)} \\
&= \frac{1}{5}\int \square^{\frac{1}{2}} \frac{d\square}{dx}\, dx, \quad \text{(now it looks like (6.5) with } r = 1/2\text{)}, \\
&= \frac{1}{5}\left(\frac{\square^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + C, \quad \text{(by (6.5))}, \\
&= \frac{1}{5}\left(\frac{2}{3}\square^{\frac{3}{2}}\right) + C, \\
&= \frac{2}{15}(5x+1)^{\frac{3}{2}} + C.
\end{aligned}$$

Okay, **but how do we know we have the right answer!?** Well, by definition of an antiderivative, *if we find the derivative of this antiderivative then we have to get the original result*, right? A simple check shows that this is true once we use the Generalized Power Rule, that is,

$$\begin{aligned}
\frac{d}{dx}\mathcal{F}(x) &= \frac{d}{dx}\left(\frac{2}{15}(5x+1)^{\frac{3}{2}} + C\right), \\
&= \frac{2}{15}\cdot\frac{3}{2}(5x+1)^{\frac{1}{2}}\cdot 5, \\
&= f(x).
\end{aligned}$$



The method just outlined in the previous example is the basis for one of many methods of integration, namely the **method of substitution**, a method which we'll see in the next chapter. The anatomy of an indefinite integral is described here:

$$\underset{\swarrow}{\int} \overbrace{f(x)}^{\text{integrand } f(x)} \underset{\searrow}{dx}$$

integral sign   free variable, x

Example 251. Evaluate $\int (2x^3+1)x^2\, dx$.

Solution Inspect this one closely. There are 'two ways' of doing this one! The first one involves recognizing the fact that the integrand is really a polynomial *in disguise* (when you multiply everything out). If you don't see this, then you can try to use (6.5). The first way of doing it is the easiest.

Method 1 We note that $(2x^3+1)x^2 = 2x^5 + x^2$. So,

$$\begin{aligned}
\int (2x^3+1)x^2\, dx &= \int (2x^5 + x^2)\, dx, \\
&= \frac{2x^6}{6} + \frac{x^3}{3} + C_1, \\
&= \frac{x^6 + x^3}{3} + C_1, \\
&= \mathcal{F}_1(x).
\end{aligned}$$

Method 2 We set $u(x) = 2x^3+1$, and find $\frac{du}{dx} = 6x^2$, which, aside from the number 6, is similar to the "other" term appearing in the integrand. So,

$$\begin{aligned}
\int (2x^3+1)x^2\ dx &= \int \frac{1}{6}(2x^3+1)(6x^2)\ dx, \\
&= \frac{1}{6}\int u(x)\frac{du}{dx}\ dx, \\
&= \frac{1}{6}\left(\frac{u(x)^2}{2}\right) + C_2, \quad \text{(by (6.4) with } r=1\text{)}, \\
&= \frac{1}{12}u(x)^2 + C_2, \\
&= \frac{1}{12}(2x^3+1)^2 + C_2, \qquad \text{where } C \text{ is some constant}, \\
&= \mathcal{F}_2(x).
\end{aligned}$$

Now you look at these two answers, $\mathcal{F}_1(x), \mathcal{F}_2(x)$ and you probably think: "This is nuts! These two answers don't *look* the same", and ... you're right! But, *believe it or not* they each represent the *same* general antiderivative. Let's have a closer look...

$$\begin{aligned}
\mathcal{F}_1(x) - \mathcal{F}_2(x) &= \left(\frac{x^6+x^3}{3} + C_1\right) - \left(\frac{1}{12}(2x^3+1)^2 + C_2\right), \\
&= \frac{x^6+x^3}{3} + C_1 - \left(\frac{1}{12}(4x^6+4x^3+1) + C_2\right), \\
&= \frac{x^6}{3} + \frac{x^3}{3} + C_1 - \left(\frac{x^6}{3} + \frac{x^3}{3} + \frac{1}{12} + C_2\right), \\
&= C_1 - \frac{1}{12} - C_2, \\
&= C \qquad \text{where } C \text{ is some NEW constant.}
\end{aligned}$$

We just showed something that we knew for a while now, namely, that any two antiderivatives always differ by a constant, (Table 6.1)! So, these two answers define the *same general antiderivative.*

Still not convinced? Let's see what happens if we were asked to find the antiderivative such that $\mathcal{F}_1(1) = -2$, say. We know that there is *only one answer* to this question, right? (no constants floating around!). OK, so we set $x = 1$ into the expression for $\mathcal{F}_1(x)$ and find $\mathcal{F}_1(1) = \frac{2}{3} + C_1$. But $\mathcal{F}_1(1) = -2$, because this is given. So, $\frac{2}{3} + C_1 = -2$, which says that $C_1 = -\frac{8}{3}$. So, the claim is that

$$\begin{aligned}
\mathcal{F}_1(x) &= \frac{x^6+x^3}{3} - \frac{8}{3}, \\
&= \frac{x^6+x^3-8}{3},
\end{aligned}$$

is the required antiderivative. We use exactly the same idea with $\mathcal{F}_2(x)$. We set $\mathcal{F}_2(1) = -2$ into the expression for $\mathcal{F}_2(x)$ and find that $C_2 = -2 - \frac{9}{12} = -\frac{11}{4}$. The expression for $\mathcal{F}_2(x)$ now looks like,

$$\begin{aligned}
\mathcal{F}_2(x) &= \frac{1}{12}(2x^3+1)^2 - \frac{11}{4}, \\
&= \frac{x^6+x^3}{3} + \frac{1}{12} - \frac{11}{4}, \\
&= \frac{x^6+x^3-8}{3},
\end{aligned}$$

just like before. So, even though the constants differ, the answer is the same. That's all.

SNAPSHOTS

Example 252. Evaluate $\int \sin^2 x \ \cos x \ dx$.

Solution We see a power, so we can start by trying $\square = \sin x$, so that we have a term that looks like $\square^2$. But we also need $\square'(x)$, right? From Chapter 3, we know that $\square'(x) = \cos x$ which is just the term we have! Okay, we were lucky. Using (6.5) with $r = 2$, we find

$$\begin{aligned}\int \sin^2 x \ \cos x \ dx &= \int \square^2 \frac{d\square}{dx} \ dx, \\ &= \frac{\square^3}{3} + C, \\ &= \frac{(\sin x)^3}{3} + C, \\ &= \frac{\sin^3 x}{3} + C.\end{aligned}$$

As a check we note that, using the Generalized Power Rule,

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin^3 x}{3} + C\right) &= \frac{3\sin^2 x}{3} D(\sin x) + 0, \\ &= \sin^2 x \cos x,\end{aligned}$$

so we found the correct form of the most general antiderivative (the one with the C in it).

Example 253. Let k be a real number (or constant). Find the most general antiderivative of f where $f(x) = e^{kx}$ where e is **Euler's number**, (Chapter 4).

Solution We want $\int e^{kx} \ dx$, right? There is a power here, but it's not easy to see what to do. How about we let $\square = e^x$? Then we can write the integrand as $\square^k$, so maybe there's some hope that we can use (6.5). But we also need $\square' = e^x$, remember (see Chapter 4)? So,

$$\begin{aligned}\int e^{kx} \ dx &= \int \square^k \frac{e^x}{e^x} \ dx, \quad \text{(because we need a } \square' \text{ in here)} \\ &= \int \square^k \frac{\square'}{\square} \ dx, \\ &= \int \square^{k-1} \frac{d\square}{dx} \ dx, \\ &= \frac{\square^k}{k} + C, \quad \text{(by (6.5) with } r = k-1\text{)}, \\ &= \frac{(e^x)^k}{k} + C, \\ &= \frac{e^{kx}}{k} + C,\end{aligned}$$

and this is the required antiderivative.

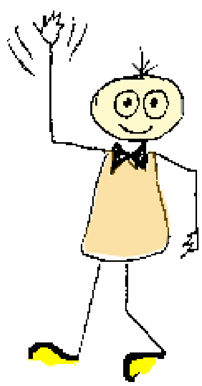
If $\mathcal{F}, \mathcal{G}$ are antiderivatives of f, g, respectively, and c is any constant (positive or negative), then the function $\mathcal{F}+c\cdot\mathcal{G}$ is an antiderivative of $f+c\cdot g$, that is,

$$\int^x (f(t)+c\cdot g(t))\ dt \quad = \quad \int^x f(t)\ dt + c\int^x g(t)\ dt.$$

If $\square, f$ represent any two differentiable functions, then

$$\begin{aligned} \int \square^r \frac{d\square}{dx}\ dx &= \frac{\square^{r+1}}{r+1}+C, \qquad r\neq -1,\\ \int \frac{d\square}{dx}\ dx &= \square + C,\\ \frac{d}{dx}\int^x f(t)\ dt &= f(x)+C, \end{aligned}$$

Table 6.2: Summary of Basic Formulae Regarding Antiderivatives



Example 254. Evaluate

$$\int x(1+x^2)^{\frac{3}{2}}\ dx.$$

Solution We want $\int x(1+x^2)^{3/2}\ dx$, right? Once again, there is a power here, and we can write the integrand as $x\ \square^{3/2}$, where $\square = 1+x^2$, so that we can recognize it as being of the form (6.5). What about the x? Well, this should be part of the $\square' = 2x$ - term, right? But we don't have the factor of '2' in the integrand ... no problem, because we can rewrite this as

$$\begin{aligned} \int (1+x^2)^{3/2}\ x\ dx &= \int \square^{3/2}\,\frac{2x}{2}\ dx, \qquad \text{(because we need a } \square' \text{ in here)}\\ &= \frac{1}{2}\int \square^{3/2}\square'\ dx,\\ &= \frac{1}{2}\int \square^{3/2}\frac{d\square}{dx}\ dx,\\ &= \frac{1}{2}\left(\frac{2}{5}\square^{5/2}\right)+C, \qquad \text{(by (6.5) with } r=3/2\text{)},\\ &= \frac{1}{5}\square^{5/2}+C,\\ &= \frac{1}{5}(1+x^2)^{5/2}+C, \end{aligned}$$

which is the required antiderivative or indefinite integral. We recap the main results of this section in Table 6.2, above.

Some simple applications

We recall that a **differential equation** is basically an equation which relates the derivatives of some unknown function, a function that we seek. If we can find it, and it is sufficiently differentiable, we call it a **solution** of the differential equation. One of the earliest uses of the indefinite integral was in solving simple (and not so simple) differential equations, like the one in the next example ...

Example 255. Solve the equation $\dfrac{dy}{dx} = \dfrac{x}{y}, \quad y > 0.$

Solution This means we seek a differentiable function $y(x)$ whose derivative at x is equal to $\frac{x}{y(x)}$, if $y(x) > 0$. To do this we "separate variables" *i.e.* put all the "y's" on one side (usually the left) and the "x's" on the other (the right side), a very old idea which can be traced right down to Johann (John) Bernoulli, 300 years ago! Thus, this unknown function $y = y(x)$ has the property that

$$y(x)\frac{dy}{dx} = x$$

and so these functions must have the same antiderivative (up to a constant), *i.e.*

$$\begin{aligned} \text{antiderivative of} \left(y\frac{dy}{dx} \right) &= \text{antiderivative of } (x), \text{ or,} \\ \int y\frac{dy}{dx}\, dx &= \int x\, dx. \end{aligned}$$

But, on the left-hand-side we have by (6.4) with $r = 1$, and $y = u$,

$$\int y\frac{dy}{dx}\, dx = \frac{y(x)^2}{2} + c_1,$$

where c_1 is a constant, while on the right-hand-side,

$$\int x dx = \frac{x^2}{2} + c_2,$$

where c_2 is another constant. Combining the last two equalities gives

$$\begin{aligned} \frac{y(x)^2}{2} + c_1 &= \frac{x^2}{2} + c_2 \\ \frac{y(x)^2}{2} &= \frac{x^2}{2} + c_3, \quad (\text{where } c_3 = c_2 - c_1) \\ y(x)^2 &= x^2 + C, \quad (C = 2c_3) \\ y(x) &= \sqrt{x^2 + C}, \quad (\text{since } \ y(x) > 0, \text{ we choose the '+' square root}). \end{aligned}$$

THE CONSTANT C IS DETERMINED BY AN INITIAL CONDITION!! That is, if we are given the value of y somewhere, then we can determine C. For example, the solution of $\frac{dy}{dx} = \frac{x}{y}$ such that $y(0) = 1$ is given by $y(x) = \sqrt{x^2+1}$ (because $1 = y(0) = \sqrt{x^2+C}\Big|_{x=0} = \sqrt{C}$ so $C = 1$ and the results follows.)

As usual, we can **check this answer** by *substituting back into the equation.* We want $\frac{dy}{dx} = \frac{x}{y}$. Now, our candidate '$y(x)$' has the following property, that,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(2x)(x^2 + C)^{-\frac{1}{2}}, \\ &= x(x^2 + C)^{-\frac{1}{2}}, \\ &= \frac{x}{\sqrt{x^2 + C}} = \frac{x}{y(x)}, \\ &= \frac{x}{y}, \end{aligned}$$

which is what is required.

Notice that constant of integration, C, which is still 'floating around'. As we pointed out when we studied *Laws of Growth and Decay* in Chapter 4.7, (where it looked like

$N(0)$ there), this constant of integration C is necessary in order to solve so-called **initial value problems**. Let's recall how this is done.

Example 256. On the moon, acceleration due to gravity is 1.6 m/sec^2. If a rock is dropped into a crevasse, how fast will it be going just before it hits the bottom 30 seconds later? How far down the crevasse is the rock after 30 seconds?

Solution From basic Physics we know $\frac{d^2s}{dt^2} = 1.6$ m/sec^2 where $s(t)$ is the distance of the rock at time t from its "original" position. We want its speed after 30 seconds, *i.e.*

$$\frac{ds}{dt}(30) = ?$$

Now,

$$\frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = 1.6, \quad \text{(where we forget about units here.)}$$

Taking the antiderivative of both sides gives,

$$\begin{aligned}
\int \frac{d}{dt}\left(\frac{ds}{dt}\right)\, dt &= \int (1.6)\, dt, \quad \text{(we'll put the } C \text{ in later ...)}\\
\int \frac{d}{dt}(\Box)\, dt &= \int (1.6)\, dt, \quad \left(\Box = \frac{ds}{dt} \text{ here}\right)\\
\Box &= \int (1.6)\, dt, \quad \text{(by (6.5) with } r = 0, \text{ since } \Box^0 = 1, \text{ yes! ...)}\\
\frac{ds}{dt} &= (1.6)t + C,
\end{aligned}$$

is the speed of the rock at any time t! Since the rock is merely "dropped", its initial velocity is zero, and mathematically this means

$$\frac{ds}{dt}(0) = 0.$$

Now, this 'initial condition' determines C, since

$$0 = \frac{ds}{dt}(0) = (1.6)\cdot 0 + C = C$$

so $C = 0$. Thus, in general,

$$\frac{ds}{dt} = (1.6)t$$

gives the speed of the rock at any time t, while after 30 seconds its speed is obtained by setting $t = 30$ into the expression for the speed, that is,

$$\frac{ds}{dt}(30) = 1.6 \text{ m/sec}^2 \cdot 30 \text{ sec} = 48 \text{ m/sec}.$$

NOTE:

We only need *ONE* initial condition in order to determine C in the above.

How far down the crevasse is the rock after 30 seconds? Well, we need *another* initial condition in order to get this, because even though we know its speed, we don't know where the rock is *initially*! So, let's say that

$$s(0) = 0,$$

i.e., its initial position is 0 units, which is the location of some arbitrary point which we decide upon before hand (e.g., the height of our outstretched horizontal arm above the lunar surface could be defined to be $s = 0$). Then

$$\begin{aligned}
\frac{ds}{dt} &= (1.6)t, \\
\int \frac{ds}{dt}\, dt &= \int (1.6)t\ dt, \\
s(t) &= (1.6)\frac{t^2}{2} + C, \quad \text{(by (6.4) with } r = 0,\ s = u,\ t = x)
\end{aligned}$$

is the "distance" at time t. Since $s(0) = 0$, we have $C = 0$ again, (just set $t = 0$, here). Thus, after 30 seconds,

$$\begin{aligned}
s(30) &= (1.6) \cdot \frac{900}{2} = (1.6) \cdot (450), \\
&= 720 \text{ m},
\end{aligned}$$

a "deep" crevasse!

MORAL: We needed 2 "initial" conditions to determine the unknown function in this "second order" differential equation and, as we have seen earlier, only *one* condition is needed for determining the antiderivative of a function explicitly. The general result says that *we need n initial conditions to explicitly determine the unknown function of a differential equation of order n.*

Example 257. The escape velocity v_0 at the surface of a star of radius R, and mass M (assumed constant) is given by

$$v_0 = \sqrt{\frac{2GM}{R}}.$$

Given that the star is "collapsing" according to the formula

$$R(t) = \frac{1}{t^2},$$

how long will it take the star to become a "black hole"!? Show that the rate of change of its 'escape velocity' is a constant for all t. [You're given that $M = 10^{31}$, (10 solar masses), $G = 6.7 \times 10^{-11}$, and that the speed of light, $c = 3 \times 10^8$, all in MKS units.]

A **black hole** is the remnant of a collapsed star. The gravitational field around a black hole is so intense that even light cannot escape from it. Thus, it appears "black" to the human eye.

Solution Since R is a function of t, the initial velocity also becomes a function of t, right? We're assuming that its mass remains constant for simplicity. It's easy to calculate $\sqrt{2GM} \approx 3.7 \times 10^{10}$ so $v_0(t) = (3.7 \times 10^{10})\ t$. Now, we want $v_0(t) > c$, because then, by definition, the *photons* inside can't escape through the surface. This means that we need to solve the inequality $v_0(t) > c = 3 \times 10^8$ which holds only if $(3.7 \times 10^{10})\ t > 3 \times 10^8$ or, solving for t, we find $t > 10^{-2}$ or $t > 0.01$ seconds! This means that after $t = 0.01$ sec. light will no longer be able to escape from its surface, and so it will appear 'black'.

In this case, $v_0(t) = \sqrt{2GM/R} = \sqrt{2GM}\ t$, because of the assumption on $R(t)$, so $v_0'(t) = D(v_0(t)) = \sqrt{2GM}$ is a constant.

Example 258. Solve the equation

$$\frac{d^3y}{dx^3} = 6$$

given that $y = 5$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = -8$ when $x = 0$.

Solution The initial conditions here are given by $y(0) = 5$, $\frac{dy}{dx}(0) = 0$ and $\frac{d^2y}{dx^2}(0) = -8$. Note that there are 3 initial conditions and the equation is of "order" 3 (because the highest order derivative which appears there is of order 3). Now,

$$\begin{aligned}
\frac{d^3y}{dx^3} &= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 6, \quad \text{(is given, so)} \\
\int \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)\, dx &= \int 6\, dx, \\
&= 6x + C, \quad \text{or, if we set } \square = y'' \text{ and use Table 6.2 ... ,} \\
\frac{d^2y}{dx^2} &= 6x + C,
\end{aligned}$$

where we've "reduced" the order of the differential equation by 1 after the integration .... We determine C using the "initial condition" $\frac{d^2y}{dx^2}(0) = -8$.

So, we set $x = 0$ in the formula $\frac{d^2y}{dx^2} = 6x + C$ to find

$$\begin{aligned}
-8 &= 6 \cdot 0 + C = C, \quad \text{and so,} \\
\frac{d^2y}{dx^2} &= 6x + C, \\
&= 6x - 8.
\end{aligned}$$

"Integrating" this latest formula for $y''(x)$ once again we find

$$\frac{dy}{dx} = 6\frac{x^2}{2} - 8x + C$$

where C is some generic constant (not related to the previous one). To determine C we set $x = 0$ in preceding formula and find

$$0 = y'(0) = 3 \cdot 0^2 - (8 \cdot 0) + C = C.$$

Now, $C = 0$ and so because of this,

$$\begin{aligned}
\frac{dy}{dx} &= 3x^2 - 8x + C, \\
&= 3x^2 - 8x.
\end{aligned}$$

Finally, one last integration (our third such integration...) gives us

$$y(x) = 3\frac{x^3}{3} - 8\frac{x^2}{2} + C$$

where C is a generic constant again. Setting $x = 0$ here, gives us $C = 5$, (since $y(0) = 5$) and so,

$$\begin{aligned}
y(x) &= x^3 - 4x^2 + C, \\
&= x^3 - 4x^2 + 5,
\end{aligned}$$

is the required answer. *Check this by differentiating and using initial conditions.*

NOTES:

Exercise Set 28.

Find the required antiderivatives or indefinite integrals. Don't forget to add a generic constant C at the end.

1. $\int -5\ dx$
2. $\int 1\ dx$
3. $\int 0\ dx$
4. $\int x^{0.6}\ dx$
5. $\int^x 3t\ dt$
6. $\int (x-1)\ dx$
7. $\int (x^2+1)\ dx$
8. $\int (2x^2+x-1)\ dx$
9. $\int^x 3u\ du$
10. $\int (4x^3+2x-1.314)\ dx$
11. $\int \sqrt{2x-2}\ dx$
12. $\int \sqrt{3x+4}\ dx$
13. $\int \sqrt{1-x}\ dx$, careful here: Look out for the minus sign!
14. $\int x\sqrt{4x^2+1}\ dx$
15. $\int x\sqrt{1-2x^2}\ dx$
16. $\int x(1+x^2)^{0.75}\ dx$
17. $\int x^2(2+x^3)^{2/3}\ dx$
18. $\int -x^3\sqrt{4+9x^4}\ dx$
19. $\int x^{1.4}\sqrt{1+x^{2.4}}\ dx$

Evaluate the following antiderivatives, $\mathcal{F}$, under the given initial condition. No generic constant should appear in your final answer.

20. $\int \sin^3 x \cos x\ dx$ given that $\mathcal{F}(0) = -1$.
21. $\int \cos^2 x \sin x\ dx$ given that $\mathcal{F}(0) = 0$.
22. $\int e^{-2x}\ dx$ given that $\mathcal{F}(1) = 0$.

Find the solution of the following differential equations subject to the indicated initial condition(s).

23. $$\frac{dy}{dx} = \frac{x^2}{y^3}, \quad y(0) = 1.$$

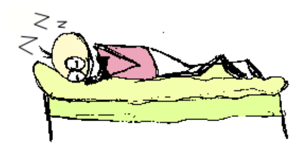
    - Leave your solution in *implicit form*, that is, don't solve for y explicitly.

24. $$\frac{d^2y}{dx^2} = 12x^2, \quad y(0) = -1, \quad y'(0) = 0.$$

25. $$\frac{d^3y}{dx^3} = 24x, \quad y(0) = -1, \quad y'(0) = 0, \quad y''(0) = 0.$$

Suggested Homework Set 21. *Do problems 4, 8, 13, 15, 18, 22, 23.*

NOTES:

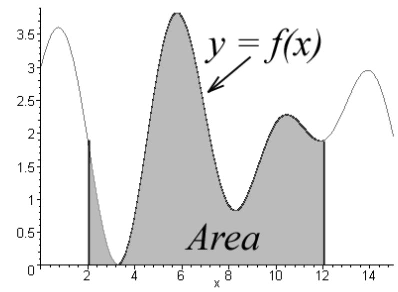


6.2 Definite Integrals

You'll remember that the principal geometric interpretation of the **derivative** of a function, f, is its representation as the slope of the tangent line to the graph of the given function. So far, our presentation of the **antiderivative** has focussed upon the actual finding of antiderivatives of given functions. In this section, we develop the machinery necessary to arrive at a **geometric interpretation of the antiderivative** of a function. For example, let $\mathcal{F}$ be any given antiderivative of f on an interval I containing the points a, b, where we assume that $f(x) \geq 0$ there. The number $\mathcal{F}(b) - \mathcal{F}(a)$, will be called the **definite integral** of f between the limits $x = a$ and $x = b$, and it will be denoted generally by the symbol on the left of (6.6),

$$\int_a^b f(x)\ dx = \mathcal{F}(b) - \mathcal{F}(a). \tag{6.6}$$

This number turns out to be the **area** of the closed region under the graph of the given function and between the lines $x = a$, $x = b$ and the $x-$axis, (see Figure 115). Even if f is not positive in the interval, there is still some area-related interpretation of this definite integral, and we'll explore this below. In order to arrive at this area interpretation of the definite integral of f, we'll use the so-called **Riemann Integral** which will define the *area* concept once and for all, along with the Mean Value Theorem of Chapter 3, which will relate this area to the difference $\mathcal{F}(b) - \mathcal{F}(a)$ of any one of its antiderivatives, $\mathcal{F}$. This will result in the Fundamental Theorem of Calculus which is the source of many applications of the integral. Let's see how this works...



The shaded area is given by $\mathcal{F}(12) - \mathcal{F}(2)$, where $\mathcal{F}$ is any antiderivative of f.

Figure 115.

Let $[a, b]$ be a given interval, f a given function with domain $[a, b]$, assumed **continuous** on its domain. The **definite integral of f over** $[a, b]$ is, by definition, an expression of the form

$$\int_a^b f(x)\ dx = \mathcal{F}(b) - \mathcal{F}(a),$$

where $\mathcal{F}' = f$, *i.e.,* $\mathcal{F}$ is some antiderivative of f. Well, the right-hand side should be some number, right? And, this number *shouldn't change* with the choice of the antiderivative (because it is a number, not a variable). So, you may be wondering why this definition involves *any* antiderivative of f! This is okay. The idea behind this is the fact that antiderivatives differ from one another by a constant, C, (see Table 6.1). So, if we let $\mathcal{F}_1, \mathcal{F}_2$ be any two antiderivatives of f, then, by the discussion following Table 6.1, there is a constant C such that $\mathcal{F}_1 - \mathcal{F}_2 = C$, and so,

$$\begin{aligned} \mathcal{F}_1(b) - \mathcal{F}_1(a) &= (\mathcal{F}_2(b) + C) - (\mathcal{F}_2(a) + C), \\ &= \mathcal{F}_2(b) - \mathcal{F}_2(a), \end{aligned}$$

and the constant C cancels out! Because of this, it follows that we can put **any** antiderivative in the right side of (6.6). Let's look at a few examples.

Example 259. Evaluate $\displaystyle\int_0^1 (2x + 1)\ dx$.

Solution Using the methods of the preceding section, *one* antiderivative of $2x + 1$ is given by

$$\int (2x + 1)\ dx = x^2 + x,$$

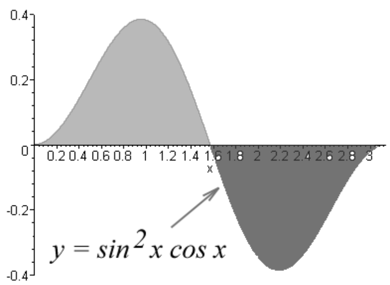
where we have chosen $C = 0$ here because we want *one* antiderivative, not the most

general one (so we can leave out the C). Thus $\mathcal{F}(x) = x^2 + x$. Hence, by definition,

$$\begin{aligned} \int_0^1 (2x+1)\, dx &= \mathcal{F}(1) - \mathcal{F}(0) \\ &= (1^2+1) - (0^2+0) \\ &= 2. \end{aligned}$$

Example 260. Evaluate $\displaystyle\int_0^\pi \sin^2 x \cos x \, dx$.

Solution Refer to Example 252. Using this Example we see that *one* antiderivative of this integrand is given by



The two areas in the shaded regions 'cancel out' because they are equal but opposite in sign. The general idea is that the area of the region above the $x-$axis is positive, while the area of the region below the $x-$axis is negative.

Figure 116.

$$\int \sin^2 x \cos x \, dx = \frac{\sin^3 x}{3},$$

where, once again, we have chosen $C = 0$ here because we want *one* antiderivative, not the most general one (so we can leave out the C). Thus,

$$\mathcal{F}(x) = \frac{\sin^3 x}{3}.$$

Hence, by definition,

$$\begin{aligned} \int_0^\pi \sin^2 x \cos x \, dx &= \mathcal{F}(\pi) - \mathcal{F}(0), \\ &= (\frac{\sin^3 \pi}{3}) - (\frac{\sin^3 0}{3}), \\ &= 0 - 0, \quad \text{(because } \sin\pi = 0 = \sin 0\text{)}, \\ &= 0. \end{aligned}$$

It's okay for a definite integral to be equal to zero because, the 'areas' we spoke of above 'cancel out', see Figure 116.

Example 261. Evaluate $\displaystyle\int_{-1}^{+1} x^2 \, dx$.

Solution In this example we can choose

$$\begin{aligned} \mathcal{F}(x) &= \int x^2 \, dx = \frac{x^3}{3}. \\ \text{Thus,} \quad \int_{-1}^{+1} x^2 \, dx &= \mathcal{F}(+1) - \mathcal{F}(-1), \\ &= \left(\frac{1^3}{3}\right) - \left(\frac{(-1)^3}{3}\right), \\ &= \left(\frac{1}{3}\right) - \left(\frac{-1}{3}\right), \\ &= \frac{2}{3.} \end{aligned}$$

In this case, the 'area' we spoke of earlier is *double* the area of the region under the graph of f between the lines $x = 0$, $x = 1$, and the $x-$axis, (see Figure 117).

The main properties of the definite integral of an arbitrary function are listed in Table 6.3. These properties can be verified by using the definition of an antiderivative,

$\mathcal{F}$, of f, and the methods in Chapters 3 and 5.

For example, in order to verify item 5 in Table 6.3, we proceed as follows: We let $\mathcal{F}(x) = \int f(x)\, dx$ and assume that $f(x) > 0$. Using the definition of the definite integral we must show that $\mathcal{F}(b) - \mathcal{F}(a) > 0$, right? But, by definition of an antiderivative, we know that $\mathcal{F}'(x) = f(x)$ and that $f(x) > 0$, (because this is given). This means that $\mathcal{F}'(x) > 0$ which implies that $\mathcal{F}$ is an increasing function over $[a, b]$. But this last conclusion means that if $a < b$ then $\mathcal{F}(a) < \mathcal{F}(b)$, which, in turn, means that $\mathcal{F}(b) - \mathcal{F}(a) > 0$. Now, if $f(x)$ is equal to zero somewhere, the argument is a little more complicated. If you're interested, see the margin on the right for the idea ...

Let $f(x) \geq 0$. We define a new function f_ε by setting its values $f_\varepsilon(x) = f(x) + \varepsilon$, where $\varepsilon > 0$ is an arbitrary small but positive number. This new function, $f_\varepsilon(x) > 0$, right? It also has an antiderivative, $\mathcal{F}_\varepsilon$, given by $\mathcal{F}_\varepsilon(x) = \mathcal{F}(x) + \varepsilon \cdot x$. We can apply the argument of the paragraph to $\mathcal{F}_\varepsilon$ instead of $\mathcal{F}$, and obtain the conclusion that $\mathcal{F}_\varepsilon(b) - \mathcal{F}_\varepsilon(a) = (\mathcal{F}(b) - \mathcal{F}(a)) + \varepsilon \cdot (b - a) > 0$. But this is true for each $\varepsilon > 0$. So, we can take the limit as $\varepsilon \to 0$ and find that $\mathcal{F}(b) - \mathcal{F}(a) \geq 0$.

Properties of the Definite Integral

1. $\displaystyle\int_a^a f(x)\, dx = 0$
2. $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
3. If k is a constant, then $\displaystyle\int_a^b kf(x)\, dx = k\int_a^b f(x)\, dx$
4. $\displaystyle\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$
5. If $f(x) \geq 0$ and $a \leq b$, then $\displaystyle\int_a^b f(x)\, dx \geq 0$
6. If $f(x) \leq g(x)$, for each x in $[a, b]$, then $\displaystyle\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$
7. $\displaystyle\left(\min_{[a,b]} f(x)\right)(b-a) \leq \int_a^b f(x)\, dx \leq \left(\max_{[a,b]} f(x)\right)(b-a),$

   where the symbols on the left (resp. right) denote the (global) minimum and maximum values of f over $[a, b]$, (see Chapter 5).
8. If c is any number (or variable) for which $\mathcal{F}(c)$ is defined, then $\displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

Table 6.3: Properties of the Definite Integral

Example 262. Evaluate $\displaystyle\int_0^1 \frac{1}{1+x^2}\, dx.$

Solution From Chapter 3, we know that

$$\frac{d}{dx}\mathrm{Arctan}\, x = \frac{1}{1+x^2}.$$

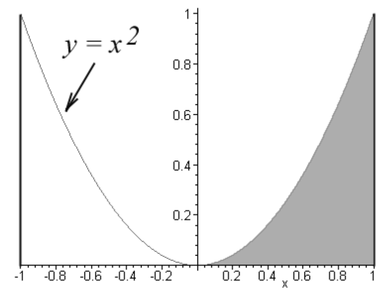
But, by definition of the antiderivative, this means that $\mathcal{F}(x) = \mathrm{Arctan}\, x$ is the

antiderivative of $(1+x^2)^{-1}$, right? That is, we can infer that

$$\int \frac{1}{1+x^2}\ dx = \operatorname{Arctan} x + C,$$

where C is a constant. Now, by definition of the definite integral, we can set $C = 0$ for convenience, and find

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}\ dx &= \mathcal{F}(1) - \mathcal{F}(0),\\ &= \operatorname{Arctan} 1 - \operatorname{Arctan} 0,\\ &= \frac{\pi}{4}, \quad \text{(since } \tan(\pi/4) = 1 \text{ and } \tan 0 = 0).\end{aligned}$$



The area of the shaded region is one-half the area under the curve $y = x^2$ between the points -1 and $+1$.

Figure 117.

$\boxed{\textbf{Example 263.}}$ Evaluate $\displaystyle\int_0^1 xe^{-x^2}\ dx$.

Solution This one is a little tricky. We have the product of a function, x, and an exponential of a function, e^{-x^2}, which makes us think about a composition of two functions. Next, we're asking for an antiderivative of such a combination of functions. So, we should be thinking about the Chain Rule for derivatives (because it deals with compositions) applied to exponentials. This brings to mind the following formula from Chapter 4, namely, if u (or $\square$), is a differentiable function then

$$\frac{d}{dx}e^{u(x)} = e^{u(x)}\frac{du}{dx}.$$

In terms of antiderivatives this is really saying,

$$\int e^{u(x)}\frac{du}{dx}\ dx = e^{u(x)} + C, \tag{6.7}$$

or the term on the right of (6.7) is the antiderivative of the integrand on the left. Now, let's apply this formula to the problem at hand. We have little choice but to use the identification $u(x) = -x^2$ so that our preceding integral in (6.7) will even remotely look like the one we want to evaluate, right? What about the $u'(x)$-term? Well, in this case, $u'(x) = -2x$ and we 'almost' have this term in our integrand (we're just missing the constant, -2). So, we use the ideas of the preceding Section and write,

$$\begin{aligned}\int xe^{-x^2}\ dx &= \int (-2x)\ (-\frac{1}{2})e^{-x^2}\ dx,\\ &= -\frac{1}{2}\int (-2x)e^{-x^2}\ dx,\\ &= -\frac{1}{2}\int e^{u(x)}\frac{du}{dx}\ dx, \quad \text{(since } u(x) = -x^2\text{, here)},\\ &= -\frac{1}{2}e^{u(x)} + C, \quad \text{(by (6.7))},\\ &= -\frac{1}{2}e^{-x^2} + C,\end{aligned}$$

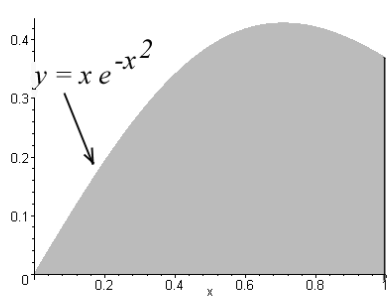


Figure 118.

and this is the most general antiderivative, $\mathcal{F}(x)$. The required definite integral is

Summary

In order to evaluate the definite integral

$$\int_a^b f(x)\ dx,$$

we must

1. Find an antiderivative, $\mathcal{F}$, (*any* antiderivative will do), and
2. Evaluate the quantity $\mathcal{F}(b) - \mathcal{F}(a)$.

Table 6.4: How to Evaluate a Definite Integral

therefore given by (let's choose $C = 0$ again)

$$\begin{aligned}
\int_0^1 xe^{-x^2}\ dx &= \mathcal{F}(1) - \mathcal{F}(0), \\
&= -\frac{1}{2}e^{-1^2} - \left(-\frac{1}{2}\right)e^{-0^2}, \\
&= -\frac{1}{2}\left(e^{-1} - 1\right), \qquad (\text{since } e^0 = 1), \\
&= \frac{1}{2}\left(1 - \frac{1}{e}\right), \\
&= \frac{e-1}{2e}.
\end{aligned}$$

This last number is the area of the shaded region in Figure 118.

NOTATION

There is a universal convention for writing down the results of a definite integral. Let $\mathcal{F}$ denote any antiderivative of f. Instead of writing

$$\begin{aligned}
\int_a^b f(x)\ dx &= \mathcal{F}(b) - \mathcal{F}(a), \quad \text{we can write this briefly as,} \\
&= \mathcal{F}(x)\big|_{x=a}^{x=b}, \qquad \text{or,} \\
&= \mathcal{F}(x)\big|_a^b
\end{aligned}$$

Next, we summarize in Tables 6.5, 6.6, 6.7 those basic formulae regarding antiderivatives which are most useful in Calculus.

NOTES:

Derivatives	**Antiderivatives**
$D(\Box^{r+1}) = (r+1)\Box^r D\Box, \quad r \neq -1.$	$\int \Box^r \ \frac{d\Box}{dx} \ dx = \frac{\Box^{r+1}}{r+1} + C,$
$D(e^{\Box}) = e^{\Box} D\Box$	$\int e^{\Box} \frac{d\Box}{dx} \ dx = e^{\Box} + C$
$D(a^{\Box}) = a^{\Box} \ \ln a \ D\Box, \quad a > 0.$	$\int a^{\Box} \frac{d\Box}{dx} \ dx = \frac{a^{\Box}}{\ln a} + C$
$D(\ln\lvert\Box\rvert) = \frac{1}{\Box} D\Box, \ \ \Box \neq 0.$	$\int \frac{1}{\Box} \frac{d\Box}{dx} \ dx = \ln\lvert\Box\rvert + C$
$D(\log_a \lvert\Box\rvert) = \frac{1}{\ln a} \ \frac{1}{\Box} D\Box, \ \ \Box \neq 0, \ a > 0.$	Not needed.

Table 6.5: Antiderivatives of Power and Exponential Functions

NOTES:

Derivatives	**Antiderivatives**
$D(\sin\Box) = \cos\Box \; D\Box$	$\int \cos\Box \; \frac{d\Box}{dx} \; dx = \sin\Box \; + C,$
$D(\cos\Box) = -\sin\Box \; D\Box$	$\int \sin\Box \; \frac{d\Box}{dx} \; dx = -\cos\Box \; + C,$
$D(\tan\Box) = \sec^2\Box \; D\Box$	$\int \sec^2\Box \; \frac{d\Box}{dx} \; dx = \tan\Box \; + C,$
$D(\cot\Box) = -\csc^2\Box \; D\Box$	$\int \csc^2\Box \; \frac{d\Box}{dx} \; dx = -\cot\Box \; + C,$
$D(\sec\Box) = \sec\Box \; \tan\Box \; D\Box$	$\int \sec\Box \tan\Box \; \frac{d\Box}{dx} \; dx = \sec\Box \; + C,$
$D(\csc\Box) = -\csc\Box \; \cot\Box \; D\Box$	$\int \csc\Box \cot\Box \; \frac{d\Box}{dx} \; dx = -\csc\Box \; + C,$

Table 6.6: Antiderivatives of Trigonometric Functions

Derivatives	**Antiderivatives**
$D(\text{Arcsin}\,\Box) = \frac{1}{\sqrt{1-\Box^2}} \; D\Box$	$\int \frac{1}{\sqrt{1-\Box^2}} \frac{d\Box}{dx} \; dx = \text{Arcsin}\,\Box + C$
$D(\text{Arccos}\,\Box) = -\frac{1}{\sqrt{1-\Box^2}} \; D\Box$	Not needed, use one above.
$D(\text{Arctan}\,\Box) = \frac{1}{1+\Box^2} \; D\Box$	$\int \frac{1}{1+\Box^2} \frac{d\Box}{dx} \; dx = \text{Arctan}\,\Box \; + C$
$D(\text{Arccot}\,\Box) = -\frac{1}{1+\Box^2} \; D\Box$	Not needed, use one above.
$D(\text{Arcsec}\,\Box) = \frac{1}{\lvert\Box\rvert\sqrt{\Box^2-1}} \; D\Box$	$\int \frac{1}{\lvert\Box\rvert\sqrt{\Box^2-1}} \frac{d\Box}{dx} \; dx = \text{Arcsec}\,\Box + C$
$D(\text{Arccsc}\,\Box) = -\frac{1}{\lvert\Box\rvert\sqrt{\Box^2-1}} \; D\Box$	Not needed, use one above.

Table 6.7: Antiderivatives Related to Inverse Trigonometric Functions

Exercise Set 29.

Evaluate the following definite integrals.

1. $\displaystyle\int_0^1 3x\ dx$
2. $\displaystyle\int_{-1}^0 x\ dx$
3. $\displaystyle\int_{-1}^1 x^3\ dx$
4. $\displaystyle\int_0^2 (x^2 - 2x)\ dx$
5. $\displaystyle\int_{-2}^2 (4 - 4x^3)\ dx$
6. $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \cos x\ dx$
7. $\displaystyle\int_0^{\pi} \cos^2 x \sin x\ dx$
8. $\displaystyle\int_{-\pi}^{\frac{\pi}{2}} \sin^3 x \cos x\ dx$
9. $\displaystyle\int_{1.5}^{1.2} (2x - x^2)\ dx$
   - *Hint:* Use Table 6.3, (2).
10. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx$
    - *Hint:* Think about the Arcsine function. Notice that the integrand approaches $+\infty$ as $x \to 1^-$. Thus the value of this integral is the area of an *unbounded region*, (*i.e.,* a region which is *infinite* in extent, see Figure 119)!
11. $\displaystyle\int_0^1 xe^{x^2}\ dx$
12. $\displaystyle\int_0^2 4xe^{-x^2}\ dx$
13. $\displaystyle\int_0^1 3^x\ dx$
    - *Hint:* Think about the derivative of the power function, 3^x, then guess its antiderivative.
14. $\displaystyle\int_{\square}^{\triangle} e^{3x}\ dx$
15. $\displaystyle\int_0^{0.5} \frac{x}{\sqrt{1-x^2}}\ dx$
    - *Hint:* Rewrite the integrand as $x(1-x^2)^{-1/2}$. Now, let $\square = 1 - x^2$, find $D\square$, and use equation (6.5).
16. $\displaystyle\int_0^1 x\ 2^{x^2+1}\ dx$
    - *Hint:* Find the derivative of a general power function.
17. $\displaystyle\int_0^{\frac{\sqrt{\pi}}{2}} x\ \sec(x^2)\ \tan(x^2)\ dx$
    - *Hint:* What is the derivative of the secant function?

18. $\displaystyle\int_{-1}^{1} \frac{x}{1+x^4}\, dx$

    • *Hint:* Let $\Box = x^2$ and think 'Arctangent'.

19. Show that $\displaystyle\frac{d}{dx}\int_0^{x^2} e^t\, dt = 2x\, e^{x^2}$ in the following steps:

    • a) Let $\mathcal{F}(t) = e^t$ be an antiderivative of f where $f(t) = e^t$. Show that the definite integral, by itself, is given by

$$\int_0^{x^2} e^t\, dt \quad = \quad \mathcal{F}(x^2) - \mathcal{F}(0).$$

    • b) Next, use the Chain Rule for derivatives and the definition of $\mathcal{F}$ to show that

$$\begin{aligned} \frac{d}{dx}\left(\mathcal{F}(x^2) - \mathcal{F}(0)\right) &= 2x\ f(x^2) - 0, \\ &= 2x\ f(x^2). \end{aligned}$$

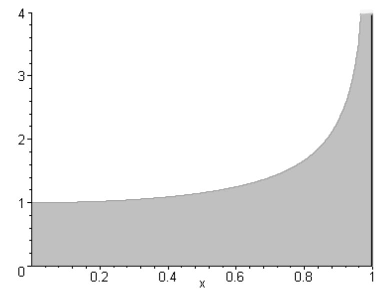
    • c) Conclude that $\displaystyle\frac{d}{dx}\int_0^{x^2} e^t\, dt = 2x\, e^{x^2}$.

    This result is special case of a more general result called **Leibniz's Rule**.

20. Recall the definitions of an *odd* and an *even* function (see Chapter 5). Use a geometrical argument and the ideas in Figures 116, 117 to convince yourself that on a given **symmetric interval**, $[-a, a]$, where $a > 0$, we have:

    • If f is even, then $\displaystyle\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$,

    • If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\, dx = 0$. (See Example 18 above with $a = 1$.)



The shaded area is actually infinite (towards the top of the page) in its extent. The area under the graph of the Arcsine function is, however, finite.

Figure 119.

Suggested Homework Set 22. *Suggested problems: 7, 11, 13, 15, 17, 18*

NOTES:

6.3 The Summation Convention

This section summarizes the basic material and notation used in the theory of the integral. The **"*summation*"** sign Σ (upper case Greek letter and corresponding to our "s" and read as "sigma") is defined by the symbols

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n, \tag{6.8}$$

where the a_i are all real numbers. This last display is read as "The sum of the numbers a_i as i runs from 1 to n is equal to ..." (whatever). Okay, what this means is that the group of symbols on the left of this equation are *shorthand* for the sum of the n numbers on the right. Here, the numbers $\{a_n\}$ represent a *finite sequence* of numbers (see Chapter 2 for more details on sequences). For example, if we are asked to find the sum of the first 15 whole numbers and obtain 120, the equivalent statement

$$120 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15, \tag{6.9}$$

would be rewritten more simply as

$$\sum_{i=1}^{15} i = 120, \tag{6.10}$$

because it takes up less room when you write it down. Here, the sequence $\{a_n\}$ is characterized by setting its n^{th}-term equal to n, that is, $a_n = n$, where $n \geq 1$. Replacing n by i we get $a_i = i$ (because i is just another symbol for an integer) and so we find that the right-side of (6.8) becomes the right-side of (6.9) which can be rewritten more briefly as (6.10).

The formula for the sum of the first n squares of the integers is given by:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

See Example 264, with $n = 5$. This formula is shown by using the *Principle of Mathematical Induction.* Briefly stated, a demonstration using this Principle goes like this: Let S_n be the n^{th} statement of a string of (mathematical) statements. Let's say that we can prove that S_1 is a true statement. Next, we assume that S_n is a 'true' statement and *on this basis* we prove that S_{n+1} is also a true statement. Then, the Principle says that each statement S_n must be a true (mathematical) statement.

Figure 120.

Example 264. Evaluate $\sum_{i=1}^{5} i^2$.

Solution By definition, this statement means that we want to find the sum of numbers of the form $a_i = i^2$ as i runs from 1 to 5. In other words, we want the value of the quantity

$$\begin{aligned}\sum_{i=1}^{5} i^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2, \\ &= 1 + 4 + 9 + 16 + 25, \\ &= 55, \quad (\textit{see Fig. } 120.)\end{aligned}$$

There may be 'other symbols' appearing in the a_i (like $x, y, ...$),and that's OK, don't worry. This is one reason that we have to specify which symbol, or variable, is the one that is being used in the summation (in our case it is 'i').

Example 265. Let $n \geq 1$ be any integer. The quantity

$$p(x) = \sum_{i=1}^{n} \frac{\sin ix}{i}$$

is called a **finite trigonometric polynomial**, and these are of central importance in areas of mathematics and engineering which deal with **Fourier Series**. What is this symbol shorthand for? What is the value of $p(0)$?

Solution By definition, this string of symbols is shorthand for

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \sum_{i=1}^{n} \frac{\sin ix}{i},\\ &= \frac{\sin 1x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots+\frac{\sin nx}{n},\\ &= \sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots+\frac{\sin nx}{n},\end{aligned}$$

so, you just leave the $x's$ alone. But if you want the value of $p(0)$, say, then substitute $x = 0$ in each term of the last display. Then $p(0) = 0 + 0 + 0 + \ldots + 0 = 0$. Sometimes, you may have these long sums of numbers or functions that you want to write more compactly, using this shorthand notation. In this case you have to *guess* the general term of the sequence of numbers whose sum you're taking and then use the summation convention, above. Let's look at such an example.

Example 266. Rewrite the sum

$$f(t_1)\Delta t_1 + f(t_2)\Delta t_2 + f(t_3)\Delta t_3 + f(t_4)\Delta t_4,$$

using the summation convention. Don't worry about what the symbols mean, right now, just concentrate on the form .

Solution We see that we're adding four terms and they can be identified by their relative position. For example the *second term* is $f(t_2)\Delta t_2$, the fourth term is $f(t_4)\Delta t_4$. If we denote the first term by the symbol a_1, the second term by the symbol a_2, etc. then we see that

$$\begin{aligned}f(t_1)\Delta t_1 + f(t_2)\Delta t_2 + f(t_3)\Delta t_3 + f(t_4)\Delta t_4 &= a_1+a_2+a_3+a_4,\\ &= \sum_{i=1}^{4} a_i,\\ &= \sum_{i=1}^{4} f(t_i)\Delta t_i,\end{aligned}$$

because the term in the i^{th} position is $f(t_i)\Delta t_i$, that's all.

Occasionally, we use the summation convention to define functions too. Maybe the function is a sum of a finite number of other, simpler looking functions, (we call this sum a **finite series**) or it may be the sum of an infinite number of different functions (and we then call this sum an **infinite series**).

Example 267. A function ζ_n (read this as 'zeta sub n'; zeta is the Greek letter for our 'z') is defined by the sum

$$\begin{aligned}\zeta_n(s) &= \sum_{i=1}^{n} \frac{1}{i^s}\\ &= 1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots+\frac{1}{n^s},\end{aligned}$$

where s is a variable (usually a **complex number**, like $\sqrt{-1}$). It can be shown that, if we take the limit as $n \to \infty$, then the resulting so-called **infinite series** is called **Riemann's Zeta Function**, denoted by $\zeta(s)$, and it is associated with a very old unsolved problem in mathematics called the **Riemann Hypothesis**. This old question asks for the location of all the zeros (see Chapter 5) of this function, whether they be real or complex numbers. Over 100 years ago, Bernhard Riemann conjectured that *all* the zeros must look like $\frac{1}{2}+\beta\sqrt{-1}$, where β is some real number.

The story goes ... When the famous mathematician **C.F. Gauss**, 1777 - 1855, was ten years young he was asked to add the first one hundred numbers (because he was apparently bored in his mathematics class?). So, he was asked to find the sum $1 + 2 + 3 + \ldots + 100$, or, using our shorthand notation,

$$\sum_{i=1}^{100} i.$$

Within minutes, it is said, he came up with the answer, ... 5050. How did he do it? Here's the idea ... Let's say you want to add the first 5 numbers (using his method). Write $S = 1 + 2 + 3 + 4 + 5$. Now write S again but the other way around, that is, $S = 5 + 4 + 3 + 2 + 1$. Add these two equalities together ... Then $2S = (1 + 5) + (2 + 4) + (3 + 3) + (4 + 2) + (5 + 1)$. But this gives $2S = 6+6+6+6+6$ and there are as many 6's as there are numbers in the original sum, right? That is, there are 5 of these 6's. So, $2S = 5\cdot 6 = 30$ and then $S = \frac{5\cdot 6}{2}$. See if you can do this in general, like Gauss did.

Figure 121.

Let a_i, b_i be any two finite sequences of numbers. Then,

$$(I)\quad \sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

$$(II)\quad \sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

If c is any quantity not depending on the index i, then

$$(III)\quad \sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i.$$

Table 6.8: Properties of the Summation Operator

So far, no one has been able to prove or disprove this result ... It's one of those unsolved mysteries whose answer would impact on many areas within mathematics.

Anyhow, let's look at the quantity

$$\begin{aligned}\zeta_{10}(2) &= \sum_{i=1}^{10} \frac{1}{i^2} \\ &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{10^2} \\ &= 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{100} \\ &= 1.54977, \quad \textit{approximately}.\end{aligned}$$

Now, the larger we take n the closer we get to $\zeta(2)$, see Figure 122. We can use the Theory of Fourier Series (mentioned above) and show that $\zeta(2) = \frac{\pi^2}{6}$, an amazing result, considering we have to 'add infinitely many terms' to get this!

Approximating the value of $\zeta(2) = \frac{\pi^2}{6} \approx 1.644934068$ using a finite sum (or finite series) consisting of 1 term, 5 terms, 10 terms, 100 terms, etc.

n	$\zeta_n(2)$
1	1.0
5	1.463611
10	1.549768
100	1.634984
1000	1.643935
10000	1.644834
100000	1.644924
...	...

Figure 122.

Since 'sums' consist of sums (or differences) of numbers, we can permute them (move them around) and not change their values, right? So, we would expect the results in Table 6.8 to hold. We can refer to the Greek letter, Σ as a **summation operator**, much like the symbol D is referred to a *differential operator* in Chapter 3. Furthermore, you don't always *have* to start with $i = 1$, you can start with any other number, too, for example $i = m$, where $m \geq 2$.

Example 268. Show that Table 6.8 (III) holds whenever c is any quantity not depending on the index i.

Solution The point is that c can be factored out of the whole expression. Now, by definition,

$$\begin{aligned}\sum_{i=m}^{n} c \cdot a_i &= c \cdot a_m + c \cdot a_{m+1} + c \cdot a_{m+2} + \ldots + c \cdot a_n, \\ &= c \cdot (a_m + a_{m+1} + a_{m+2} + \ldots + a_n), \\ &= c \cdot \sum_{i=m}^{n} a_i.\end{aligned}$$

Example 269. Find an approximate value of

$$\sum_{k=2}^{5} \frac{(-1)^k}{k^2}.$$

Solution Refer to Table 6.8. Watch out, but $m = 2$ here, right ? And we have replaced the symbol i by a new symbol, k. This is OK. By definition, we are looking for the value of a sum of numbers given by

$$\begin{aligned}\sum_{k=2}^{5} \frac{(-1)^k}{k^2} &= \frac{(-1)^2}{2^2} + \frac{(-1)^3}{3^2} + \frac{(-1)^4}{4^2} + \frac{(-1)^5}{5^2}, \\ &= \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2}, \\ &= \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25}, \\ &= 0.161389, \quad approx.\end{aligned}$$

NOTE: You can't factor out the $(-1)^k$ term out of the original expression because it is raised to a power which depends on the index, k. However, since $(-1)^k = (-1) \cdot (-1)^{k-1}$, we can write (using Table 6.8, (III), with $c = -1$),

$$\sum_{k=m}^{n} \frac{(-1)^k}{k^2} = (-1) \cdot \sum_{k=m}^{n} \frac{(-1)^{k-1}}{k^2}$$

because now, the extra (-1) does not involve k. More results on Bernoulli numbers may be found in books on **Number Theory**, see also Figure 123. In fact, the main result in Figure 120 involves Bernoulli numbers too! The exact formula for the sum of the k^{th} powers of the first n integers is also known and involves Bernoulli numbers.

Finally, we note that, as a consequence of Table 6.8, (III), if $n \geq m$, then we must have

$$\boxed{\sum_{k=m}^{n} c = c \cdot (n - m + 1),}$$

because there are $n - m + 1$ symbols 'c' appearing on the right.

Approximating the value of $\zeta(3) \approx 1.202056903$ using a finite sum (or finite series) consisting of 1 term, 5 terms, 10 terms, 100 terms, etc. There is **no known 'nice' explicit formula for the values of ζ(*odd number*)**, compare this with Example 267. On the other hand, there *is* an explicit formula for ζ(*even number*) which involves mysterious numbers called **Bernoulli numbers**.

n	$\zeta(3)$
1	1.0
5	1.185662
10	1.197532
100	1.202007
1000	1.202056
...	...

Indeed, if we denote the $2n^{th}$ Bernoulli number by B_{2n}, then the value of the zeta function evaluated at an even number, $2n$, is given by

$$\zeta(2n) = \frac{(2\pi)^{2n}|B_{2n}|}{2(2n)!}$$

Figure 123.

Exercise Set 30.

Write the following sums in terms of the summation operator, Σ, and some index, i.

1. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$
2. $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1$
3. $\sin \pi + \sin 2\pi + \sin 3\pi + \sin 4\pi + \sin 5\pi$
4. $\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \frac{4}{n} + ... + \frac{n}{n}$

Expand and find the value of each of the following sums (using a calculator, if you wish).

5. $\displaystyle\sum_{k=1}^{5} \frac{(-1)^k}{k^2}$
6. $\displaystyle\sum_{k=2}^{6} \frac{1}{k^3}$
7. $\displaystyle p(x) = \sum_{k=1}^{3} \frac{\sin(k\pi x)}{k^2}$, at $x = 1$
8. $\displaystyle\sum_{i=1}^{50} i$
   • See Figure 121.
9. $\displaystyle\sum_{i=1}^{100} i^2$
   • See Figure 120.
10. $\displaystyle\sum_{i=1}^{n} \frac{i}{n}$
   • See the method outlined in Figure 121, and factor out the $\frac{1}{n}$ term because it doesn't depend on the index i.
11. $\displaystyle\sum_{i=1}^{n} 6\left(\frac{i}{n}\right)^2$
   • See Figure 120.
12. Show that $\displaystyle\sum_{i=1}^{6}(a_i - a_{i-1}) = a_6 - a_0$ for any 7 numbers, $\{a_0, a_1, a_2, a_3, ..., a_6\}$.
13. Use the idea in the previous Exercise to conclude that

$$\boxed{\sum_{i=1}^{n}(a_i - a_{i-1}) = a_n - a_0,}$$

for any finite sequence of numbers $\{a_i\}$. This is called a **telescoping sum**.

14. Use Figure 120 and your knowledge of limits to show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{3}.$$

15. This problem is **Really, really, hard**, but not impossible! A **prime number** is a positive integer whose only proper divisors are 1 and itself. For example, 2, 3, 5, 11, are primes while $4, 8, 9$ are not primes. It was proved by **Euclid of Alexandria** over 2000 years ago, that there is an infinite number of such primes. Let's label them by $\{p_1, p_2, p_3, ...\}$ where, for the purpose of this Exercise, we take it that $p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 5, p_5 = 7, p_6 = 11, p_7 = 13, p_8 = 17$, *etc.* Show that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{n^3}{n^4 + in^3 + p_n} = \ln 2.$$

Hint: Use the fact the $n^{th}-$prime can be estimated by $p_n < 36n\ln n$, for every $n \geq 2$. This estimate on the $n^{th}-$prime number is called **Sierpinski's Estimate**. Then see Example 279 in the next section.

NOTES:

6.4 Area and the Riemann Integral

In this section we produce the construction of the so-called **Riemann Integral** developed by Gauss' own *protégé*, G. F. Bernhard Riemann, 1826-1866, over 100 years ago. When this integral is combined with our own definition of the 'definite integral' we'll be able to calculate areas under curves, areas between curves, and even areas of arbitrary closed regions, etc. This will identify the antiderivative and the subsequent definite integral with the notion of area, thus giving to the definite integral this powerful geometric interpretation.

Let $[a, b]$ be a closed interval of real numbers i.e., $[a, b] = \{x : a \leq x \leq b\}$. By a **partition $\mathcal{P}$ of $[a, b]$** we mean a subdivision (or splitting, or breaking up) of the interval by a finite number of points $a = x_0, x_1, x_2, ..., x_{n-1}, x_n = b$, into a finite number of smaller intervals, where these points are arranged in ascending order as

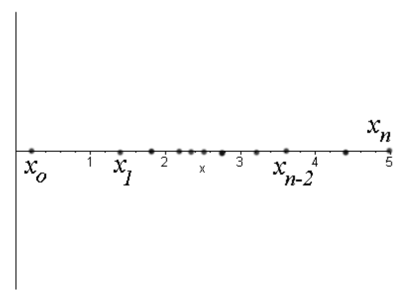
$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b. \tag{6.11}$$

Example 270. For example, the points $a = 3 < 3.1 < 3.2 < 3.3 < 4.0 < 4.2 < b = 5$ define a partition of the closed interval $[3, 5]$ where the points are given by $x_0 = 3, x_1 = 3.1, x_2 = 3.2, x_3 = 3.3, x_4 = 4.0, x_5 = 4.2$ and $x_6 = 5$. Note that these points **do not have to be equally spaced** along the interval $[3, 5]$.

Example 271. Another example of a partition is given by selecting any finite sequence of numbers in the interval $[0, 1]$, say. Let's choose

$$\begin{aligned} a = x_0 &= 0, \\ x_1 &= \frac{1}{19}, \quad x_2 = \frac{1}{17}, \\ x_3 &= \frac{1}{13}, \quad x_4 = \frac{1}{11}, \\ x_5 &= \frac{1}{7}, \quad x_6 = \frac{1}{5}, \\ x_7 &= \frac{1}{3}, \quad x_8 = \frac{1}{2}, \\ x_9 = b &= 1. \end{aligned}$$

Can you guess where these numbers come from? Well, this is a partition of the interval $[0, 1]$ into 9 subintervals and the partition consists of 10 points (because we include the end-points).



A typical partition of the interval $[0.2, 5]$ by $(n+1)$ points, $0.2 = x_0 < x_1 < \ldots < x_n = 5$.

Figure 124.

Now you can see that **any** partition of $[a, b]$ breaks up the interval into a finite number of subintervals so that $[a, b]$ is the union of all these subintervals $[a, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$,..., $[x_{n-1}, b]$, right? O.K., now the **length** of each subinterval of the form $[x_i, x_{i+1}]$ is, of course, equal to $x_{i+1} - x_i$. We use the notation **"Δx_i"** (read this symbol Δx_i, as "**delta-x-eye**") to represent this length, so, for any subscript i, $i = 0, 1, 2, 3, ..., n-1$,

$$\Delta x_i = x_{i+1} - x_i. \tag{6.12}$$

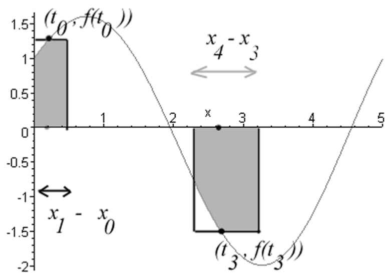
The symbol, Δ, just defined is also called a **forward difference operator**, and it is used extensively in applications of Calculus in order to estimate the size of a derivative, when we're not given the actual function, but we just have some of its values. For the partition in Example 270, $\Delta x_0 = x_1 - x_0 = 3.1 - 3 = 0.1$, $\Delta x_1 = x_2 - x_1 = 3.2 - 3.1 = 0.1$, $\Delta x_4 = x_5 - x_4 = 4.2 - 4.0 = 0.2$, etc.

Finally, for a **given** partition $\mathcal{P}$ of an interval $[a, b]$ we define its **norm** by the symbol $|\mathcal{P}|$, to be the **length of the largest subinterval making it up.** The norm of a partition is used to measure the *size* or *fineness* of a partition.

Example 272. For example, the norm of our partition $\mathcal{P}$, in Example 270, is by definition, the length of the largest subinterval, which, in this case, is the largest of all the numbers

$$\begin{aligned}
\Delta x_0 &= 0.1,\\
\Delta x_1 &= 0.1,\\
\Delta x_2 &= 0.1,\\
\Delta x_3 &= 0.7,\\
\Delta x_4 &= 0.2,\\
\Delta x_5 &= 0.8, \quad \longleftarrow \text{ largest length!}
\end{aligned}$$

which, of course, is 0.8. So $|\mathcal{P}| = 0.8$, by definition.



The area of the rectangle with base-length $x_1 - x_0$ and height $f(t_0)$ approximates the actual shaded area 'under the graph' on the interval $[x_0, x_1]$. The same thing can be said for the rectangle $x_4 - x_3$ and height $f(t_3)$.

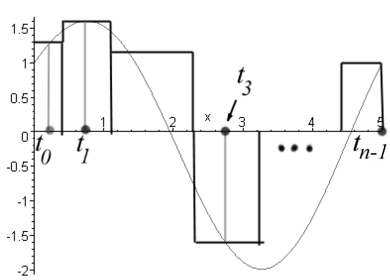
Figure 125.

Example 273. We let $I = [0, 1]$ and define $\mathcal{P}$ to be the partition defined by the subintervals $[0, 1/5], [1/5, 1/3], [1/3, 1/2], [1/2, 7/8], [7/8, 1]$. Then the norm, $|\mathcal{P}|$, of this partition is equal to $7/8 - 1/2 = 3/8 = 0.375$, which is the length of the largest subinterval contained within $\mathcal{P}$.

Now, let f be some function with domain, $Dom(f) = [a, b]$. For a given partition $\mathcal{P}$ of an interval $[a, b]$, we define a quantity of the form

$$\sum_{i=0}^{n-1} f(t_i)\Delta x_i = f(t_0)\ (x_1 - x_0) + f(t_1)\ (x_2 - x_1) + \ldots + f(t_{n-1})\ (x_n - x_{n-1}). \quad (6.13)$$

where t_i is some number between x_i and x_{i+1}. This quantity is called a **Riemann sum** (pronounced **Ree-man**), and it is used in order to approximate the 'area under the curve $y = f(x)$' between the lines $x = a$ and $x = b$, see Figures 124, 125, 126, 127.



A typical Riemann sum consists of the sum of the areas of each rectangle in the figure, with due regard taken for rectangles which lie below the $x-$axis. For such rectangles, the area is interpreted as being **negative**. Note that in this Figure, the point $t_{n-1} = x_n = 5$, is the endpoint of the interval. This is OK.

Figure 126.

Now, we define another type of limit. The string of symbols

$$\lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{n-1} f(t_i)\Delta x_i = L \quad (6.14)$$

means the following: Let's say we are given some small number $\varepsilon > 0$, (called 'epsilon'), and we have this guess, L, and for a given partition, $\mathcal{P}$, we can make

$$\left| \sum_{i=0}^{n-1} f(t_i)\ \Delta x_i - \quad L \right| < \varepsilon. \quad (6.15)$$

but **only if** our partition is 'small enough', that is, if $|\mathcal{P}| < \delta$, where the choice of this new number δ will depend on the original ε.

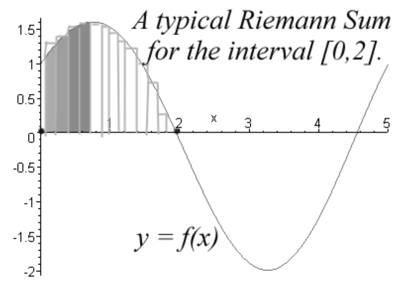
If we can do this calculation (6.15) for **every possible** $\boldsymbol{\varepsilon > 0}$, provided the partition is small enough, then we say that **limit of the Riemann sum as the norm of the partition $\mathcal{P}$ approaches 0 is L.** These ideas involving these curious numbers ε, δ are akin to those presented in an optional chapter on rigorous methods. The limit definition included here is for completeness, and not intended to cause any stress.

Generally speaking, this inequality (6.15) depends upon the function f, the partition, $\mathcal{P}$, and the actual points t_i chosen from the partition, in the sense that if we change any one of these quantities, then we expect the value of L to change too, right? This is natural. But now comes the definition of the **Riemann Integral**.

Let's assume that f is a function defined on $[a, b]$ with the property that the limit, L, in (6.14) exists and the number L so obtained is **independent** of the choice of the partition (so long as its norm approaches zero), and L is also independent of the choice of the interior points, t_i, ... In this case, we say that f is **Riemann Integrable** over the interval $[a, b]$ and L is the **value of the Riemann Integral of f over $[a, b]$.** We will denote this value of L by the quick and descriptive notation,

$$L = \mathcal{R}\int_a^b f(x)\,dx = \lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{n-1} f(t_i)\Delta x_i$$

where the symbols on the right make up what we'll call the Riemann integral, or equivalently, the **$\mathcal{R}$-integral** of f over $[a, b]$.



This Riemann sum shows that the smaller the rectangles, (or the closer together the x_i), the better is this sum's approximation to the area 'under the curve'. Remember that if the curve is below the $x-$axis, its area is negative in that portion.

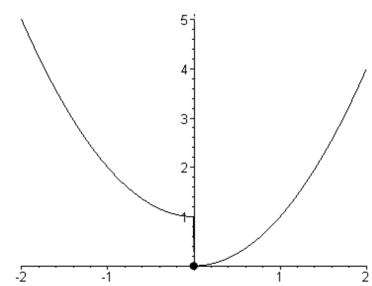
Figure 127.

Seems hard to believe that this integral will exist at all, right? Well, lucky for us this Riemann integral does exist when f is continuous over $[a, b]$ and even if f is continuous over a finite number of pieces whose union is the whole of the interval $[a, b]$, see Figure 128. We then call f **piecewise continuous**.

At this point in the discussion we will relate the definite integral of a function (assumed to have an antiderivative) with its Riemann integral (assumed to exist). We will then be in a position to define the area under the graph of a given continuous or piecewise continuous function and so we will have related the definite integral with the notion of 'area', its principal geometric interpretation. Thus, the concepts of a derivative and an antiderivative will each have a suitable and wonderful geometric interpretation.

In order to arrive at this geometric result, we'll make use of the **Mean Value Theorem** which we will recall here (see Chapter 3). If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one c between a and b at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



The graph of a piecewise continuous function. Note that the function is continuous at every point except at a finite number of points where its graph makes a 'jump!' In this case, its only jump is at $x = 0$.

Figure 128.

Let $\mathcal{P}$ denote an arbitrary partition of the interval $[a, b]$, so that (6.11) holds. Let $\mathcal{F}$ denote an antiderivative of f. Note that for any increasing sequence of points $x_0 = a, x_1, x_2, \ldots, x_{n-1}, x_n = b$, in $[a, b]$ we have,

$$\begin{aligned}
\mathcal{F}(b) - \mathcal{F}(a) &= \mathcal{F}(x_n) - \mathcal{F}(x_0) \\
&= \mathcal{F}(x_n) - \mathcal{F}(x_{n-1}) + \mathcal{F}(x_{n-1}) - \mathcal{F}(x_{n-2}) + \ldots \\
&\quad + \mathcal{F}(x_2) - \mathcal{F}(x_1) + \mathcal{F}(x_1) - \mathcal{F}(x_0) \\
&= \left(\frac{\mathcal{F}(x_n) - \mathcal{F}(x_{n-1})}{x_n - x_{n-1}}\right)(x_n - x_{n-1}) \\
&\quad + \left(\frac{\mathcal{F}(x_{n-1}) - \mathcal{F}(x_{n-2})}{x_{n-1} - x_{n-2}}\right)(x_{n-1} - x_{n-2}) + \ldots \\
&\quad + \left(\frac{\mathcal{F}(x_1) - \mathcal{F}(x_0)}{x_1 - x_0}\right)(x_1 - x_0).
\end{aligned}$$

Now, we apply the Mean Value Theorem to $\mathcal{F}$ over each interval of the form $[x_i, x_{i+1}]$,

For $f(x) \geq 0$,

$$\begin{aligned}\text{Area under } f \text{ between the two lines at } a \text{ and } b &= \int_a^b f(x)\ dx,\\ &= \mathcal{F}(b) - \mathcal{F}(a),\\ &= \mathcal{F}(x)\big|_a^b .\end{aligned}$$

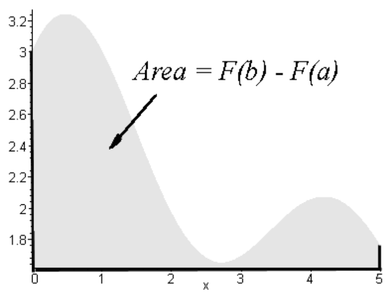
Table 6.9: The Area Formula for a Positive Integrable Function

$(i = 0, 1, 2, ..., n-1)$ to find that,

$$\begin{aligned}\frac{\mathcal{F}(x_{i+1}) - \mathcal{F}(x_i)}{x_{i+1} - x_i} &= \mathcal{F}'(c_i) \text{ where } c_i \text{ is in } (x_i, x_{i+1})\\ &= f(c_i) \quad (\text{because } \mathcal{F}' = f),\end{aligned}$$

where the existence of the $c_i's$ is guaranteed by the Mean Value Theorem. Note that these depend on the initial partition (or on the choice of the points $x_0, x_1, \ldots, x_n$). So, we just keep doing this construction every time, for every interval of the form (x_i, x_{i+1}), $i = 0, 1, ..., n-1$, until we've found the particular string of numbers, $c_0, c_1, ..., c_{n-2}, c_{n-1}$, and we'll see that

$$\begin{aligned}\mathcal{F}(b) - \mathcal{F}(a) &= \sum_{i=0}^{n-1} \frac{\mathcal{F}(x_{i+1}) - \mathcal{F}(x_i)}{x_{i+1} - x_i} \Delta x_i,\\ &= \sum_{i=0}^{n-1} \mathcal{F}'(c_i)\Delta x_i,\\ &= \sum_{i=0}^{n-1} f(c_i)\Delta x_i.\end{aligned}$$



The shaded area under the graph of this positive function is given by its definite integral from $x = 0$ to $x = 5$, that is,

$$Area = \int_0^5 f(x)\ dx$$

.

Figure 129.

Now, the left-hand side, namely the number $\mathcal{F}(b) - \mathcal{F}(a)$, is just a constant, right? But the sum on the right is a Riemann sum for the given partition, and the given partition is *arbitrary*. So, we can make it as fine as we like by choosing $|\mathcal{P}|$ as small as we want, and then finding the $c_i's$. Then we can pass to the limit as $|\mathcal{P}| \to 0$ and find (since f is continuous and its Riemann integral exists),

$$\begin{aligned}\mathcal{F}(b) - \mathcal{F}(a) &= \lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{n-1} f(c_i)\Delta x_i,\\ &= \mathcal{R}\int_a^b f(x)\ dx, \quad (\text{by definition of the } \mathcal{R}\text{-integral}),\\ &= \int_a^b f(x)\ dx \quad (\text{by definition of the definite integral}).\end{aligned}$$

So, we have just shown that the $\mathcal{R}-$integral and the definite integral coincide. This means that we can use antiderivatives to evaluate Riemann integrals! Okay, but because of the nice geometric interpretation of the Riemann integral as some sort of area (see Figure 127), we can use this notion to **define the area under a curve whose equation is given by $y = f(x) \geq 0$ and between the vertical lines $x = a, x = b$ and the $x-$axis,** by Table 6.9 if, say, f is continuous (or even piecewise continuous) on $[a, b]$.

Recall that we had already done area calculations earlier, in the section on *Definite integrals*. The results from Table 6.9 and the $\mathcal{R}-$integral above justify what we were

doing then, (see Figure 129).

Let's recap. (... contraction of the word *recapitulate*). If f has an antiderivative $\mathcal{F}$ over $[a,b]$, then for x in (a,b), we will have

$$\begin{aligned}\frac{d}{dx}\int_a^x f(t)\ dt &= \frac{d}{dx}\left(\mathcal{F}(x)-\mathcal{F}(a)\right),\\ &= \mathcal{F}'(x)-0,\quad \text{(since } \mathcal{F}(a) \text{ is a constant)},\\ &= f(x),\quad \text{(by definition of the antiderivative)}.\end{aligned}$$

This means that we can consider an indefinite integral as a function in its own right, a function defined on the *same* domain as the original integrand from where it was defined. Now, let f be a continuous function over $[a,b]$ and differentiable over (a,b). Let's also assume that this derivative function is itself continuous over $[a,b]$. Then the function f is an antiderivative of the function f', right? But this means that we can set $\mathcal{F}=f$ in the expression for the definite integral of f', and find,

The notion of **area** as introduced here is only ONE possible definition of this concept. It is the most natural at this point in your studies in mathematics. This *area* topic has been defined by mathematicians for functions which do not have the property of continuity stated here. Words like **Lebesgue measure, Jordan content** are more general in their description of area for much wider classes of functions. The interested reader may consult any book on **Real Analysis** for further details.

$$\begin{aligned}\int_a^b f'(x)\ dx &= \mathcal{F}(b)-\mathcal{F}(a),\\ &= f(b)-f(a),\qquad \text{(since } f=\mathcal{F} \text{ here)}.\end{aligned}$$

This is one of the many versions of the **Fundamental Theorem of Calculus**. Summarizing all this we get,

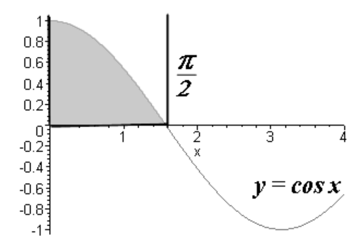
The Fundamental Theorem of Calculus

If f' is continuous over $[a,b]$, then

$$\int_a^b f'(x)\ dx = f(b)-f(a),$$

while if f is continuous over $[a,b]$, then

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x).$$



The area of the shaded region under the graph of the cosine function is given by its definite integral from $x=0$ to $x=\frac{\pi}{2}$, that is,

$$\text{Shaded Area} = \int_0^{\frac{\pi}{2}} \cos x\ dx$$

.

Figure 130.

One of the important consequences of the Fundamental Theorem of Calculus is **Leibniz's Rule** which is used for differentiating an integral with variable limits of integration. For example, if we assume that a,b are differentiable functions whose ranges are contained within the domain of f, then we can believe the next formula, namely that

$$\int_{a(x)}^{b(x)} f(t)\ dt = \mathcal{F}(b(x))-\mathcal{F}(a(x)),$$

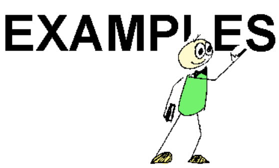
and from this formula, if we differentiate both sides and use the Chain Rule on the right, we find

$$\begin{aligned}\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\ dt &= \frac{d}{dx}(\mathcal{F}(b(x)) - \mathcal{F}(a(x))),\\ &= \mathcal{F}'(b(x))b'(x) - \mathcal{F}'(a(x))a'(x),\\ &= f(b(x))\frac{db}{dx} - f(a(x))\frac{da}{dx},\end{aligned}$$

which is Leibniz's Rule, see Table 6.10. Summarizing this we get

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\ dt = f(b(x))\frac{db}{dx} - f(a(x))\frac{da}{dx}.}$$

Table 6.10: Leibniz's Rule for Differentiating a Definite Integral



Example 274. Show that the area under the curve defined by $f(x) = \cos x$ on the interval $[0, \pi/2]$ is given by the definite integral (see Figure 130),

$$\int_0^{\frac{\pi}{2}} \cos x\ dx = 1,$$

Solution First we must check that the given function is positive on the given interval. But this is true by trigonometry, right? (or you can see this from its graph too, Figure 130). Now, we see that the area *under the curve* defined by $f(x) = \cos x$ on the interval $[0, \pi/2]$ is (by definition) the same as the area of the region bounded by the vertical lines $x = 0$, $x = \frac{\pi}{2}$, the $x-$axis, and the curve $y = \cos x$. Using Table 6.9, we see that the required area is given by

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos x\ dx &= \sin x\Big|_0^{\pi/2},\\ &= \sin\frac{\pi}{2} - \sin 0 = 1.\end{aligned}$$

In the next example we'll explore what happens if the region whose area we want to find lies above and below the $x-$axis.

Example 275. Evaluate $\displaystyle\int_0^6 (\sin(x\sqrt{2}) + \cos x)\ dx$, and interpret your result geometrically.

Solution The integration is straightforward using the techniques of the previous sections. Thus,

$$\int_0^6 (\sin(x\sqrt{2}) + \cos x)\ dx = \int_0^6 \sin(x\sqrt{2})\ dx + \int_0^6 \cos x\ dx.$$

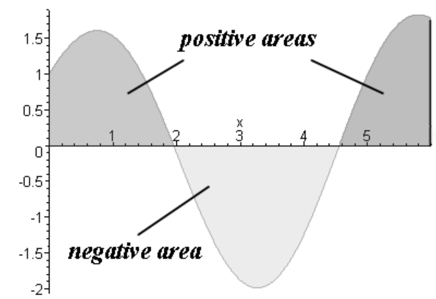
In order to evaluate the first of these integrals, we use the fact that if $\square = x\sqrt{2}$,

then $\square'(x) = \sqrt{2}$, and so

$$\begin{aligned}
\int \sin(x\sqrt{2})\,dx &= \int \sin(\square)\frac{d\square}{dx}\frac{1}{\sqrt{2}}\,dx,\\
&= \frac{1}{\sqrt{2}}\int \sin(\square)\frac{d\square}{dx}\,dx,\\
&= \frac{1}{\sqrt{2}}(-\cos\square)+C, \quad \text{(by Table 6.6)},\\
&= -\frac{1}{\sqrt{2}}\cos(x\sqrt{2})+C,\\
&= -\frac{\cos(x\sqrt{2})}{\sqrt{2}}+C.
\end{aligned}$$

So, [remember to switch to radian mode on your calculator], we find

$$\begin{aligned}
\int_0^6 \sin(x\sqrt{2})\,dx + \int_0^6 \cos x\,dx &= \left(-\frac{\cos(x\sqrt{2})}{\sqrt{2}}+\sin x\right)\Bigg|_0^6,\\
&= \left(-\frac{\cos(6\sqrt{2})}{\sqrt{2}}+\sin 6\right)-\left(-\frac{1}{\sqrt{2}}+\sin 0\right),\\
&= \frac{1-\cos(6\sqrt{2})}{\sqrt{2}}+\sin 6,\\
&= .84502.
\end{aligned}$$



The value of the definite integral is the sum of the areas of the three shaded regions above, where the region under the $x-$axis is taken to have a negative area. Even though we don't know the area of each region individually, we can still find their area sum, which is the number 0.84502.

Figure 131.

For a geometric interpretation see the graph of the integrand in Figure 131. Notice that if we had found the zeros of this function (using *Newton's method*, Chapter 3) then we could have obtained the areas of *each one* of the shaded regions in Figure 131 by integrating over each interval separately. But we don't have to find these zeros to get the answer!

Example 276. Evaluate $\displaystyle\frac{d}{dx}\int_a^x e^{-t^2}\,dt.$

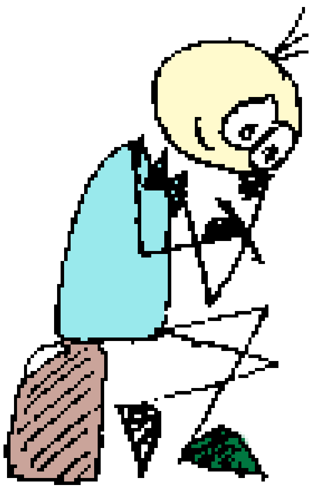
Solution We set $f(t) = e^{-t^2}$ in the Fundamental Theorem of Calculus (FTC). Then,

$$\begin{aligned}
\frac{d}{dx}\int_a^x f(t)\,dt &= f(x),\\
&= e^{-x^2}.
\end{aligned}$$

Amazingly enough, we don't have to compute the antiderivative first and *then* find the derivative, as this would be almost impossible for this type of exponential function. This, so-called, **Gaussian function** is a common occurrence in **Probability Theory** and, although it *does* have an antiderivative, we can't write it down with a finite number of symbols (this is a known fact). Reams of reference tables abound where the definite integral of this function is calculated numerically using sophisticated methods (such as one called **Simpson's Rule**) a few of which we'll see later.

Example 277. Evaluate $\displaystyle\frac{d}{dx}\int_{-x^2+1}^4 \cos(t^2\ln t)\,dt.$

Solution Because of the variable limit $-x^2+1$, we think about using Leibniz's Rule, Table 6.10, on the *function* defined by the definite integral. Next, we set $f(t) = \cos(t^2\ln t)$, $a(x) = -x^2+1$, $b(x) = 4$ into the expression for the Rule. Then,



$a'(x) = -2x$, $b'(x) = 0$ and so,

$$
\begin{aligned}
\frac{d}{dx}\int_{-x^2+1}^{4} \cos(t^2 \ln t)\ dt &= \frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\ dt, \\
&= f(b(x))\frac{db}{dx} - f(a(x))\frac{da}{dx}, \\
&= f(4)(0) - f(-x^2+1)(-2x), \\
&= 2x\cos\left((-x^2+1)^2 \ln(-x^2+1)\right),
\end{aligned}
$$

where we could have used the 'Box' method of Chapter 1 to evaluate $f(-x^2+1)$ explicitly.

Example 278. Evaluate

$$\frac{d}{dx}\int_{x^2}^{\sin x} 3t^2\ dt$$

in *two* different ways, first by integrating and then finding the derivative, and secondly by using Leibniz's Rule. Compare your answers.

Solution First, an antiderivative of the integrand $3t^2$ is easily seen to be $\mathcal{F}(t) = t^3$. Thus,

$$
\begin{aligned}
\int_{x^2}^{\sin x} 3t^2\ dt &= \mathcal{F}(\sin x) - \mathcal{F}(x^2), \\
&= (\sin x)^3 - (x^2)^3, \\
&= \sin^3 x - x^6.
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{d}{dx}\int_{x^2}^{\sin x} 3t^2\ dt &= \frac{d}{dx}\left(\sin^3 x - x^6\right), \\
&= 3\,(\sin x)^2 \cos x - 6(x^5), \\
&= 3\ \sin^2 x \cos x - 6x^5.
\end{aligned}
$$

Next, we use Leibniz's Rule, as in Table 6.10. We set $f(t) = 3t^2$, $a(x) = x^2$, and $b(x) = \sin x$, to find $a'(x) = 2x$, $b'(x) = \cos x$. So,

$$
\begin{aligned}
\frac{d}{dx}\int_{x^2}^{\sin x} 3t^2\ dt &= f(\sin x)\cos x - f(x^2)2x, \\
&= 3(\sin x)^2 \cos x - (3x^4)(2x), \\
&= 3\sin^2 x \cos x - 6x^5.
\end{aligned}
$$

We get the same answer, as expected (but Leibniz's Rule is definitely less tedious!).

The next example shows another powerful application of the Riemann integral. It will be used to find a very difficult looking limit!

Example 279. Show that

$$\lim_{n\to+\infty} \sum_{i=0}^{n-1} \frac{1}{n+i} = \ln 2.$$

Solution This is not obvious! First we check the numerical evidence supporting this claim, For example, let's note that $\boxed{\ln 2 \approx 0.693147}$ to 7 significant digits.

For $n = 100$,

$$\sum_{i=0}^{99} \frac{1}{100+i} = \frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \ldots + \frac{1}{199}, \approx \boxed{0.69565}.$$

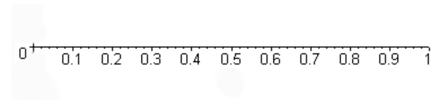
For $n = 10,000$,

$$\sum_{i=0}^{9999} \frac{1}{10000+i} = \frac{1}{10000} + \frac{1}{10001} + \frac{1}{10002} + \ldots + \frac{1}{19999}, \approx \boxed{0.693172}.$$

For $n = 100,000$,

$$\sum_{i=0}^{99999} \frac{1}{100000+i} = \frac{1}{100000} + \frac{1}{100001} + \frac{1}{100002} + \ldots + \frac{1}{199999}, \approx \boxed{0.6931497},$$

so the result is believable. We are only off by about 0.000002 in the latest estimate of the required limit. But how do we show that this limit *must* converge to this number, $\ln 2$? The idea is that sums of this type should remind us of Riemann sums for a particular function f, a given interval, and a corresponding partition. But which ones? We make a few standard assumptions...



This figure shows a regular partition, $\mathcal{P}$, of the interval $[0, 1]$ into $n = 10$ pieces where the x_i's are numbered. Each subinterval has length 0.1 or, equivalently, the distance between successive points $x_i = i/n = i/10$, where $i = 0, 1, 2, ..., 10$, is 0.1. As the integer $n \to +\infty$ the partition gets smaller and smaller, and its norm (*i.e.,* the length of the largest subinterval), approaches zero. The converse is also true, that is if the norm of the partition approaches zero, then $n \to \infty$, too. In this way we can use definite integrals to approximate or even evaluate complicated looking sums involving limits at infinity.

Figure 132.

Let's assume that the partition is *regular*, that is all the numbers Δx_i appearing in a typical Riemann sum are *equal*! This means that the points x_i subdivide the given interval into an equal number of subintervals, right? Let's also assume that the interval is $[0, 1]$. In this case we get $x_i = \frac{i}{n}$, (see Figure 132). We want to convert the original sum into a Riemann sum so we'll have to untangle and re-interpret the terms in the original sum somehow!? Note that $\Delta x_i = 1/n$. Now, here's the idea ...

$$\begin{aligned}
\frac{1}{n+i} &= \Delta x_i \cdot \ (\textit{something else}), \\
&= \left(\frac{1}{n}\right)\left(\frac{n}{n+i}\right), \\
&= \left(\frac{1}{n}\right)\left(\frac{1}{1+\frac{i}{n}}\right), \\
&= \left(\frac{1}{1+c_i}\right)\left(\frac{1}{n}\right), \\
&= f(c_i)\Delta x_i,
\end{aligned}$$

if we define f by $f(t) = \dfrac{1}{1+t}$, and the c_i are chosen by setting $c_i = x_i = \frac{i}{n}$, for $i = 0, 1, 2, ..., n-1$.

Now, we simply take the limit as the norm (*i.e.,* the length of the largest subinter-

val), of our regular partition approaches zero to find,

$$\begin{aligned}
\sum_{i=0}^{n-1} \frac{1}{n+i} &= \sum_{i=0}^{n-1} f(c_i)\Delta x_i, \\
\lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{n-1} \frac{1}{n+i} &= \lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{n-1} f(c_i)\Delta x_i, \\
\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n+i} &= \int_0^1 \frac{1}{1+t}\, dt, \quad \text{(because } |\mathcal{P}| = 1/n \ \to 0 \text{ means } n \to \infty\text{)}, \\
&= \ln(1+t)\Big|_0^1, \quad \text{(see the second column of Table 6.5 with } \square = 1+t\text{)}, \\
&= \ln 2 - \ln 1, \\
&= \ln 2.
\end{aligned}$$

Example 280. Let

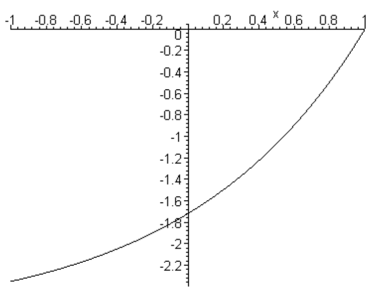
$$\mathcal{F}(x) = \int_0^x f(t)\ dt$$

where $f(1) = 0$ and $f'(x) > 0$ for every value of x (see the margin). Which of the following statements is true?

a) $\mathcal{F}$ is a differentiable function of x?

b) $\mathcal{F}$ is a continuous function of x?

c) The graph of $\mathcal{F}$ has a horizontal tangent at $x = 1$?



d) $\mathcal{F}$ has a local maximum at $x = 1$?

e) $\mathcal{F}$ has a local minimum at $x = 1$?

f) The graph of $\mathcal{F}$ has an inflection point at $x = 1$?

g) The graph of $\frac{d\mathcal{F}}{dx}$ crosses the x-axis at $x = 1$?

Solution a) Yes, by the FTC.

b) Yes, because $\mathcal{F}$ is differentiable by the FTC, so it must be continuous too!

c) Yes, since $\mathcal{F}$ is differentiable and $\mathcal{F}'(x) = f(x)$ by the FTC, we see that $\mathcal{F}'(1) = f(1) = 0$, by the given assumption on f. Since $\mathcal{F}'(1) = 0$ this means that the graph of $\mathcal{F}$ has a horizontal tangent at $x = 1$.

d) No, since the Second Derivative Test applied to $\mathcal{F}$ gives us $\mathcal{F}''(1) = f'(1) > 0$, by the given assumption on the derivative of f. For a local maximum at $x = 1$ we ought to have $\mathcal{F}''(1) < 0$.

e) Yes, see the reason in (d), above.

f) No, since $\mathcal{F}''(x)$ must change its sign around $x = 1$. In this case, $\mathcal{F}''(x) = f'(x) > 0$, for each x including those near $x = 1$. So, there is no change in concavity around $x = 1$ and so $x = 1$ cannot be a point of inflection.

g) Yes, since $\mathcal{F}'(1) = f(1) = 0$, and so, by definition, $\mathcal{F}'(x) = 0$ at $x = 1$ and its graph must touch the $x-$axis there. That the graph of $\mathcal{F}'$ *crosses* the $x-$axis at

$x = 1$ follows since $f'(x) > 0$ for any x, and so $x = 1$ cannot be a **double root** of f. Recall that a root $x = c$ of f is called a double root of f if $f(c) = 0$ and $f'(c) = 0$.

SNAPSHOTS

Example 281. Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin 2x}{\cos^2 2x}\, dx$. Interpret your result geometrically.

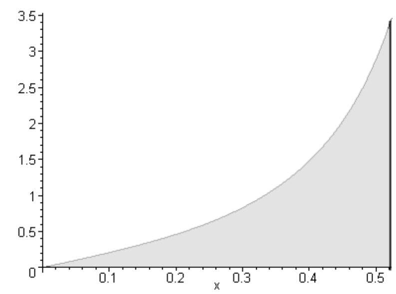
Solution First we need an antiderivative $\mathcal{F}(x) = \displaystyle\int f(x)\, dx = \int \frac{\sin 2x}{\cos^2 2x}\, dx$. We really want to use one of the Tables 6.5 6.6, or 6.7. We *do* have a power here, and it's in the denominator. So, let $\square = \cos 2x$, then $\square'(x) = -2\sin 2x$ from Table 6.6. Now, an antiderivative looks like

$$\begin{aligned}
\mathcal{F}(x) &= -\frac{1}{2}\int \frac{1}{\square^2}\frac{d\square}{dx}\, dx, \\
&= -\frac{1}{2}\int \square^{-2}\frac{d\square}{dx}\, dx, \\
&= -\frac{1}{2}\left(-\frac{1}{\square}\right) + C, \\
&= \frac{1}{2\cos 2x} + C,
\end{aligned}$$

where we have used the first of Table 6.5 with $r = -2$ in the second equation, above. Choosing $C = 0$ as usual, we see that

$$\begin{aligned}
\int_0^{\frac{\pi}{6}} \frac{\sin 2x}{\cos^2 2x}\, dx &= \mathcal{F}\left(\frac{\pi}{6}\right) - \mathcal{F}(0) = \mathcal{F}(x)\Big|_0^{\frac{\pi}{6}}, \\
&= \frac{1}{2\cos\frac{\pi}{3}} - \frac{1}{2\cos 0} = 1 - \frac{1}{2}, \\
&= \frac{1}{2}.
\end{aligned}$$

We note that the integrand $f(x) \geq 0$ for $0 \leq x \leq \frac{\pi}{6}$, because the denominator is the square of something while the numerator $\sin 2x \geq 0$ on this interval. Thus, the value of $1/2$ represents the area of the closed region bounded by the curve $y = f(x)$, the $x-$axis, and the vertical lines $x = 0$ and $x = \pi/6$, see Figure 133.



The area (equal to 1/2) under the curve $y = f(x)$, the $x-$axis, and the vertical lines $x = 0$ and $x = \pi/6$, is the area of the shaded region where

$$f(x) = \frac{\sin 2x}{\cos^2 2x}.$$

Figure 133.

Example 282. Given that

$$I(x) = \int_1^{x^2} \frac{1}{1 + \sqrt{1 - t}}\, dt,$$

find $I'(x)$ using any method.

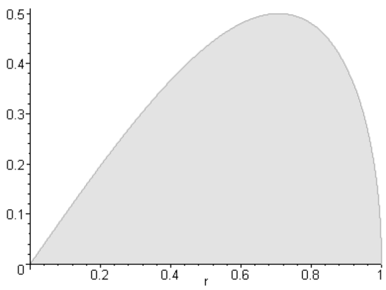
Solution Integrating directly and then finding the derivative looks difficult, so we use Leibniz's Rule, Table 6.10 where we set $a(x) = 1$, $b(x) = x^2$, and

$$f(t) = \frac{1}{1 + \sqrt{1 - t}}.$$

Then,

$$
\begin{aligned}
I'(x) &= \frac{d}{dx}\int_1^{x^2} \frac{1}{1+\sqrt{1-t}}\, dt,\\
&= f(b(x))b'(x) - f(a(x))a'(x),\\
&= \frac{2x}{1+\sqrt{1-x^2}},
\end{aligned}
$$

and we didn't have to find the antiderivative first!



The area (equal to 1/3) under the curve $y = f(t)$, the $t-$axis, and the vertical lines $t = 0$ and $t = 1$, is the area of the shaded region where

$$f(t) = t\sqrt{1-t^2}.$$

Figure 134.

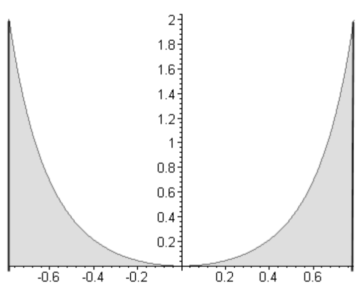
Example 283. Find the area under the curve $y = f(t)$ over the interval $[0, 1]$, where $f(t) = t\sqrt{1-t^2}$.

Solution Hopefully, the given function satisfies $f(t) \geq 0$ on the given interval, otherwise we will have to draw a graph (using the methods of Chapter 5, see Figure 134). However, we see directly (without using the graph) that $f(t) \geq 0$ on the interval $0 \leq t \leq 1$. OK, now we need an antiderivative. Since we see a square root, we think of a power, and so we try to use the first of Table 6.5, with $r = 1/2$. Then with $\square = 1 - t^2$, $\square'(t) = -2t$, we find

$$
\begin{aligned}
\mathcal{F}(t) &= \int t\sqrt{1-t^2}\, dt = -\frac{1}{2}\int \frac{d\square}{dt}\square^{\frac{1}{2}}\, dt,\\
&= -\frac{1}{3}\square^{3/2} + C,\\
&= -\frac{1}{3}(1-t^2)^{3/2} + C.
\end{aligned}
$$

So we can choose $C = 0$ as usual and then the required area is simply given by the definite integral in Table 6.9, or

$$
\begin{aligned}
Area &= -\frac{1}{3}(1-t^2)^{3/2}\Big|_0^1,\\
&= \frac{1}{3}.
\end{aligned}
$$



The area (equal to 2/3) under the curve $y = f(x)$, the $x-$axis, and the vertical lines $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$, is the area of the shaded region where

$$f(x) = \tan^2 x \sec^2 x.$$

In fact, since this function is an *even function*, its area is *double* the area of the single shaded region to the right or left of the $y-$axis.

Figure 135.

Example 284. Evaluate

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x\, dx,$$

and interpret your result geometrically.

Solution The given function satisfies $f(x) \geq 0$ on the given interval, because both the trigonometric terms in the integrand are 'squared'. So, the natural interpretation of our result will be as an area under the curve, etc. Now we need an antiderivative. Since we see powers, we think about using the first of Table 6.5, with $r = 2$. OK, but what is $\square$? If we let $\square = \tan x$, then $\square'(x) = \sec^2 x$, (table 6.6). This is really good because now

$$
\begin{aligned}
\mathcal{F}(x) &= \int \tan^2 x \sec^2 x\, dx = \int \square^2 \frac{d\square}{dx}\, dx,\\
&= \frac{1}{3}\square^3 + C,\\
&= \frac{1}{3}(\tan x)^3 + C.
\end{aligned}
$$

Choosing $C = 0$ as usual the required area is again simply given by the definite integral in Table 6.9, or (see Figure 135 for the region under consideration),

$$\begin{aligned} Area &= \frac{1}{3}(\tan x)^3 \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} . \\ &= \frac{2}{3}. \end{aligned}$$

NOTES:

6.5 Chapter Exercises

Find the most general antiderivative of the following functions. Let k be real constant.

1. $(x+1)^{26}$
2. $\cos 2x$
3. $\sqrt{2x+1}$
4. $x\sqrt{1-4x^2}$
5. $\sin x + \cos x$
6. $\sqrt{5-2x}$
7. $\sin 2x$
8. $x^{1.5} - \sin(1.6x)$
9. $3\sec^2 x$
10. $x(x^2+1)^{99}$
11. $\csc 3x \ \cot 3x$
12. xe^{-3x^2}
13. e^{-kx}
14. $\cos kx, \quad k \neq 0$
15. $\sin kx, \quad k \neq 0$

Evaluate the following definite integrals using any method.

16. $\int_0^1 (2x+1)\ dx,$
17. $\int_{-1}^1 x^3\ dx,$
18. $\int_0^2 (3x^2+2x-1)\ dx,$
19. $\int_0^{\pi/2} \sin^4 x\ \cos x\ dx,$
20. $\int_0^1 x3^{x^2}\ dx,$
21. $\int_0^1 2^{-x}\ dx,$
22. $\int_0^{\pi} \cos^2 x\ \sin x\ dx,$
23. $\int_{-2}^2 (x^2+1)\ dx,$
24. $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\ dx,$
25. $\int_0^2 xe^{x^2}\ dx,$

26. Expand and find the value of the following sum, $\sum_{i=1}^{n} 12\left(\frac{i}{n}\right)^2$
27. Use the method of Example 279 to show that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n}e^{\frac{i}{n}} = e-1.$$

- *Hint:* Let $f(x) = e^x$.

28. **Hard.** Show that

$$\lim_{n\to\infty}\left(\frac{n}{n^2+0^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+...+\frac{n}{n^2+(n-1)^2}\right)=\frac{\pi}{4}.$$

   - *Hint:* Factor out n^2 from each denominator and let $f(x) = (1 + x^2)^{-1}$.

29. This problem is **Really, really, hard**, but not impossible! A **prime number** is a positive integer whose only proper divisors are 1 and itself. For example, 2, 3, 5, 11, are primes while $4, 8, 9$ are not primes. It was proved by **Euclid of Alexandria** over 2000 years ago, that there is an infinite number of such primes. Let's label them by $\{p_1, p_2, p_3, ...\}$ where, for the purpose of this Exercise, we take it that $p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 5, p_5 = 7, p_6 = 11, p_7 = 13, p_8 = 17,$ *etc.* Show that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{n^3}{\sqrt{n^8-i^2n^6-2ip_n-{p_n}^2}}=\frac{\pi}{2}.$$

   Hint: Use the fact the $n^{th}-$prime can be estimated by $p_n < 36n\ln n$, for every $n \geq 2$. This estimate on the $n^{th}-$prime number is called **Sierpinski's Estimate**.

   This crazy-looking limit can be verified theoretically using the Riemann integral and the fundamental idea of Example 279. You can actually verify the stated convergence using your calculator (or a simple C++ program) and you'll realize that the convergence is very slow, that is, even after $n = 20,000$ you may only get 4 or 5 significant digits!

Show that the integrand is positive, calculate the following definite integrals, and interpret your result as an area.

30. $\displaystyle\int_0^{\pi}\sin x\,dx$

31. $\displaystyle\int_0^{\frac{\pi}{2}}|\cos x\ -\ \sin x|\,dx$

32. $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{8}}\frac{\cos 2x}{\sin^2 2x}\,dx$

33. $\displaystyle\int_0^1 t^2\sqrt{1+t^3}\,dt$

34. $\displaystyle\int_0^1\frac{x}{1+x^4}\,dx$

Use Leibniz's Rule and/or L'Hospital's Rule to justify the values of the following limits

35. Show that $\displaystyle\lim_{x\to 0+}\frac{d}{dx}\int_1^{x^2}\frac{\sin t}{t^{3/2}}\,dt = 2.$

36. Show that $\displaystyle\lim_{x\to\infty}\frac{d}{dx}\int_{\sqrt{3}}^{\sqrt{x}}\frac{r^3}{(r+1)(r-1)}\,dr = \frac{1}{2}.$

37. Show that $\displaystyle\lim_{x\to\infty}\frac{d}{dx}\int_x^{x^2}e^{-t^2}\,dt = 0.$

38. Show that $\displaystyle\lim_{x\to 0^+}\frac{d}{dx}\int_1^{\sqrt{x}}\frac{\sin(y^2)}{2y}\,dy = \frac{1}{4}.$

39. Show that $\lim_{x\to 0^+} \frac{d}{dx} \int_1^{\sin x} \frac{\ln t}{\ln(\text{Arcsin}\, t)}\, dt = 1.$

40. If c is a constant, show that $\lim_{t\to 0+} \frac{d}{dt} \int_{2\pi-ct}^{2\pi+ct} \frac{\sin x}{cx}\, dx = 0.$

41. Show that $\lim_{h\to 0+} \frac{d}{dx} \left(\frac{1}{h} \int_{x-h}^{x+h} \sqrt{t}\, dt \right) = \frac{1}{\sqrt{x}}.$

42. Show that $\lim_{x\to 0} \frac{1}{2x} \int_{-x}^{x} \cos t\, dt = 1.$

Find the solution of the following differential equations subject to the given initial condition. Leave your solutions in implicit form.

43. $\frac{dy}{dx} = \frac{x^3}{y^4}, \quad y(0) = 1.$

44. $\frac{dy}{dx} = \frac{\sin x}{\cos y}, \quad y(0) = \frac{\pi}{2}.$

45. $\frac{dy}{dx} = (1+y^2)\, e^{2x}, \quad y(0) = 1.$

46. $\frac{d^2y}{dx^2} = 24x^2 + 8x, \qquad y(0) = 0, y'(0) = 1.$

47. $\frac{d^3y}{dx^3} = -24x, \qquad y(0) = 0, y'(0) = 0, y''(0) = -1.$

48. $\frac{d^2y}{dx^2} = e^x, \qquad y(0) = 0, y'(0) = 0.$

49. $\frac{d^4y}{dx^4} = 2, \qquad y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 2.$

50. A company producing hand held computers has a marginal cost per computer of
$$MC(x) = 60 + \frac{40}{x+10}$$
dollars per computer at production level x per week.
    (a) Find the increase in total costs resulting from an increase in production from $x = 20$ to $x = 40$ computers per week.
    (b) If fixed costs of production are $C(0) = 5000$ dollars per week, find the total cost of producing $x = 20$ hand held computers per week.

51. An investment is growing at the rate of $\frac{500e^{\sqrt{t}}}{\sqrt{t}}$ dollars per year. Find the value of the investment after 4 years if its initial value was \$ 1000 (**Hint:** Let $\square = \sqrt{t}$ and Table 6.5.)

Suggested Homework Set 23. *Do problems 19, 25, 27, 34, 38, 45*

6.6 Using Computer Algebra Systems

Use your favorite Computer Algebra System (CAS), like Maple, MatLab, etc., or even a graphing calculator to answer the following questions:

1. Estimate the value of the definite integral $\int_0^1 e^{x^2}\,dx$ to six significant digits.
2. Show by differentiating that $\int e^{x^2}\,dx \neq \frac{e^{x^2}}{2x}$ as many would like to believe!
3. Let n be a given integer, $n \geq 1$. Calculate the value of the sum

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2.$$

   Verify that for any given integer $n \geq 1$ we actually get

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

   Can you prove this formula using the method of mathematical induction?
4. Find a formula for the sum $\sum_{k=1}^{n} 3k + 2k^2$.
5. Let $n \geq 1$ be an integer. Prove the trigonometric identity

$$\sec((n+1)x)\ \sec(nx) = \frac{\tan((n+1)x) - \tan(nx)}{\sin x},$$

   for all values of x for which the denominator is defined.
6. Find a formula for the sum $\sum_{k=1}^{n} k^4$. Can you prove it using mathematical induction?
7. Confirm the values of the limits obtained in Exercises 35-42 above using your CAS.
8. Try to verify the result in Exercise 28 above using your CAS.
9. Evaluate $\int \frac{\sin 2x}{1+\cos x}\,dx$.

NOTES:

Chapter 7

Techniques of Integration

The Big Picture

In this chapter we describe the main techniques used in evaluating an indefinite or definite integral. Many such integrals cannot be evaluated simply by applying a formula so one has to look at the integrand carefully, move terms around, simplify, and ultimately use something more elaborate like a substitution or a, so-called, Integration by Parts, among many other possibilities. However, with the **Substitution Rule** and **Integration by Parts** we can evaluate many integrals. Other techniques for evaluation depend on the actual *form* of the integrand, whether it has any "squared" expressions, trigonometric functions, etc. This Chapter is at the core of *Integral Calculus*. Without it, the Theory of the Integral developed in Chapter 7 would remain just a theory, devoid of any practical use.

Review

Review Chapter 6, especially the examples involving the evaluation of specific definite and indefinite integrals. Also, always keep in mind Tables 6.5, 6.6, and Table 6.7. Many of the integrals can be simplified or transformed to one of the basic forms in these Tables and so, the better you remember them, the faster you'll get to the final answers. You should also review Chapter 3, in particular, the Chain Rule.

7.1 Trigonometric Identities

Before we start this new chapter on *Techniques of Integration* let's review some trigonometric identities, identities which we reproduce here, for convenience. Don't forget, **you should remember this stuff!** It is true that: For any symbols θ, x, y representing real numbers, (or angles in radians),

$$\sin^2\theta + \cos^2\theta = 1 \tag{7.1}$$

$$\sec^2\theta - \tan^2\theta = 1 \tag{7.2}$$

$$\csc^2\theta - \cot^2\theta = 1 \tag{7.3}$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta \tag{7.4}$$

$$2\sin\theta\cos\theta = \sin 2\theta \tag{7.5}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \tag{7.6}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{7.7}$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y \tag{7.8}$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \tag{7.9}$$

$$\sin(-\theta) = -\sin\theta \tag{7.10}$$

$$\cos(-\theta) = \cos\theta \tag{7.11}$$

If you know these basic formulae you can deduce almost every other basic trigonometric formula that you will need in your study of Calculus. For example, if we let $y = -z$ in (7.9) and use (7.10) and (7.11) with $\theta = z$ we get a "*new*" (7.9) formula (see (7.12) below).

$$\begin{aligned} \cos(x-z) &= \cos x\cos(-z) - \sin x\sin(-z), \\ &= \cos x\cos z + \sin x\sin z. \end{aligned}$$

And since x, z are any two **arbitrary** angles, it doesn't matter what "*symbol*" we use to describe them, so we can also write:

$$\cos(x-y) = \cos x\cos y + \sin x\sin y. \tag{7.12}$$

Similarly, (by setting $y = -z$ in (7.8) and use (7.10) and (7.11), once again) we can show that:

$$\sin(x-y) = \sin x\cos y - \cos x\sin y. \tag{7.13}$$

Combining the expressions (7.8) and (7.13) we get

$$\sin(x+y) + \sin(x-y) = 2\sin(x)\cos(y)$$

(since the other two terms cancel each other out), or

$$\sin x\cos y = \frac{\sin(x+y) + \sin(x-y)}{2}. \tag{7.14}$$

We can use the same idea to show that when (7.9) and (7.12) are combined we get

$$\cos x\cos y = \frac{\cos(x+y) + \cos(x-y)}{2}. \tag{7.15}$$

Of course, once we know (7.14) and (7.15) you don't really have to worry about memorizing the formula for "$\cos x\sin y$" because we can **interchange x and y** in (7.14) to obtain

$$\sin y\cos x = \cos x\sin y = \frac{\sin(x+y) + \sin(y-x)}{2}.$$

Note that from (7.10) we can write $\sin(-\theta) = -\sin\theta$ for $\theta = y - x$, right? (Because this 'θ' is just another angle). This means that $-\sin(y-x) = \sin(x-y)$ or, combining this with the last display, we see that

$$\cos x\sin y = \frac{\sin(x+y) - \sin(x-y)}{2}. \tag{7.16}$$

Finally, if we combine (7.12) and (7.9) we find that

$$\sin x\sin y = \frac{\cos(x-y) - \cos(x+y)}{2} \tag{7.17}$$

(since the other two terms cancel each other out, right?)

Lucky for us, with (7.1) to (7.7) we can perform many simplifications in the *integrands*, which we'll introduce below, and these will result in a simpler method for evaluating the integral!

7.2 The Substitution Rule

The evaluation of indefinite and corresponding definite integrals is of major importance in Calculus. In this section we introduce the method of substitution as a possible rule to be used in the evaluation of indefinite or definite integrals. Actually, we SAW and actually USED this Rule earlier, in Section 6.2 (without knowing its name). In this section we are simply elaborating on what we did earlier. It is based on a *change of variable formula*, cf. (7.19) below, for integrals which we now describe. Given a definite integral of f over $I = [a, b]$ we know that

$$\int_a^b f(x)\ dx = \mathcal{F}(b) - \mathcal{F}(a), \tag{7.18}$$

where $\mathcal{F}$ is any antiderivative of f. The substitution $u(x) = t$, where we assume that u has a differentiable inverse function $x = u^{-1}(t)$ inside the integral, corresponds to the **change of variable formula**

$$\mathcal{F}(b) - \mathcal{F}(a) = \int_{u(a)}^{u(b)} f(u^{-1}(t))\ \left(\frac{d}{dt}u^{-1}(t)\right)\ dt. \tag{7.19}$$

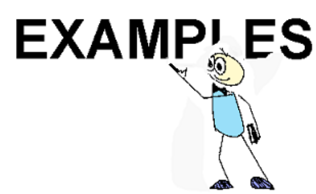
This formula is a consequence of the following argument: By the Chain Rule of Chapter 3, (setting $u^{-1}(t) = \square$),

$$\begin{aligned} \frac{d}{dt}\ \mathcal{F}(u^{-1}(t)) &= \mathcal{F}'(u^{-1}(t))\ \frac{d}{dt}\ u^{-1}(t), & (7.20)\\ &= f(u^{-1}(t))\ \frac{d}{dt}\ u^{-1}(t),\ \text{(since } \mathcal{F}' = f\text{)}. & (7.21) \end{aligned}$$

Integrating both sides of (7.21) over the interval $u(a)$, $u(b)$ and using the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} \int_{u(a)}^{u(b)} f(u^{-1}(t))\ \frac{d}{dt}\ u^{-1}(t)\ dt &= \int_{u(a)}^{u(b)} \frac{d}{dt}\ \mathcal{F}(u^{-1}(t))\ dt,\\ &= \mathcal{F}(u^{-1}(u(b))) - \mathcal{F}(u^{-1}(u(a))),\\ &= \mathcal{F}(b) - \mathcal{F}(a), \end{aligned}$$

which is (7.19).



In practice, we proceed as follows.

Example 285. Evaluate $\displaystyle\int_a^b f(x)\ dx = \int_0^2 2x\ e^{x^2}\ dx.$

Solution We see an exponential to a power so we think, $e^{\square}$, right? That is, we're hoping that we can apply the third formula in Table 6.5 to find an antiderivative. We make the *substitution*

$$u(x) = x^2 = t,$$

whose inverse (that is, whose inverse function) is given by

$$x = \sqrt{t} = u^{-1}(t).$$

Using this substitution we see that the *old* limits, $x = 0$ and $x = 2$, correspond to the *new* limits, $u(0) = 0$, and $u(2) = 4$. Since $f(u^{-1}(t)) = f(\sqrt{t}) = 2\sqrt{t}\ e^t$, and the derivative of $u^{-1}(t)$ is $1/(2\sqrt{t})$, (7.19) becomes, in this case,

$$\begin{aligned} \int_0^2 2x\ e^{x^2}\ dx &= \int_0^4 2\sqrt{t}\ e^t\ \frac{1}{2\sqrt{t}}\ dt,\\ &= \int_0^4 e^t\ dt,\\ &= (e^4 - 1). \end{aligned}$$

Shortcut

The **shortcut to Integration by Substitution**, which amounts to the same thing as an answer can be summarized, in the case of this example, by setting $t = x^2$, $dt = 2x\ dx$ with the limits being changed according to the rule $t = 0$ when $x = 0$, $t = 4$ when $x = 2$. We then write

$$\int_0^2 2x\ e^{x^2}\ dx = \int_0^4 e^t\ dt$$

as before, but more directly. The point is we can leave out all the details about the inverse function and its derivative.

For convenience, let's recall some of the most basic antiderivatives and results from Tables 6.5, 6.6, 6.7. Let $\square$ represent any differentiable function with its derivative denoted by $\square'$. Think of $\square$ as a generic symbol for any other symbol like, u, t, x, etc. Then

$$\int \cos\square \cdot \square'\ dx = \sin\square + C, \tag{7.22}$$

where C is our generic *constant of integration* which follows every indefinite integral.

Next, for $r \neq -1$,

$$\int \square^r dx = \frac{\square^{r+1}}{r+1} + C, \tag{7.23}$$

while, if $r = -1$, we get, provided $\square \neq 0$,

$$\int \frac{\square'}{\square} dx = \ln|\square| + C. \tag{7.24}$$

Formula 7.31 with $\square = x$ is obtained by noticing that $\tan x = \sin x / \cos x$ and then using the substitution $u = \cos x$, $du = -\sin x\ dx$, so that the integral becomes

$$\begin{aligned} \int \tan x\ dx &= \int \frac{-du}{u}, \\ &= -\ln|u| + C, \\ &= -\ln|\cos x| + C, \\ &= \ln\left(|\cos x|^{-1}\right) + C, \\ &= \ln|\sec x| + C. \end{aligned}$$

Furthermore, if e is **Euler's number** $e = 2.71828....$, (see Chapter 4),

$$\int e^{\square}\square' dx = e^{\square} + C \tag{7.25}$$

and since $a^{\square} = e^{\ln a^{\square}} = e^{\square \ln a}$ we get, if $a \neq 0$,

$$\int a^{\square}\square' dx = \frac{a^{\square}}{\ln a} + C \tag{7.26}$$

Next from our *differentiation* formulae for trigonometric functions we obtain our *integration* formulae,

$$\int \sin\square\ \ \square' dx = -\cos\square + C \tag{7.27}$$

$$\int \cos\square\ \ \square' dx = \sin\square + C \tag{7.28}$$

$$\int \sec^2\square\ \ \square' dx = \tan\square + C \tag{7.29}$$

$$\int \csc^2 \Box \quad \Box' dx = -\cot \Box + C \tag{7.30}$$

$$\begin{aligned} \int \tan \Box \quad \Box' dx &= -\ln|\cos \Box| + C, & (7.31) \\ &= \ln|\sec \Box| + C. & (7.32) \end{aligned}$$

$$\int \cot \Box \quad \Box' dx = \ln|\sin \Box| + C \tag{7.33}$$

In some applications you may see functions called **hyperbolic functions**. These functions are denoted by $\sinh x, \cosh x$, etc. in agreement with convention. These hyperbolic functions, analogous to the usual **circular functions** or *trigonometric functions*, are defined by

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2}, \\ \cosh x &= \frac{e^x + e^{-x}}{2}, \\ \tanh x &= \frac{\sinh x}{\cosh x}, \ \textit{etc}, \end{aligned}$$

just as one would expect them to be defined. For these functions it's not hard to see that,

$$\int \sinh \Box \quad \Box' dx = \cosh \Box + C, \tag{7.34}$$

$$\int \cosh \Box \quad \Box' dx = \sinh \Box + C, \tag{7.35}$$

where we don't have to worry about that "minus sign" in front of the sine integral!

Finally, you should remember that

$$\int \sec \Box \tan \Box \quad \Box' dx = \sec \Box + C, \tag{7.36}$$

$$\int \csc \Box \cot \Box \quad \Box' dx = -\csc \Box + C, \tag{7.37}$$

Let $a \neq 0$ be any number (NOT a variable), and let $\Box$ represent some differentiable function. Then the following two formulae may be verified by finding the derivative of the corresponding right-hand sides along with the Chain Rule:

$$\int \frac{\Box'}{\Box^2 + a^2}\, dx = \frac{1}{a}\text{Arctan}\, \frac{\Box}{a} + C \tag{7.38}$$

$$\int \frac{\Box'}{\sqrt{a^2 - \Box^2}}\, dx = \text{Arcsin}\, \frac{\Box}{a} + C \tag{7.39}$$

Example 286. Evaluate the definite integral, $\displaystyle\int_0^{\sqrt{\pi}} x \sin x^2 \ dx$.

Solution

The study of these hyperbolic functions began when someone noticed that the area under a hyperbola was given by an integral of the form

$$\int \sqrt{x^2 - a^2}\ dx$$

whereas the area under a circle was given by an integral of the form

$$\int \sqrt{a^2 - x^2}\ dx.$$

Since the two expressions differ by a factor of $\sqrt{-1}$ and the first integral can be evaluated using the substitution $x = a \sin \theta$, and the area under the hyperbola is related to the natural logarithm, it was guessed that there should be a relation between the logarithm and the circular functions. This led to the birth of these hyperbolic functions whose first comprehensive treatment was given by J.H. Lambert in 1768. Note that, for example, $\sin(\theta\sqrt{-1}) = \sqrt{-1}\sinh\theta$ is one of the fundamental relationships.

Method 1 We need an antiderivative $\mathcal{F}(x)$, first, right? We see the sine of "something", so we try *something* $= \Box$, and hope that this will lead to one of the formulae above, like, maybe, (7.27).

O.K., this is where we use the substitution $\Box = x^2$. Well, if $\Box = x^2$, then $\Box' = 2x$. So, rewriting the integrand in terms of $\Box$ and $\Box'$ we get

$$\int x \sin x^2 \, dx \quad = \quad \int \frac{\Box'}{2} \cdot \sin \Box \, dx,$$

because $x = \frac{\Box'}{2}$ and $\sin(x^2) = \sin(\Box)$. So,

$$\begin{aligned}
\mathcal{F}(x) &= \int x \sin(x^2) dx, \\
&= \frac{1}{2} \int \sin \Box \;\; \Box' \, dx, \\
&= \frac{1}{2} \; (-\cos \Box) + C, \quad \text{(by (7.27))}, \\
&= -\frac{1}{2} (\cos \Box) + C, \\
&= -\frac{1}{2} \cos(x^2) + C, \quad \text{(by \textbf{back substitution}, i.e., setting } \Box = x^2).
\end{aligned}$$

Next, it follows that

$$\begin{aligned}
\int_0^{\sqrt{\pi}} x \sin x^2 \, dx &= -\frac{1}{2} \cos(x^2) \Big|_0^{\sqrt{\pi}}, \\
&= -\frac{1}{2} \cos \pi + \frac{1}{2} \cos(0), \\
&= 1.
\end{aligned}$$

WATCH OUT! When evaluating an indefinite integral you must always *back-substitute* after a change of variable, i.e., always replace your "Box terms" or your "last variable" by the variable in the original integral.

This means that you *start and end your integration with the same symbol!*.

Method 2 This amounts up to the same reasoning as Method 1, above, but it is somewhat shorter to write down. You can use it in case you don't like boxes. Let $t = x^2$. Now think of t as a differentiable function of x. Then we have $\frac{dt}{dx} = 2x$ from which we write the symbolic relation $dt = 2x \, dx$. Don't worry about what this means, right now. Now, let's see "what's left over" after we substitute this t into the integral (and forget about the limits of integration for now).

$$\begin{aligned}
\int x \sin x^2 \, dx &= \int \sin x^2 \, (x \, dx), \\
&= \int \sin t \, (x \, dx),
\end{aligned}$$

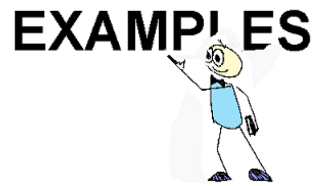
where we have to write the stuff inside the parentheses (i.e., the stuff that is "left-over") in terms of dt. But there is only the term $x \, dx$ that is "left over". So, we need to solve for the symbol, $x \, dx$ in the expression for dt. Since $dt = 2x \, dx$, this gives us

$$x \, dx = \frac{dt}{2}.$$

The right-hand side of the previous formula depends only on the new variable t and this is good. There shouldn't be any x's floating around on the right-side, just t's.

Using this, we see that

$$\begin{aligned}\int x\sin x^2\ dx &= \int \sin t\,(x\ dx),\\ &= \int \sin t\ \left(\frac{dt}{2}\right),\\ &= \frac{1}{2}\int \sin t\ dt.\end{aligned}$$



Now, we put the *limits of integration* back in on the left. The ones on the right are to be found using the substitution formula, $t = x^2$. When $x = 0$, $t = 0$ (because $t = x^2$). Next, when $x = \sqrt{\pi}$, $t = x^2 = (\sqrt{\pi})^2 = \pi$. So, according to our *Change of Variable* formula we get,

$$\begin{aligned}\int_0^{\sqrt{\pi}} x\sin x^2\ dx &= \frac{1}{2}\int_0^{\pi} \sin t\ dt,\\ &= -\frac{1}{2}\cos t\Big|_0^{\pi},\\ &= 1,\end{aligned}$$

as before.

NOTE: If this Example seems long it is because *we put in all the details.* Normally, you can skip many of these details and get to the answer faster. See the *Snapshots* later on for such examples.

Example 287. Evaluate $\mathcal{G}(y) = \int e^y \sec e^y \tan e^y\ dy$.

Solution We want an antiderivative, $\mathcal{G}(y)$, right? We see a bunch of trig. functions acting on *the same* symbol, namely, e^y. So, let's just replace this symbol by $\square$ or t, or "whatever" and see what happens ... Maybe we'll get lucky and this will look like maybe, (7.36), with something in the Box, (x or t or whatever.)

So, let $x = e^y$. Then, proceeding as above, $dx = e^y\ dy$ and now, (do you see it?),

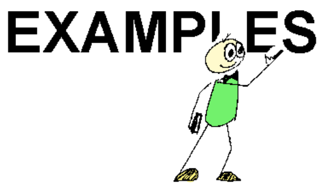
$$\begin{aligned}\mathcal{G}(y) &= \int \sec e^y\ \tan e^y\ (e^y\ dy),\\ &= \int \sec x\ \tan x\ \ dx,\end{aligned}$$

which is (7.36) with $\square$ **replaced by** $\boldsymbol{x}$. So, we get

$$\begin{aligned}\mathcal{G}(y) &= \sec x\ + C,\\ &= \sec(e^y)\ + C, \quad \text{(after the \textbf{back-substitution})}.\end{aligned}$$

Example 288. Evaluate $\int \frac{\tan(\ln x)}{x}\ dx$.

Solution Here we see the tangent of *something* and an x in the denominator. Remember that we really don't know what to do when we're starting out, so we have to make "good guesses". What do we do? Let's try $t = \ln x$ and hope that this will lead to an easier integral involving $\tan t$, hopefully something like (7.31), with



$\square = t$. By substituting $t = \ln x$, we get $dt = \frac{1}{x}\ dx$ so, now what? Well,

$$\begin{aligned}
\int \frac{\tan(\ln x)}{x}\ dx &= \int \tan(\ln x) \cdot \left(\frac{1}{x}\ dx\right), \\
&= \int \tan t\ dt, \\
&= -\ln|\cos t| + C, \quad \text{(by (7.31) with } \square = t), \\
&= -\ln|\cos(\ln x)| + C, \quad \text{(since } t = \ln x),
\end{aligned}$$

and this is the answer.

Note that $\ln(A^{-1}) = -\ln A$ for any $A > 0$, by a property of logarithms. So, replacing A by $\cos(\ln x)$ and using the fact that the reciprocal of the cosine function, cos, is the secant function, sec, we can also write this answer as

$$\int \frac{\tan(\ln x)}{x}\ dx = \ln|\sec(\ln x)| + C.$$

Example 289. Evaluate $\displaystyle\int \frac{\sinh\sqrt{z}}{\sqrt{z}}\ dz$.

Solution Recall the definition of the sinh function from before and its basic property in (7.34). We see a "square root " in both the numerator and denominator so, what if we try to replace the square root by a new variable, like, x, or box, $\square$? Will this simplify the integrand? Let's see if we can get this integral into the more familiar form, (7.34), if possible. So, we set $x = \sqrt{z}$. Then, $dx = \frac{1}{2\sqrt{z}}\ dz$. Let's see "what's left over" after we substitute this $x = \sqrt{z}$ into the integral.

$$\int \frac{\sinh\sqrt{z}}{\sqrt{z}}\ dz = \int \sinh x \left(\frac{1}{\sqrt{z}}\ dz\right),$$

where we have to write the stuff inside the parentheses in terms of dx. Now there is only $\frac{1}{\sqrt{z}}\ dz$ that's "left over". So, we need to solve for the symbol, $\frac{1}{\sqrt{z}}\ dz$ in the expression for dx. This gives us

$$\frac{1}{\sqrt{z}}\ dz = 2\ dx.$$

The right-hand side of the previous formula depends only on the new variable x and this is good. There shouldn't be any z's floating around on the right-side, just x's. Now,

$$\begin{aligned}
\int \frac{\sinh\sqrt{z}}{\sqrt{z}}\ dz &= \int \sinh\sqrt{z}\left(\frac{1}{\sqrt{z}}\ dz\right), \\
&= \int \sinh x \cdot (2\ dx) \quad \text{(since } 2\ dx = \frac{1}{\sqrt{z}}\ dz), \\
&= 2\int \sinh x\ dx, \\
&= 2\cosh x + C, \quad \text{(by (7.34) with } \square = x), \\
&= 2\cosh\sqrt{z} + C, \quad \text{(since } x = \sqrt{z} \text{ to begin with)},
\end{aligned}$$

and we're done!

Example 290. Evaluate $\displaystyle\int_0^1 x^2 \sec^2 x^3\ dx$.

Solution This complicated-looking expression involves a $\sec^2$, right? So, we set $u = x^3$, and hope that this will make the integrand look like, say, (7.29), with

$\square = u$. In this case, $u = x^3$ means that $du = 3x^2\,dx$. Now, let's see "what's left over" after we substitute this into the integral.

$$\begin{aligned}\int x^2 \sec^2 x^3\,dx &= \int \sec^2 x^3\,(x^2\,dx),\\ &= \int \sec^2 u\,(x^2\,dx),\end{aligned}$$

where we have to write the stuff inside the parentheses in terms of du. But there is only $x^2\,dx$ that's "left over". So, we need to solve for the symbol, $x^2\,dx$ in the expression for du, namely $du = 3x^2\,dx$. This gives us

$$x^2\,dx = \frac{du}{3}.$$

The right-hand side of the previous formula depends only on the new variable u and this is good. There shouldn't be any x's floating around on the right-side, just u's. Now,

$$\begin{aligned}\int x^2 \sec^2 x^3\,dx &= \int \sec^2 u\,(x^2\,dx),\\ &= \int \sec^2 u\left(\frac{du}{3}\right),\\ &= \frac{1}{3}\int \sec^2 u\,du.\end{aligned}$$

When $x = 0$ we have $u = 0$, while if $x = 1$ we have $u = x^3 = (1)^3 = 1$. So, the new limits of integration for u are the same as the old limits for x. OK, we were just *lucky*, that's all. It follows that

$$\begin{aligned}\int_0^1 x^2 \sec^2 x^3\,dx &= \frac{1}{3}\int_0^1 \sec^2 u\,du,\\ &= \frac{1}{3}\tan u\Big|_0^1, \quad (\text{ from (7.29), with } \square = u),\\ &= \frac{1}{3}\tan 1 - 0,\\ &\approx 0.5191.\end{aligned}$$

NOTE: These example serve to reinforce the technique of **substitution** which is used in cases where we feel that a particular substitution might simplify the **look** of an integrand, thereby allowing easy evaluation of the integral by means of formulae like those in (7.22 - 7.35) and the ones in Tables 6.5, 6.6, 6.7.

As a check: You can always check your answer by *differentiating* it, and this is easy when you know the Chain Rule really well (Chapter 3.5).

SNAPSHOTS

Example 291. Evaluate $\displaystyle\int x^2\sqrt{1+x^3}\,dx$.

Solution Let $\boxed{u = 1 + x^3}$. Then $du = 3x^2\,dx$ and

$$x^2\,dx = \frac{du}{3}.$$

So,

$$\begin{aligned}
\int x^2\sqrt{1+x^3}\,dx &= \int \sqrt{1+x^3}\,(x^2\,dx),\\
&= \frac{1}{3}\int \sqrt{u}\,du,\\
&= \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)u^{\frac{3}{2}}+C,\\
&= \frac{2}{9}(1+x^3)^{\frac{3}{2}}+C.
\end{aligned}$$

$\boxed{\text{Check your answer by differentiation!}}$

Example 292. Evaluate $\displaystyle\int_{-1}^{0}\frac{x^3\,dx}{\sqrt[4]{1+x^4}}$.

Solution

Let $\boxed{u=1+x^4}$. Then $du=4x^3\,dx$, or, solving for the symbols $x^3\,dx$ (which appear in the numerator of the integrand) we get

$$x^3\,dx = du/4.$$

Substituting this information back into the integral we get

$$\begin{aligned}
\int\frac{x^3\,dx}{\sqrt[4]{1+x^4}} &= \frac{1}{4}\int\frac{du}{\sqrt[4]{u}},\\
&= \frac{1}{4}\int u^{-\frac{1}{4}}\,du,\\
&= \frac{1}{4}\cdot\frac{4}{3}u^{\frac{3}{4}}+C,
\end{aligned}$$

or,

$$\int\frac{x^3\,dx}{\sqrt[4]{1+x^4}} = \frac{1}{3}(1+x^4)^{\frac{3}{4}}+C.$$

It follows that

$$\begin{aligned}
\int_{-1}^{0}\frac{x^3\,dx}{\sqrt[4]{1+x^4}} &= \frac{1}{3}(1+x^4)^{\frac{3}{4}}\Big|_{-1}^{0},\\
&= \frac{1}{3}-\frac{1}{3}2^{\frac{3}{4}},\\
&= \frac{1}{3}\left(1-\sqrt[4]{8}\right).
\end{aligned}$$

Equivalently, we could use the *Change of Variable* formula, and find (since $u=2$ when $x=-1$ and $u=1$ when $x=0$),

$$\begin{aligned}
\int_{-1}^{0}\frac{x^3\,dx}{\sqrt[4]{1+x^4}} &= \frac{1}{4}\int_{2}^{1}u^{-\frac{1}{4}}\,du,\\
&= \frac{1}{3}u^{\frac{3}{4}}\Big|_{2}^{1},\\
&= \frac{1}{3}\left(1-\sqrt[4]{8}\right),
\end{aligned}$$

as before.

Example 293. Evaluate $\displaystyle\int\frac{x+1}{2\sqrt{x+1}}\,dx$.

Solution *Always try to simplify the integrand whenever possible!* In this case we don't really have to use a complicated substitution since

$$\frac{x+1}{2\sqrt{x+1}} = \frac{\sqrt{x+1}}{2},$$

so, all we really want to do is to evaluate

$$\begin{aligned}\int \frac{\sqrt{x+1}}{2}\,dx &= \frac{1}{2}\int \sqrt{x+1}\,dx,\\ &= \frac{1}{2}\cdot\frac{2}{3}(x+1)^{\frac{3}{2}} + C,\\ &= \frac{1}{3}(x+1)^{\frac{3}{2}} + C,\end{aligned}$$

by the Generalized Power Rule for integrals, (Table 6.5 with $r = 1/2$, $\square = x+1$).

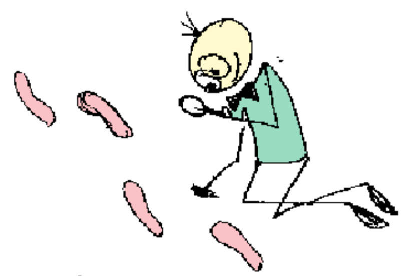
Example 294. Evaluate $\displaystyle\int \frac{y^2+1}{y^3+3y+1}\,dy$.

Solution Let $u = y^3+3y+1$. Then $du = (3y^2+3)\,dy = 3(y^2+1)\,dy$, and it contains the term, $(y^2+1)\,dy$, which also appears in the numerator of the integrand. Solving for this quantity and rewriting we get

$$\frac{(y^2+1)\,dy}{y^3+3y+1} = \left(\frac{du}{3}\right)\frac{1}{u},$$

right? This means that the integral looks like

$$\begin{aligned}\int \frac{y^2+1}{y^3+3y+1}\,dy &= \int \frac{du}{3u},\\ &= \frac{1}{3}\int \frac{du}{u},\\ &= \frac{1}{3}\ln|u| + C,\ \text{(by the second entry in Table 6.5, } \square = u),\\ &= \frac{1}{3}ln\left|y^3+3y+1\right| + C.\end{aligned}$$



Example 295. Evaluate $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$.

Solution The *square root sign* makes us think of the substitution $u = x-1$. Then $du = 1\,dx = dx$. Okay, now solve for x (which we need as it appears in "what's left over" after we substitute in $u = x-1$). So, $x = u+1$. When $x = 1$, $u = 0$ and when $x = 2$, $u = 1$. These give the new limits of integration and the integral becomes,

$$\begin{aligned}\int_1^2 x\sqrt{x-1}\,dx &= \int_0^1 (u+1)\sqrt{u}\,du,\\ &= \int_0^1 \left(u^{3/2}+u^{1/2}\right)\,du,\\ &= \left.\left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right)\right|_0^1,\\ &= \frac{2}{5}+\frac{2}{3},\\ &= \frac{16}{15}.\end{aligned}$$

Notice that the opening equation in the last display had only u's on the right: You always try to do this, so you may have to solve for the "original" variable (in this case, x, in terms of u), sometimes.

Example 296. Evaluate $\displaystyle\int (\ln x)^3 \frac{dx}{x}$.

Solution We see a power so we let $u = \ln x$ and see... In this case, $du = \frac{dx}{x}$ which is precisely the term we need! So,

$$\begin{aligned}\int (\ln x)^3 \frac{dx}{x} &= \int u^3\, du \\ &= \frac{u^4}{4} + C \\ &= \frac{(\ln x)^4}{4} + C,\end{aligned}$$

because of the "back substitution", $u = \ln x$.

Example 297. Evaluate $\displaystyle\int \frac{e^{-x}}{1+e^{-x}}\, dx$.

Solution Trying $u = e^{-x}$ gives $du = -e^{-x}\, dx$, which we have to write in terms of u. This means that $du = -u\, dx$ or $dx = -du/u$. The integral now looks like,

$$\begin{aligned}\int \frac{e^{-x}}{1+e^{-x}}\, dx &= -\int \frac{u\, du}{(1+u)u}, \\ &= -\int \frac{1}{1+u}\, du, \\ &= -\ln|1+u| + C, \\ &= \ln|1+e^{-x}| + C, \\ &= \ln(1+e^{-x}) + C.\end{aligned}$$

Example 298. Evaluate $\displaystyle\int_0^{\frac{1}{3}} \frac{dx}{1+9x^2}$.

Solution The natural guess $u = 9x^2$ leads nowhere, since $du = 18x\, dx$ and we can't solve for dx easily, (just in terms of u and du). The denominator does have a square term in it, though, and so this term looks like $1 + (\mathit{something})^2$, which reminds us of an Arctan integral, (third item in Table 6.7, with $\square = 3x$). Okay, this is our clue. If we let $\mathit{something} = u = 3x$, then the denominator looks like $1 + u^2$ and the subsequent expression $du = 3\, dx$ is not a problem, as we can easily solve for dx in terms of du. Indeed, $dx = du/3$. This looks good so we try it and find,

$$\begin{aligned}\int \frac{dx}{1+9x^2} &= \int \frac{du}{3(1+u^2)}, \\ &= \frac{1}{3}\int \frac{du}{1+u^2}, \\ &= \frac{1}{3}\text{Arctan}\, u + C, \\ &= \frac{1}{3}\text{Arctan}\, 3x + C.\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^{\frac{1}{3}} \frac{dx}{1+9x^2} &= \frac{1}{3}\text{Arctan}\, 3x\Big|_0^{\frac{1}{3}}, \\ &= \frac{1}{3}\text{Arctan}\, 1 - \frac{1}{3}\text{Arctan}\, 0, \\ &= \frac{1}{3}\cdot\frac{\pi}{4} - 0, \\ &= \frac{\pi}{12}.\end{aligned}$$

Exercise Set 31.

Evaluate the following integrals

1. $\int (2x-1)^{99}\ dx$
2. $\int 3(1+x)^{5.1}\ dx$
3. $\int_0^1 \frac{1}{(3x+1)^5}\ dx$
4. $\int \frac{dx}{(x-1)^2}$
5. $\int x(1-x^2)^{100}\ dx$
6. $\int x\ 2^{x^2}\ dx$
   - *Hint:* See Table 6.5.
7. $\int_0^{\frac{\pi}{4}} \tan x\ dx$
8. $\int z^2\ e^{z^3}\ dz$
9. $\int \sqrt[3]{2-x}\ dx$
10. $\int_{-2}^{2} \cos(2x+4)\ dx$
11. $\int \frac{\cos t\ dt}{1+\sin t}$
12. $\int \frac{x}{\sqrt{1-x^2}}\ dx$
13. $\int \frac{y+1}{y^2+2y}\ dy$
14. $\int \frac{dx}{\cos^2 x\sqrt{1+\tan x}}$
15. $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x}\ dx$
16. $\int \sec x\ dx$
   - *Hint:* Multiply the numerator and denominator by the same expression, namely, $\sec x + \tan x$, use some identity, and then a substitution.
17. $\int (z^4+z)^4 \cdot (4z^3+1)\ dz$
18. $\int \frac{\sin x}{1+\cos^2 x}\ dx$
19. $\int_0^1 \frac{t}{t^4+1}\ dt$
20. $\int x\ \sin^3(x^2+1)\cos(x^2+1)\ dx$
   - *Hint:* This one requires *two* substitutions.
21. $\int \frac{3x-1}{x^2+1}\ dx$
   - *Hint:* Separate this integral into two parts and apply appropriate substitutions to each one, separately.

22. $\displaystyle\int_e^{e^2} \frac{dx}{x\ln x}$, where $e = 2.71828...$ is Euler's constant.

23. $\displaystyle\int \frac{(\operatorname{Arctan} x)^2}{1+x^2}\,dx$

24. $\displaystyle\int \frac{\cosh(e^t)}{e^{-t}}\,dt$

25. $\displaystyle\int \frac{ds}{\sqrt{1-25s^2}}$

26. $\displaystyle\int_{\pi^2}^{4\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

27. $\displaystyle\int x\,e^{x^2}\,dx$

28. $\displaystyle\int \frac{1+y}{\sqrt{1-y^2}}\,dy$

29. $\displaystyle\int \sec(\ln x)\tan(\ln x)\,\frac{dx}{x}$

30. $\displaystyle\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx$

31. $\displaystyle\int_0^1 e^t e^{e^t}\,dt$

32. $\displaystyle\int x\,(1.5^{x^2+1})\,dx$

Suggested Homework Set 24. *Work out problems 1, 5, 7, 12, 15, 19, 21, 26, 29*

NOTES:

7.3 Integration by Parts

When you you've tried everything in the evaluation of a given integral using the Substitution Rule, you should resort to **Integration by Parts**. This procedure for the possible evaluation of a given integral (it doesn't always work, though) is the "reverse operation" of the Product Rule for derivatives (see Chapter 3). Recall that the Product Rule states that if u, v are each differentiable functions, then

$$\frac{d}{dx}(uv) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx},$$

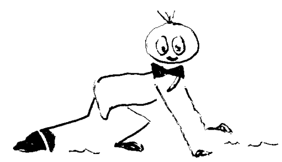
where the "·" represents the usual product of two functions. Use of the Fundamental Theorem of Calculus (Chapter 7), tells us that, on integrating both sides and rearranging terms, we find

$$\boxed{\int \left(u \frac{dv}{dx} \right) \, dx \quad = \quad u(x)v(x) - \int \left(v \frac{du}{dx} \right) \, dx + C, \qquad (7.40)}$$

where C is a constant of integration. This technique is useful when "nothing else seems to work". In fact, there is this old saying in Calculus, that says, " If you don't know what to do, try *integrating by parts*".

This formula is more commonly written as:

$$\int u \, dv = uv - \int v \, du + C, \qquad (7.41)$$



where u, v are functions with the property that the symbols $u \, dv$ make up all the terms appearing to the right of the integral sign, but we have to determine what these u, v actually look like!

Example 299. Evaluate $\displaystyle\int x \sin x \, dx.$

Solution This business of integration can be time consuming. First, we'll do it the *long* way. It is completely justifiable but it has the disadvantage of just being long. After this we'll do it the *normal way* and, finally, we'll show you the *lightning fast* way of doing it.

The long way

We need to rewrite the integrand as a product of two functions, u and $\frac{dv}{dx}$ and so WE have to decide how to break up the terms inside the integrand! For example, here, we let

$$u = x, \quad \frac{dv}{dx} = \sin x.$$

Then, in accordance with (7.40), we need to find $\frac{du}{dx}$ and $v(x)$, right? But it is easy to see that

$$\frac{du}{dx} = 1, \quad v(x) = -\cos x,$$

since we are being asked to provide the antiderivative of $\sin x$ and the derivative of x. Combining these quantities into the general formula (7.40), we obtain

$$\begin{aligned}
\int x \sin x \, dx &= \int (u \frac{dv}{dx}) \, dx, \\
&= u(x)v(x) - \int (v \frac{du}{dx}) \, dx + C, \\
&= x \, (-\cos x) - \int (-\cos x) \cdot 1 \, dx + C, \\
&= -x \cos x + \int \cos x \, dx + C, \\
&= -x \cos x + \sin x + C,
\end{aligned}$$

where C is our usual constant of integration. This answer can be checked as usual by differentiating it and showing that it can be brought into the form of "$x \ \sin x$". But this merely involves a simple application of the Product Rule. So, we have shown that,

$$\int x \sin x \, dx = -x \cos x + \sin x + C.$$

The normal way

For decades integrals involving the use of Integrations by Parts have been evaluated using this slighly faster and more convenient method. Let's apply it to the problem at hand. As before, we write

$$u = x, \quad dv = \sin x \, dx.$$

We'll be using the modified version of the formula, namely, (7.41). The principle is the same, we just write

$$du = 1 \, dx, \quad v = -\cos x,$$

and now use (7.41) directly. This gives,

$$\begin{aligned}
\int u \, dv &= uv - \int v \, du, \\
&= -x \cos x - \int (-\cos x) \, dx \\
&= -x \cos x + \sin x + C,
\end{aligned}$$

just like before.

Now comes the **Table Method.** It dates from before the early 1980's and it was made famous by the late high school teacher Jaime Escalante (see the movie *Stand and Deliver*, 1984). We improve it here.

The answer follows by just "**looking**" at the Table below:

x	$\sin x$
1	$-\cos x$
0	$-\sin x$

This last method is essentially **a more rapid way of setting up the Integration by Parts environment** and it is completely justifiable (and also much faster, in most cases). This is how it works: Let's say that we want to evaluate

$$\int u \frac{dv}{dx}\ dx.$$

We set up a special table whose cells contain the entries as described:

$u(x)$	$v'(x)$
Derivatives of $u(x)$ $\downarrow$	Integrals of $v'(x)$ $\downarrow$

so that the **first column (on the left) contains the successive derivatives of *u*** while the **second column (on the right) contains the successive antiderivatives (or integrals) of *v***, (see Table 7.1).

Here, u', u'', u''' denote the successive derivatives of u (they appear in the *first column*) and $v_{(1)}(x), v_{(2)}(x), \ldots$ the successive antiderivatives of v' (without the constants of integration), so that

$$v_{(1)}(x) = \int v(x)\ dx$$

$$v_{(2)}(x) = \int v_{(1)}(x)\ dx$$

$$v_{(3)}(x) = \int v_{(2)}(x)\ dx$$

$$\ldots$$

In Table 7.1, the arrows indicate that the entries connected by such arrows are multiplied together and their product is preceded by the '+' or '−' sign directly above the arrow, in an alternating fashion as we move down the Table. Don't worry, many examples will clarify this technique. Remember that this technique we call the Table Method must give the *correct answer* to any integration by parts problem as it is simply a re-interpretation of the normal method outlined at the beginning of this Section.

Remember: Functions in the **left column are differentiated** while those in the **right column are integrated!**

So, when do we stop calculating the entries in the Table?

Normally, we try to stop when one of the two columns contains a zero entry (normally the one on the left). In fact, this is the general idea in using this Method. But the general rule of thumb is this:

The Table Method

$u(x)$	$+$	$\dfrac{dv}{dx}$
$u'(x)$	$-$	$v(x)$
$u''(x)$	$+$	$v_{(1)}(x)$
$u'''(x)$	$-$	$v_{(2)}(x)$
$u^{(4)}(x)$		$v_{(3)}(x)$
$\ldots$		$\ldots$

The arrows indicate that the entries connected by such arrows are multiplied together and their product is preceded by the '+' or '−' sign directly above the arrow, in an alternating fashion as we move down the Table.

Table 7.1: Schematic Description of the Table Method

Look at every row of the table and see if any of the products of the two quantities in that row can be integrated without much effort. If you find such a row, stop filling in the table and proceed as follows:

Let's say you decide to stop at row "n". Then the **last term in your answer must be the integral of the PRODUCT of the two functions in that row, multiplied by the constant $(-1)^{n-1}$ (which is either ± 1 depending on the whether n is even or odd). We can stop anytime and read a result such as the one in Table 7.2.**

Example 300. Evaluate $I = \displaystyle\int x \sin^{-1}(\frac{1}{x})\, dx, \qquad |x| \geq 1.$

Solution We can see that it if we place the Arcsine term on the right column of our Table, it will be very difficult to integrate, so we can't get to the second row! OK, just forget about it and place it on the left!

$\sin^{-1}(\dfrac{1}{x})$	$+$	x
$-\dfrac{1}{x\sqrt{x^2-1}}$	$-$	$\dfrac{x^2}{2}$
$\cdots$		$\dfrac{x^3}{6}$
$\ldots$		$\ldots$

$u(x)$	$+$	$\dfrac{dv}{dx}$
$u'(x)$	$-$	$v(x)$
$u''(x)$	$+$	$v_{(1)}(x)$
$u'''(x)$	$-$	$v_{(2)}(x)$
$u^{(4)}(x)$	$\xleftarrow{+}$	$v_{(3)}(x)$
$STOP$		$STOP$

$$\int u\frac{dv}{dx}dx = u(x)v(x) - u'(x)v_{(1)}(x) + u''(x)v_{(2)}(x) - u'''v_{(3)}(x) + \int u^{(4)}v_{(3)}dx$$

where the last integral is obtained by taking the antiderivative of the product of the two entries in the given row (here, $v_{(3)}u^{(4)}$), producing an arrow from right to left, and multiplying the product by either "+1" (if the row number is ODD) or "-1" (if the row number is EVEN).

Here, the row number is 5 so we multiply the product by +1 which means that the product stays unchanged. We show this by placing a "+" sign above the last arrow to remind us.

Table 7.2: Example of the Table Method: Stopping at the 5^{th} Row

Okay, now you're looking at this table and you're probably starting to worry!? Things are not getting any easier and there is the calculation of the derivative to go in row three (on the left) which will be a nightmare ... But wait! find the product of the two functions in row 2. This gives us

$$\frac{-x}{2\sqrt{x^2-1}},$$

which is a function which we CAN integrate easily (if we use the substitution $u = x^2 - 1$).

Okay, this is really good so we can STOP at row 2 and use Table 7.2 to get the modified table,

$\sin^{-1}(\frac{1}{x})$	$+$	x
$-\frac{1}{x\sqrt{x^2-1}}$	$-$	$\frac{x^2}{2}$
$\cdots$		$\cdots$

The answer is easily read off as:

$$\begin{aligned}
\int x\sin^{-1}(\frac{1}{x})\,dx &= \frac{x^2}{2}\sin^{-1}(\frac{1}{x}) - \int \frac{x^2}{2}\left(-\frac{1}{x\sqrt{x^2-1}}\right)\,dx,\\
&= \frac{x^2}{2}\sin^{-1}(\frac{1}{x}) + \int \frac{x}{2\sqrt{x^2-1}}\,dx,\\
&= \frac{x^2}{2}\sin^{-1}(\frac{1}{x}) + \frac{1}{4}\int \frac{du}{\sqrt{u}},\ (u = x^2-1,\ du = 2x\,dx, \text{etc.}),\\
&= \frac{x^2}{2}\sin^{-1}(\frac{1}{x}) + \frac{1}{4}(2u^{\frac{1}{2}}),\\
&= \frac{x^2}{2}\sin^{-1}(\frac{1}{x}) + \frac{(x^2-1)^{\frac{1}{2}}}{2} + C.
\end{aligned}$$

7.3.1 The Product of a Polynomial and a Sine or Cosine

The previous method can be used directly to evaluate any integrals of the form

$$\boxed{\int (\textit{polynomial in } x)\ (\textit{sine/cosine function in } x)\ dx}$$

where the integrand is a product of a polynomial and either a sine or a cosine function.

Example 301. Evaluate $\int x\sin x\,dx$, using the method outlined above or summarized in Table 7.3.

Solution We apply the technique in Table 7.3 to our original question, Example 299. Using the rules described we find the table,

SUMMARY We put all the successive derivatives on the left and all the successive antiderivatives (or indefinite integrals) on the right until we reach a "zero term" on the left, or until we decide we want to stop!

$u(x)$	$v'(x)$
Derivatives $\downarrow$	*Integrals* $\downarrow$

Now, find the product of the terms joined by arrows, with the sign in front of that product being the sign appearing over that arrow. Add up all such products to get the answer! In other words, terms connected by arrows are multiplied together and their product is preceded by the '+" or '−' sign directly above the arrow, in an alternating fashion as we move down the Table.

$u(x)$	$\overset{+}{\searrow}$	$\dfrac{dv}{dx}$
u'	$\overset{-}{\searrow}$	$v(x)$
u''	$\overset{+}{\searrow}$	$v_{(1)}$
u'''	$\overset{-}{\searrow}$	$v_{(2)}$
$u^{(4)}$		$v_{(3)}$
$\cdots$		$\cdots$

If you decide to STOP then you must use SOMETHING LIKE the NEXT formulae (for example) to get your answer:

$$\int u\frac{dv}{dx}dx = u(x)v(x) - u'(x)v_{(1)}(x) + \int u''v_{(1)}\ dx, \tag{7.42}$$

or,

$$\int u\frac{dv}{dx}dx = u(x)v(x) - u'(x)v_{(1)}(x) + u''(x)v_{(2)}(x) - \int u'''v_{(2)}\ dx,$$

or,

$$\int u\frac{dv}{dx}dx = u(x)v(x) - u'(x)v_{(1)}(x) + u''(x)v_{(2)}(x) - u'''v_{(3)}(x) + \int u^{(4)}v_{(3)}dx,$$

and so on.

Table 7.3: Efficient Integration by Parts Setup

x	$+$	$\sin x$
1	$-$	$-\cos x$
0	$+$	$-\sin x$

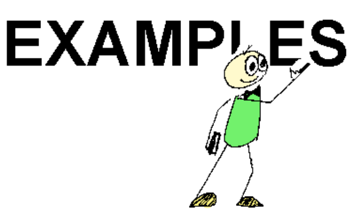
and the answer is read off as follows:

JUST remember to STOP when you see a zero in the LEFT column

$$(+1)\cdot x\cdot(-\cos x)+(-1)\cdot(1)\cdot(-\sin x) \quad = \quad -x\cos x+\sin x,$$

to which we add our constant of integration, C, at the very end!

Example 302. Evaluate $\int 2x^3 \cos 2x \; dx$.



Solution If done the "long way", this example would require a few coffees. Using the method in Table 7.3, however, we can obtain the answer fairly quickly. All that needs to be observed is that, if $a \neq 0$ is a number, then an antiderivative of $\cos ax$ is given by

$$\int \cos ax \; dx \quad = \quad \frac{\sin ax}{a},$$

(from the Substitution Rule) while, if $a \neq 0$, an antiderivative of $\sin ax$ is given by

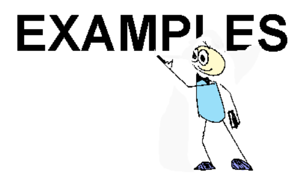
$$\int \sin ax \; dx \quad = \quad -\frac{\cos ax}{a}.$$

The Table now takes the form,

x^3	$+$	$\cos 2x$
$3x^2$	$-$	$\frac{\sin 2x}{2}$
$6x$	$+$	$-\frac{\cos 2x}{2^2}$
6	$-$	$-\frac{\sin 2x}{2^3}$
0		$\frac{\cos 2x}{2^4}$
$\ldots$		$\ldots$

We conclude with the answer in the form,

$$\begin{aligned}
\int 2x^3 \cos 2x \, dx &= 2\int x^3 \cos 2x \, dx, \\
&= 2\left\{\frac{x^3 \sin 2x}{2} - (-1)\frac{3x^2 \cos 2x}{2^2} + (-6x)\frac{\sin 2x}{2^3} - 6\frac{\cos 2x}{2^4}\right\}, \\
&= x^3 \sin 2x + \frac{3}{2}\cdot x^2 \cos 2x - \frac{3}{2}\cdot x \sin 2x - \frac{3}{4}\cdot \cos 2x + C.
\end{aligned}$$



Example 303. Evaluate $\int (x^3 + 2x^2 - x + 3)\ \sin(3x+4)\ dx$.

Solution We proceed as in Example 302. The Table now takes the form,

$x^3 + 2x^2 - x + 3$	$+$	$\sin(3x+4)$
$3x^2 + 4x - 1$	$-$	$-\dfrac{\cos(3x+4)}{3}$
$6x + 4$	$+$	$-\dfrac{\sin(3x+4)}{3^2}$
6	$-$	$\dfrac{\cos(3x+4)}{3^3}$
0		$\dfrac{\sin(3x+4)}{3^4}$
	$\cdots$	$\cdots$

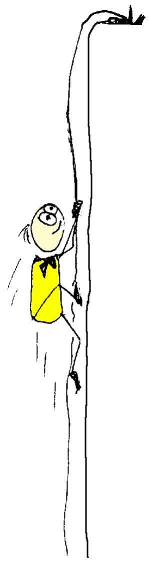
We can conclude with the answer in the form,

$$\begin{aligned}
\int (x^3 + 2x^2 - x + 3)\ \sin(3x+4)\ dx \quad = \quad & -\frac{(x^3 + 2x^2 - x + 3)\ \cos(3x+4)}{3} + \\
& \frac{(3x^2 + 4x - 1)\ \sin(3x+4)}{3^2} + \\
& \frac{(6x+4)\ \cos(3x+4)}{3^3} - \frac{6\sin(3x+4)}{3^4} \\
& +C.
\end{aligned}$$

7.3.2 The Product of a Polynomial and an Exponential

The same idea is used to evaluate an integral involving the product of a polynomial and an exponential term. We set the tables up as before and place the polynomial on the left and differentiate it until we get the "0" function while, on the right, we keep integrating the exponentials only to STOP when the corresponding entry on the left is the 0 function.

Example 304. Evaluate $\int xe^x \ dx$.



Solution In this case, the Table is

x	$+$	e^x
1	$-$	e^x
0		e^x
$\cdots$		$\cdots$

and the final answer can be written as,

$$\int xe^x\,dx \quad = \quad xe^x - e^x + C.$$

> Observe that, if $a \neq 0$ is any number (positive or negative), then an antiderivative of e^{ax} is given by
>
> $$\int e^{ax}\,dx \quad = \quad \frac{e^{ax}}{a},$$
>
> (from the Substitution Rule).

Example 305. Evaluate $\displaystyle\int x^2 e^{-2x}\,dx$.

Solution In this case, the Table is

x^2	$+$	e^{-2x}
$2x$	$-$	$-\dfrac{e^{-2x}}{2}$
2	$+$	$\dfrac{e^{-2x}}{2^2}$
0		$-\dfrac{e^{-2x}}{2^3}$
$\cdots$		$\cdots$

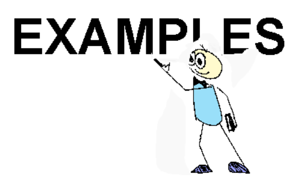
and the final answer can be written as,

$$\begin{aligned}\int x^2 e^{-2x}\,dx &= -\frac{x^2e^{-2x}}{2} - \frac{2xe^{-2x}}{2^2} - \frac{2e^{-2x}}{2^3},\\ &= -\frac{e^{-2x}}{2}\left\{x^2 + x + \frac{1}{2}\right\} + C.\end{aligned}$$

Example 306. Evaluate $\int x^2 e^{3x}\, dx$.

Solution We use the method in Table 7.3. The Table now takes the form,

x^2	$+$	e^{3x}
$2x$	$-$	$\frac{e^{3x}}{3}$
2	$+$	$\frac{e^{3x}}{3^2}$
0		$\frac{e^{3x}}{3^3}$.
$\cdots$		$\cdots$



and the final answer can be written as,

$$\int x^2 e^{3x}\, dx = \frac{e^{3x}}{3}\left\{x^2 - \frac{2x}{3} + \frac{2}{9}\right\} + C.$$

Example 307. Evaluate $\int_0^1 x^3\ 2^{-2x}\, dx$.

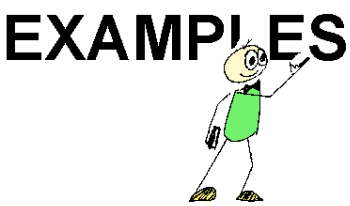
Solution This one looks mysterious but we can bring it down to a more recognizable form. For example, let's recall (from Chapter 4) that

$$\begin{aligned} 2^{-2x} &= e^{\ln 2^{-2x}}, \\ &= e^{(-2\ln 2)x}, \\ &= e^{-(ln 4)x}, \\ &= e^{ax}, \end{aligned}$$

if we set $a = -\ln 4$. So, we can set up the Table

x^3	$+$	e^{ax}
$3x^2$	$-$	$\frac{e^{ax}}{a}$
$6x$	$+$	$\frac{e^{ax}}{a^2}$
6	$-$	$\frac{e^{ax}}{a^3}$
0		$\frac{e^{ax}}{a^4}$
$\cdots$		$\cdots$

from which the answer can be read off as (remember that $a = -\ln 4$, and so $e^a = \frac{1}{4}$),



$$
\begin{aligned}
\int_0^1 x^3\ 2^{-2x}\ dx &= \int_0^1 x^3\ e^{ax}\ dx, \\
&= \left\{ \frac{x^3\ e^{ax}}{a} - \frac{3x^2\ e^{ax}}{a^2} + \frac{6x\ e^{ax}}{a^3} - \frac{6e^{ax}}{a^4} \right\}\Bigg|_0^1, \\
&= \left\{ \frac{e^a}{a} - \frac{3e^a}{a^2} + \frac{6e^a}{a^3} - \frac{6e^a}{a^4} \right\} - \left\{ -\frac{6}{a^4} \right\}, \\
&= \frac{1}{4a} - \frac{3}{4a^2} + \frac{3}{2a^3} - \frac{3}{2a^4} + \frac{6}{a^4}, \quad (\text{as } e^a = \frac{1}{4}), \\
&= \frac{a^3 - 3a^2 + 6a + 18}{4a^4}, \\
&\approx 0.08479
\end{aligned}
$$

since $a = -\ln 4 \approx -1.3863$.

As one can see, this method is very efficient for evaluating general integrals of the form

$$\boxed{\int (\textit{polynomial in } x)\ (\textit{exponential in } x)\ dx.}$$

Basically, we set up a table with the *polynomial function on the left* and then find all its derivatives until we reach the "zero" function and this is where we STOP. See the previous examples.

7.3.3 The Product of a Polynomial and a Logarithm

The case where the integrand is a product of a poynomial and a logarithmic function can be handled by means of a *simple substitution* which effectively replaces the integrand involving a logarithmic term by an integrand with an exponential term so that we can use the table method of Section 7.3.2. Let's look at a few examples.

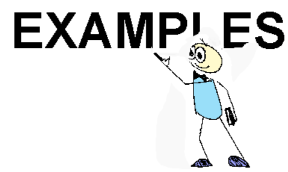
Example 308. Evaluate $\displaystyle\int \ln x\ dx.$

Solution In this example, the presence of the "logarithmic term" complicates the situation and it turns out to be *easier to integrate the polynomial and differentiate the logarithm* than the other way around. Note that "1" is a polynomial (of degree 0).

The Table method as used in Example 300 gives us easily

$\ln x$	$+$	1
$\frac{1}{x}$	$-$	x
STOP		*STOP*

and from this we can write down the answer as,



$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \left(\frac{1}{x}\right) dx, \\ &= x \ln x - x + C.\end{aligned}$$

Example 309. Evaluate $\int x^2 \ln x \, dx$.

Solution Let's write this one out using both the Table Method and the *normal way*: Note that an application of the Table method to the first two rows of Table 7.3 just gives the ordinary Integration by Parts Formula.

In other words, using the method described in Example 300 we find the table

$\ln x$	$+$	x^2
$\frac{1}{x}$	$-$	$\frac{x^3}{3}$
STOP		*STOP*

From this we see that

$$\begin{aligned}\int x^2 \ln x \, dx &= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \frac{1}{x} \, dx, \\ &= \frac{x^3 \ln x}{3} - \frac{1}{3} \int x^2 \, dx, \\ &= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C.\end{aligned}$$

Had we done this using the *normal way*, we would set $u = \ln x$, $dv = x^2 \, dx$ which means that $du = \frac{1}{x} \, dx$, and $v = \frac{x^3}{3}$. Thus,

$$\begin{aligned}
\int x^2 \ln x \, dx &= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3}\,\frac{1}{x}\, dx,\\
&= \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2\, dx,\\
&= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C.
\end{aligned}$$

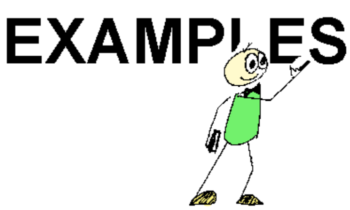
OK, so the moral is: " If you see a logarithm by itself (not a power of such) then try putting it on the left of the Table ...", so that you can differentiate it. The situation where the logarithm has a power attached to it is more delicate. Sometimes, there are examples where this idea of differentiating the logarithmic term just doesn't work easily, but our Table idea *does* work!

Example 310. Evaluate $\int x^4 (\ln x)^3 \, dx$.

Solution This is a problem involving the power of a logarithm. Any "normal method" will be lengthy. The basic idea here is to **transform out** the logarithmic term by a **substitution which converts the integrand to a product of a polynomial and an exponential** (so we can use Section 7.3.2). This is best accomplished by the *inverse function* of the logarithm (the exponential function). We let

$$x = e^t, \quad \ln x = t, \quad dx = e^t \, dt,$$

which will convert the given integral to one of the types that we have seen ... that is,



$$\begin{aligned}
\int x^4 (\ln x)^3 \, dx &= \int e^{4t} t^3 (e^t \, dt),\\
&= \int t^3 e^{5t}\, dt.
\end{aligned}$$

Setting up the table we find,

t^3	$+$	e^{5t}
$3t^2$	$-$	$\frac{e^{5t}}{5}$
$6t$	$+$	$\frac{e^{5t}}{5^2}$
6	$-$	$\frac{e^{5t}}{5^3}$
0		$\frac{e^{5t}}{5^4}$

which gives the answer in terms of the "t" variable, that is,

$$\begin{aligned}
\int x^4(\ln x)^3\,dx &= \int t^3 e^{5t}\,dt, \\
&= \frac{t^3e^{5t}}{5} - \frac{3t^2e^{5t}}{5^2} + \frac{6te^{5t}}{5^3} - \frac{6e^{5t}}{5^4} + C, \\
&= \frac{e^{5t}}{5}\left\{t^3 - \frac{3t^2}{5} + \frac{6t}{5^2} - \frac{6}{5^3}\right\} + C, \\
&= \frac{x^5}{5}\left\{(\ln x)^3 - \frac{3(\ln x)^2}{5} + \frac{6(\ln x)}{5^2} - \frac{6}{5^3}\right\} + C,
\end{aligned}$$

after our final *back substitution.*

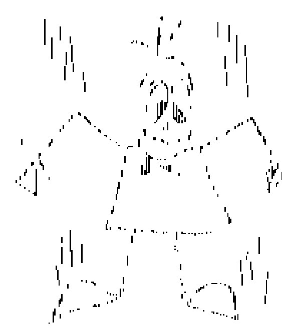
7.3.4 The Product of an Exponential and a Sine or Cosine

The next technique involves the product of an exponential with a sine or cosine function. These integrals are common in the scientific literature so we'll present another method based on Table 7.3 for adapting to this situation.

Example 311. Evaluate $I = \displaystyle\int e^{2x}\sin 3x\;dx$.

Solution We set up a table based on our usual Table 7.3.

$\boxed{e^{2x}}$	$+$	$\boxed{\sin 3x}$
$2e^{2x}$	$-$	$-\dfrac{\cos 3x}{3}$
$4\boxed{e^{2x}}$	$\leftarrow +$	$-\dfrac{\boxed{\sin 3x}}{3^2}$
$\ldots$		$\ldots$



In this case we note that the **the functions appearing in the third row are the same as the first** (aside from the coefficients and their signs). So, we **STOP at the third row**, (this is the rule), and realize that according to Table 7.3, equation (7.42),

$$I = \frac{-e^{2x}\cos 3x}{3} + \frac{2e^{2x}\sin 3x}{3^2} - \frac{4}{9}I$$

or, solving for I, we find

$$\frac{13}{9}I = \frac{e^{2x}}{3}\left(\frac{2\sin 3x}{3} - \cos 3x\right)$$

or

$$I = \frac{3e^{2x}}{13}\left(\frac{2\sin 3x}{3} - \cos 3x\right),$$

i.e., the most general antiderivative is given by

$$I = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C, \tag{7.43}$$

where C is a constant.

The MyCar Method

NOTE: OK, OK, but is there a way of getting from our Table right down to the answer without having to calculate the I symbol like we did? YES! Here's the **MyCar** ... method:

• Use the usual method of Table 7.3 and STOP at the third row. Now, refer to Figure 136 in the adjoining margin. Then write down the following expression

$$I = \square\left(\frac{-e^{2x}\cos 3x}{3} + \frac{2e^{2x}\sin 3x}{3^2}\right), \tag{7.44}$$

where **the "number" in the box**, $\square$, is given explicitly by:

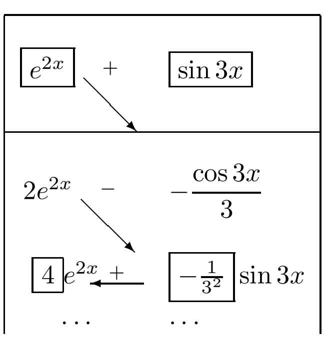
- **a) Multiplying**, "My" (for short), the coefficients in the third row (with their signs!)
  Here: $\mathbf{(4)(-\frac{1}{9}) = -\frac{4}{9}}$
- **b) Changing**, "C" (for short), the sign of the number in a)
  Here: $\mathbf{+\frac{4}{9}}$
- **c) Adding**, "A" for short, the number "**1**" to b)
  Here: $\mathbf{1 + \frac{4}{9} = \frac{13}{9}}$
- **d)** Finding the **Reciprocal**, "R" for short, of the number in c)
  Here: $\mathbf{\frac{9}{13}}$
- **e) Inserting the number in d) into the BOX in equation (7.44), above.** This is your answer! (aside from the usual constant of integration). Here:

$$\boldsymbol{I = \boxed{\frac{9}{13}}\left(\frac{-e^{2x}\cos 3x}{3} + \frac{2e^{2x}\sin 3x}{3^2}\right),}$$

which gives the SAME answer as before, see equation (7.43).

This process of finding the number in the box, $\square$, can be remembered through the steps, see Figure 136,

<u>M</u>ULTIPL<u>Y</u>, <u>C</u>HANGE SIGN, <u>A</u>DD 1, find the <u>R</u>ECIPROCAL



Briefly said, you

- MULTIPLY,
- CHANGE SIGN,
- ADD 1 and find the,
- RECIPROCAL

Figure 136.

Example 312. Evaluate $I = \int e^{3x}\cos 4x\; dx$.

Solution We proceed as per Figure 136. We set up our table as usual,

e^{3x}	$+$	$\cos 4x$
$3e^{3x}$	$-$	$\dfrac{\sin 4x}{4}$
$9e^{3x}$	$+$	$-\dfrac{\cos 4x}{4^2}$
$\ldots$		$\ldots$

The asnwer is of the form:

$$I = \square \left\{ \frac{e^{3x} \sin 4x}{4} + \frac{3e^{3x} \cos 4x}{4^2} \right\} + C,$$

where the number in the box is found easily if you ...

- **MULTIPLY:** $-\frac{9}{16}$
- **CHANGE SIGN:** $\frac{9}{16}$
- **ADD** 1**:** $\frac{25}{16}$
- **find theRECIPROCAL:** $\frac{16}{25}$.

MULTIPLY, CHANGE SIGN, ADD 1, find the RECIPROCAL

The answer is

$$\begin{aligned} I &= \frac{16}{25} \left\{ \frac{e^{3x} \sin 4x}{4} + \frac{3e^{3x} \cos 4x}{4^2} \right\} + C, \\ &= \frac{1}{25} \left\{ 4e^{3x} \sin 4x + 3e^{3x} \cos 4x \right\} + C. \end{aligned}$$

The same method can be used for other such **three-row problems**. We give them this name because our table only requires *three* rows!

Example 313. Evaluate $I = \displaystyle\int \sin 3x \ \cos 4x \ dx$.

Solution **Normal method** This is *normally done* using a trigonometric identity, namely,

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2},$$

where we set $A = 3x$ and $B = 4x$. This gives,

$$\int \sin 3x \;\; \cos 4x \; dx = \int \frac{\sin 7x + \sin(-x)}{2} \; dx, \tag{7.45}$$

$$= \int \frac{\sin 7x}{2} \; dx - \int \frac{\sin x}{2} \; dx, \tag{7.46}$$

$$= -\frac{\cos 7x}{14} + \frac{\cos x}{2} + C, \tag{7.47}$$

where we used the basic fact that the sine function is an odd function (see Chapter 5), that is, $\sin(-x) = -\sin(x)$, an identity which is valid for any x.

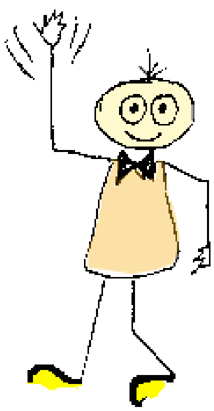


Table method Refer to the preceding examples. We set up the table,

$\sin 3x$	$+$	$\cos 4x$
$3\cos 3x$	$-$	$\frac{\sin 4x}{4}$
$-9\sin 3x$	$+$	$-\frac{\cos 4x}{4^2}$
$\cdots$		$\cdots$

to find that

$$I = \square \left\{ \frac{\sin 3x \sin 4x}{4} + \frac{3\cos 3x \cos 4x}{16} \right\} + C,$$

where the factor in the box, $\square$, is given by

- **MULTIPLY:** $\frac{9}{16}$
- **CHANGE SIGN:** $-\frac{9}{16}$
- **ADD 1:** $\frac{7}{16}$
- **find theRECIPROCAL:** $\frac{16}{7}$,

and so, we put $\frac{16}{7}$ in the Box, as the factor. The final answer is,

$$I = \frac{16}{7} \left\{ \frac{\sin 3x \sin 4x}{4} + \frac{3\cos 3x \cos 4x}{16} \right\} + C, \tag{7.48}$$

$$= \frac{1}{7} \left\{ 4\sin 3x \sin 4x + 3\cos 3x \cos 4x \right\} + C. \tag{7.49}$$

NOTE: Although this answer appears to be VERY different from the one given in (7.47) they must be the same *up to a constant*, right? This means that their difference must be a constant! In fact, repeated use of trigonometric identities show that **equations (7.47), (7.49) are equal**.

As a final example we emphasize that it is "generally" easier to convert an integral containing a natural logarithm to one involving an exponential for reasons that were described above.

Example 314. Evaluate $I = \int \sin(\ln x)\ dx$.

Solution Let $x = e^u$, $u = \ln x$ and $dx = e^u\ du$. This is the substitution which "takes out" the natural logarithm and replaces it by an exponential term. Then

$$\begin{aligned} I &= \int \sin(\ln x)\ dx, \\ &= \int e^u\ \sin u\ du. \end{aligned}$$

Now use the idea in Figure 136. We get the table,

e^u	$+$	$\sin u$
e^u	$-$	$-\cos u$
e^u	$+$	$-\sin u$
$\cdots$		$\cdots$

The answer now takes the form,

$$\begin{aligned} I &= \frac{1}{2}\{e^u \sin u - e^u \cos u\} + C, \\ &= \frac{1}{2}\{x \sin(\ln x) - x \cos(\ln x)\} + C, \end{aligned}$$

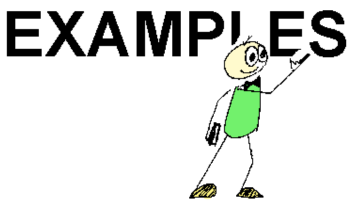
after the usual 'back-substitution'.

SNAPSHOTS

Example 315. Evaluate $\int_0^1 x^2 e^{-4x}\ dx$.

Solution The table looks like,

x^2	$+$	e^{-4x}
$2x$	$-$	$-\frac{e^{-4x}}{4}$
2	$+$	$\frac{e^{-4x}}{(-4)^2}$
0		$\frac{e^{-4x}}{(-4)^3}$.
$\cdots$		$\cdots$



and the indefinite integral can be written as

$$\int x^2 e^{-4x}\, dx = -\frac{x^2 e^{-4x}}{4} - \frac{x e^{-4x}}{8} - \frac{e^{-4x}}{32} + C,$$

from which we obtain the definite integral

$$\begin{aligned}\int_0^1 x^2 e^{-4x}\, dx &= -\frac{x^2 e^{-4x}}{4} - \frac{x e^{-4x}}{8} - \frac{e^{-4x}}{32}\Bigg|_0^1, \\ &= \left\{-\frac{e^{-4}}{4} - \frac{e^{-4}}{8} - \frac{e^{-4}}{32}\right\} - \left\{-\frac{1}{32}\right\}, \\ &= -\frac{13}{32}e^{-4} + \frac{1}{32}, \\ &\approx 0.02381.\end{aligned}$$

Example 316. Evaluate $I = \int_0^{\frac{\pi}{2}} t^2 \cos(2t)\, dt$

Solution We set up our table,

t^2	$+$	$\cos 2t$
$2t$	$-$	$\frac{\sin 2t}{2}$
2	$+$	$-\frac{\cos 2t}{4}$
0		$-\frac{\sin 2t}{8}$

From this we find,

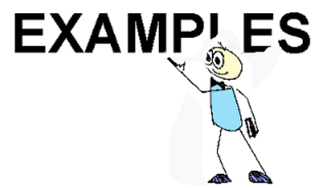
$$I = \frac{t^2 \sin 2t}{2} + \frac{2t \cos 2t}{4} - \frac{2 \sin 2t}{8} + C$$

$$= \frac{t^2 \sin 2t}{2} + \frac{t \cos 2t}{2} - \frac{\sin 2t}{4} + C$$

and so the definite integral is given by,

$$\begin{aligned} I &= \left\{ \frac{t^2 \sin 2t}{2} + \frac{t \cos 2t}{2} - \frac{\sin 2t}{4} \right\}\Bigg|_0^{\frac{\pi}{2}}, \\ &= -\frac{\pi}{4}. \end{aligned}$$

Example 317. Evaluate $I = \int x^3 (\ln x)^2 \, dx$.



Solution Let $x = e^t,\ \ln x = t,\ dx = e^t \, dt$. Then

$$I = \int e^{3t} t^2 e^t \, dt = \int t^2 e^{4t} \, dt,$$

and we obtain the table,

t^2	$+$	e^{4t}
$2t$	$-$	$\frac{e^{4t}}{4}$
2	$+$	$\frac{e^{4t}}{4^2}$
0		$\frac{e^{4t}}{4^3}$

from which we find,

$$\begin{aligned} I &= \frac{t^2 e^{4t}}{4} - \frac{2te^{4t}}{4^2} + \frac{2e^{4t}}{4^3} + C, \\ &= \frac{x^4 (\ln x)^2}{4} - \frac{2x^4 \ln x}{4^2} + \frac{2x^4}{4^3} + C, \\ &= \frac{x^4 (\ln x)^2}{4} - \frac{x^4 \ln x}{8} + \frac{x^4}{32} + C. \end{aligned}$$

Example 318. Evaluate $I = \int \cos(3x) \cos(5x) \, dx$, *a "three-row problem".*

Solution We set it up according to Figure 136. In this example, it doesn't matter which term goes on the right or left, as both can be easily differentiated or integrated. The table is,

$\cos 3x$	$+$	$\cos 5x$
$-3\sin 3x$	$-$	$\frac{\sin 5x}{5}$
$-9\cos 3x$		$-\frac{\cos 5x}{25}$

So,

$$I = \square\left(\frac{\cos 3x \sin 5x}{5} - \frac{3\sin 3x \cos 5x}{25}\right),$$

where the factor in $\square$ is obtained by the **MyCar** method:

a) $\frac{9}{25}$

b) $-\frac{9}{25}$

c) $\frac{16}{25}$

d) $\frac{25}{16}$, in the Box, above.

Finally,

$$\begin{aligned} I &= \frac{25}{16}\left(\frac{\cos 3x \sin 5x}{5} - \frac{3\sin 3x \cos 5x}{25}\right), \\ &= \frac{1}{16}\left(5\cos 3x \sin 5x - 3\sin 3x \cos 5x\right). \end{aligned}$$

NOTES:

Exercise Set 32.

Evaluate the following integrals.

1. $\int x \cos x \; dx$
2. $\int x \sin x \; dx$
3. $\int_0^{\frac{\pi}{2}} x \cos 2x \; dx$
4. $\int x^2 \sin x \; dx$
5. $\int x \sec^2 x \; dx$
6. $\int x \sec x \tan x \; dx$
7. $\int x^2 e^x \; dx$
8. $\displaystyle\lim_{T\to\infty} \int_0^T x^2 e^{-3x} \; dx$
9. $\int x^4 \ln x \; dx$
10. $\int x^3 e^{-3x} \; dx$
11. $\int \sin^{-1} x \; dx$
12. $\int \tan^{-1} x \; dx$
13. $\int x^2 (\ln x)^5 \; dx$
14. $\int x \sec^{-1} x \; dx$, for $x > 1$.
15. $\int (x-1)^2 \sin x \; dx$
16. $\int e^{-2x} \sin 3x \; dx$
17. $\int e^x \cos 4x \; dx$
18. $\int \sin 3x \cos 2x \; dx$
19. $\int \sin 2x \cos 4x \; dx$
20. $\int \cos 3x \cos 4x \; dx$
21. $\int x^5 e^{2x} \; dx$
22. $\int \cos \ln x \; dx$

Suggested Homework Set 25. *Problems 4, 8, 9, 13, 16, 19, 21, 22*

NOTES:

7.4 Partial Fractions

In this section we study a method for integrating rational functions, that is, functions which are the quotient of two polynomials, (see Chapter 5.2). For example,

$$3, \quad x-1, \quad \frac{x^2-1}{x+2}, \quad \frac{x^3+2x-1}{x^4+2x+6}$$

are all rational functions. Of course, polynomials are rational functions as we can always take the denominator to be equal to the polynomial, 1. On the other hand, polynomials are easy to integrate so we'll look at those rational functions whose denominator is not a constant function.

Review

Study the procedure of long division (next section) very carefully if you've not seen it before. Look over Chapter 5.1-2, in particular, the definition of quadratic irreducible polynomials (Type II factors).

The study of polynomials is an important area of Calculus due to their widespread applications to mathematics and the pure and applied sciences and engineering. In this section we're going to review the method for dividing one polynomial by another of the same or lower degree (and you'll get a *rational function*!). The method is called **long division**. This technique of dividing polynomials is especially useful in dealing with the **method of partial fractions** which occurs when we consider the problem of integrating a given rational function.

Specifically, let $p(x), q(x)$ be two polynomials in x with real coefficients and assume that the degree of $p(x)$, denoted by *deg* $p(x)$, satisfies

$$deg\ p(x) \geq deg\ q(x).$$

For example, $p(x) = x^2 - 1$ and $q(x) = -2x^2 + 2x - 1$ have the same degree, namely, 2. On the other hand, $p(x) = x^3$ and $q(x) = x^2 + 1$ have different degrees and $3 = deg\ p(x) > deg\ q(x) = 2$. If $deg\ p(x) < deg\ q(x)$ we can't apply long division, as such, but we can do our best at factoring both and this situation leads naturally to the subject of **partial fractions**. A partial fraction consists of a special representation of a rational function. We just rewrite the function in another way. For example, the right-side of

$$\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1},$$

is the partial fraction decomposition of the function on the left. The function on the right is the *same* function, it just looks different. This difference, however, will make it very convenient when it comes to integrating the function on the left because the antiderivative of anyone of the two functions on the right gives a natural logarithm. In other words,

$$\begin{aligned}\int \frac{5x-7}{x^2-3x+2}\,dx &= \int \frac{3}{x-2}\,dx + \int \frac{2}{x-1}\,dx, \\ &= 3\ln|x-2| + 2\ln|x-1| + C, \\ &= \ln\left|(x-2)^3(x-1)^2\right| + C.\end{aligned}$$

We start this section by studying (or reviewing) the method of long division.

7.4.1 Review of Long Division of Polynomials

The study of polynomials is an important area of Calculus due to their widespread applications to mathematics and the pure and applied sciences and engineering. In this section we're going to review the method for dividing one polynomial by another of the same or lower degree (and you'll get a *rational function*!). This technique is especially useful in dealing with the **method of partial fractions** which occurs when we consider **methods of integration**. . . which we'll see later.

Specifically, let $p(x), q(x)$ be two polynomials in x with real coefficients and assume that the degree of $p(x)$, denoted by *deg* $p(x)$, satisfies

$$deg\ p(x) \geq deg\ q(x).$$

For example, $p(x) = x^2 - 1$ and $q(x) = -2x^2 + 2x - 1$ have the same degree, namely, 2. On the other hand, $p(x) = x^3$ and $q(x) = x^2 + 1$ have different degrees and $3 = deg\ p(x) > deg\ q(x) = 2$. If $deg\ p(x) < deg\ q(x)$ there is no need to apply long division, as such, but we do our best at factoring the denominator.

The method for dividing polynomials of equal or different degrees by one another is similar to the procedure of dividing integers by one another, but the *long way*, that is, without a calculator. Let's start off with an example to set the ideas straight.

Example 319. Simplify the following rational function using long division:

$$\frac{x^3 + x - 1}{x^3 - 2}$$

.

Solution In this example, $deg\ p(x) = 3 = deg\ q(x)$, where $p(x)$ is the numerator, and $q(x)$ is the denominator. The result of long division is given in Figure 137 and is written as

$$\frac{x^3 + x - 1}{x^3 - 2} = 1 + \frac{x + 1}{x^3 - 2}.$$

The polynomial quantity "$x + 1$" left over at the bottom, whose degree is **smaller** than the degree of the denominator, is called the **remainder** of the long division procedure, and, after all is said and done, we can write the answer in the form

$$\frac{p(x)}{q(x)} = (new\ polynomial) + \frac{(remainder)}{(denominator)}.$$

We can summarize the procedure here:

$$\begin{array}{r|l} & 1 \\ \hline x^3 - 2 & x^3 \quad + x \quad - 1 \\ & \underline{x^3 \qquad\qquad - 2} \\ & \qquad\quad x \quad + 1 \end{array}$$

Figure 137.

Example 320. Use long division to simplify the rational function

$$\frac{x^4 + 3x^2 - 2x + 1}{x + 1}.$$

Solution Okay, we know from Table 7.4 that the answer will have a remainder whose degree is less than 1, and so it must be 0 (a constant polynomial), and the "new polynomial" part will have degree equal to $deg\ p(x) - deg\ q(x) = 4 - 1 = 3$. The procedure is shown in Figure 138 in the margin.

$$\begin{array}{r|l} & x^3 - x^2 + 4x - 6 \\ \hline x + 1 & x^4 \qquad + 3x^2 - 2x + 1 \\ & \underline{x^4 + x^3} \\ & \quad -x^3 + 3x^2 - 2x + 1 \\ & \quad \underline{-x^3 - x^2} \\ & \qquad\quad +4x^2 - 2x + 1 \\ & \qquad\quad \underline{4x^2 + 4x} \\ & \qquad\qquad\quad -6x + 1 \\ & \qquad\qquad\quad \underline{-6x - 6} \\ & \qquad\qquad\qquad\quad + 7 \end{array}$$

Figure 138.

So we see from this that the remainder is just the number (constant polynomial), 7, and the answer is

$$\frac{x^4 + 3x^2 - 2x + 1}{x + 1} = x^3 - x^2 + 4x - 6 + \frac{7}{x + 1}$$

Let $deg\ p(x) \geq deg\ q(x)$. The result of a long division of $p(x)$ by $q(x)$ looks like

$$\frac{p(x)}{q(x)} = (new\ polynomial)\ +\ \frac{(remainder)}{q(x)}$$

where the degree of the remainder is smaller than the degree of the denominator, $q(x)$, and where the degree of the new polynomial is equal to the degree of the numerator *minus* the degree of the denominator. Mathematically, this can be summarized by saying that,

$$deg\ (remainder) < deg\ (denominator)$$

$$deg\ (new\ polynomial) = deg\ (numerator) - deg\ (denominator)$$

Table 7.4: The Result of a Long Division

a result which can be easily verified by finding a common denominator for the expression on the right and expanding the result.

$$\begin{array}{r|l} & x^2 + x + \frac{1}{2} \\ \hline 2x^2+1 & 2x^4 + 2x^3 + 2x^2 + 2x + 2 \\ & \underline{2x^4 \qquad\quad + x^2} \\ & \qquad 2x^3 + x^2 + 2x + 2 \\ & \qquad \underline{2x^3 + \qquad +x} \\ & \qquad\qquad x^2 + x + 2 \\ & \qquad\qquad \underline{x^2 \qquad + 1/2} \\ & \qquad\qquad\qquad x + 3/2 \end{array}$$

Figure 139.

Example 321. Use long division and simplify

$$\frac{p(x)}{q(x)} = \frac{x^4 + x^3 + x^2 + x + 1}{2x^2 + 1}.$$

Solution As before, Table 7.4 tells us that the answer will have a remainder whose degree is less than 2, and so it must be 1 *or* 0 (either a linear or a constant polynomial), and the "new polynomial" will have degree equal to $deg\ p(x) - deg\ q(x) = 4 - 2 = 2$. Now, **in order to try and avoid fractions in our calculations as much as possible** we'll multiply both $p(x), q(x)$ by "2", the leading coefficient of the polynomial, $q(x)$. You can always do this and you'll still be OK in your answer. Then, **we'll forget about the "extra 1/2" until the end** because ...

$$\frac{x^4 + x^3 + x^2 + x + 1}{2x^2 + 1} = \boxed{\frac{1}{2}} \left(\frac{2x^4 + 2x^3 + 2x^2 + 2x + 2}{2x^2 + 1} \right).$$

This procedure is shown in Figure 139 in the margin. So, we see from this that the remainder is the linear polynomial $x + 3/2$, and

$$\frac{x^4 + x^3 + x^2 + x + 1}{2x^2 + 1} = \boxed{\frac{1}{2}} \left(x^2 + x + \frac{1}{2} + \frac{x + 3/2}{2x^2 + 1} \right).$$

or

$$\frac{p(x)}{q(x)} = \frac{x^2}{2} + \frac{x}{2} + \frac{1}{4} + \left(\frac{2x + 3}{8x^2 + 4} \right).$$

$$\begin{array}{r|l} & 3 \\ \hline x^2(x+1) & 3x^3 + 3x^2 + 3x + 2 \\ & \underline{3x^3 + 3x^2} \\ & \qquad\qquad 3x + 2 \end{array}$$

Figure 140.

Example 322. Simplify the rational function

$$\frac{p(x)}{q(x)} = \frac{3x^3 + 3x^2 + 3x + 2}{x^2(x + 1)}.$$

Solution In this case, $deg\ q(x) = 3$, $deg\ p(x) = 3$ and $deg\ (remainder) = 1 < deg\ q(x)$. It follows from Figure 140 that

$$\frac{3x^3 + 3x^2 + 3x + 2}{x^2(x + 1)} = 3\ +\ \frac{3x + 2}{x^3 + x^2},$$

and we can't do any better than this. Sometimes you can look for patterns ... For example, in this case we could notice that

$$\begin{aligned}\frac{3x^3+3x^2+3x+2}{x^2(x+1)} &= \frac{3x^3+3x^2}{x^3+x^2}+\frac{3x+2}{x^3+x^2}\\ &= 3 + \frac{3x+2}{x^3+x^2},\end{aligned}$$

in which case you wouldn't have to use long division!

Example 323. Simplify

$$\frac{p(x)}{q(x)} = \frac{3x^4-8x^3+20x^2-11x+8}{x^2-2x+5}.$$

Solution Here, $deg\ q(x) = 2$, $deg\ p(x) = 4$ and $deg\ (remainder) = 1 < deg\ q(x)$. Figure 141 shows that

$$\frac{3x^4-8x^3+20x^2-11x+8}{x^2-2x+5} = 3x^2-2x+1 + \frac{x+3}{x^2-2x+5}.$$

$$\begin{array}{r|l} & \;3x^2-2x+1 \\ \hline x^2-2x+5 & \;3x^4-8x^3+20x^2-11x+8 \\ & \;\underline{3x^4-6x^3+15x^2} \\ & \qquad -2x^3+5x^2-11x+8 \\ & \qquad \underline{-2x^3+4x^2-10x} \\ & \qquad\qquad x^2 \;\; -x \;\; +8 \\ & \qquad\qquad \underline{x^2 \;\; -2x \;\; +5} \\ & \qquad\qquad\qquad x \;\; +3 \end{array}$$

Figure 141.

Exercise Set 33.

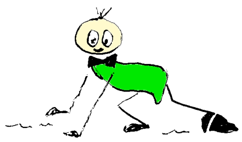
Use long division to simplify the following rational functions.

1. $\dfrac{x^2-2x+1}{x+1}$
2. $\dfrac{2x^3-3x^2+3x-1}{x^3+2x+1}$
3. $\dfrac{x^4-x^2+1}{3x^2-1}$
4. $\dfrac{x^4+1}{x^2+1}$
5. $\dfrac{x^5+x^3+1}{x-1}$
6. $\dfrac{3x^3+5x+6}{2x^2+2x+1}$

NOTES:

7.4.2 The Integration of Partial Fractions

The method of *partial fractions* applies to the case where the integrand is a *rational function* with real coefficients. It is known from Algebra that every polynomial with real coefficients can be factored into a product of linear factors (e.g., products of factors of the form $(x-r)^p$, or Type I factors), and a product of quadratic factors called **quadratic irreducibles** or Type II factors, (e.g., ax^2+bx+c where $b^2-4ac<0$, i.e., a quadratic with no real roots). For example, $x^4-1=(x^2+1)(x-1)(x+1)$, is the product of two Type I factors $((x-1),(x+1))$ and one Type II factor (x^2+1). Since the numerator and denominator of every rational function is a polynomial, it follows that the numerator and denominator of every rational function can also be factored in this way. In order to factor a given polynomial in this way one can use Newton's method (Chapter 3.11) in order to find all its real roots successively.



Now, in order to evaluate an expression of the form

$$\int \frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0}\,dx,$$

where m,n are integers and the coefficients are assumed real, there are two basic cases:

- $\boldsymbol{n \geq m}$.

  In this case we apply the *method of long division*, see Section 7.4. So, we divide the numerator by the denominator and this results in a **polynomial plus a remainder term**. This remainder term is a rational function whose numerator has degree less than the degree of the denominator.

  For example, long division gives us that

  $$\frac{x^4}{x^2-1}=x^2\ +\ 1\ +\frac{1}{x^2-1}.$$

  Here, the remainder is the rational function on the right of the last display (whose numerator, the function 1, has degree 0 and whose denominator, the function x^2-1, has degree 2).

  Since the leading term after long division is a polynomial it is easily integrated. OK, but how do we integrate the remainder term? The remainder term may be integrated either by *completing the square* (as in the preceding section), or by using the idea in the next item.

- $\boldsymbol{n < m}$.

  We **factor the denominator completely into a product of linear and irreducible factors and their powers**. Next, we **decompose this quotient into partial fractions** in the following sense:

  If the denominator has a linear factor of the form $(x-r)^p$ where p is the highest such power, there corresponds a sum of terms of the form

  $$\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\frac{A_3}{(x-r)^3}+\cdots+\frac{A_p}{(x-r)^p}$$

  where the constants, A's, are to be found. If the denominator has a quadratic irreducible factor (*i.e.,* $b^2-4ac<0$) of the form $(ax^2+bx+c)^q$, where q is the highest such power, there corresponds in its *partial fraction decomposition*, a sum of terms of the form

  $$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_qx+C_q}{(ax^2+bx+c)^q}$$

Idea Behind Integrating a Rational Function

1. Break up the denominator into its factors (so find all its roots or zeros).

e.g.: $x^2 - 2x + 1 = (x-1)^2 \leftarrow$ linear (or Type I) factors.
$x^2 + 1$ cannot be factored into "linear" factors (it is a Type II factor).

2. Use these factors to decompose the rational function into a **sum of reciprocals of these factors**.

3. Integrate each term in item 2 separately using the methods of this Chapter.

Table 7.5: Idea Behind Integrating a Rational Function

where the constants, B's and C's, are to be found, as well. The method for finding the A's, B's, and C's is best described using various examples, see Table 7.5 for a quick summary of the procedure.

In the following examples we assume that the degree, n, of the numerator satisfies $n < m$, where m is the degree of the denominator. Otherwise, we have to use *long division* BEFORE we proceed. These problems involving partial fractions can be quite long because of this additional long division which must be performed in some cases.

Example 324. Find the form of the partial fraction decomposition of the following functions:

1. $f(x) = \dfrac{2x}{(x-1)(x+3)}$
2. $f(x) = \dfrac{x-4}{x(x-1)^2}$
3. $f(x) = \dfrac{3x+2}{x^4-1}$
4. $f(x) = \dfrac{x^3+2x^2-1}{(x-1)^3(x^3-1)}$
5. $f(x) = \dfrac{x^5+1}{(x+1)(x-2)^2(x^4+1)^2}$

Solution **1)**. The factors of the denominator are $(x-1)$ and $(x+3)$, and each one is linear and simple (the highest power for each one is 1). So, the partial fraction decomposition has one term corresponding to $(x-1)$ and one term corresponding to $(x+3)$. Thus,

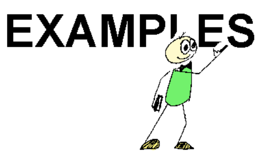
$$\frac{2x}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3},$$

where A, B are constants to be determined using the methods of this section.

2). In this case, the factors of the denominator are $x, (x-1)$ where the factor x is simple and linear. The factor $(x-1)$ occurs with a (highest) power of 2, so it is NOT simple. In this case, there are two terms in the partial fraction decomposition of f which correspond to this factor. That is,

$$\frac{x-4}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2},$$

where the constants A, B, C, are to be determined.



3). The factors of the denominator are given by $x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$ where the last factor, namely, x^2+1 is irreducible (or Type II). Each factor appears with a highest power of 1 so,

$$\frac{3x+2}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1},$$

since, according to the theory of partial fractions, to every Type II factor there corresponds a term of the form $mx+b$ in its partial fraction decomposition.

4). In this example, the denominator needs to be factored completely BEFORE we apply the method of partial fractions (because of the cubic term, x^3-1). The factors of x^3-1 are given by $x^3-1 = (x-1)(x^2+x+1)$, where $x-1$ is linear and x^2+x+1 is irreducible (Type II). So the denominator's factors are given by $(x-1)^3(x^3-1) = (x-1)^4(x^2+x+1)$. This list of factors contains the terms $x-1$, with a highest power of 4 (not 3, as it seems), and x^2+x+1 (with a highest power of 1). So, the decomposition looks like,

$$\frac{x^3+2x^2-1}{(x-1)^3(x^3-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4} + \frac{Ex+F}{x^2+x+1},$$

where $A, B, C, ..., F$ are to be determined, (of course, it might take quite a few coffees to actually find them!).

5). In this final example we see the need to factor the polynomial x^4+1 into Type I and Type II factors. This isn't easy, but it can be done. The idea is to write its factors as $x^4+1 = (x^2+ax+b)(x^2-ax+b)$ where the right-side is expanded and coefficients are compared to those on the left to give the values $a = \sqrt{2}$ and $b = 1$. So, the factors of the denominator are given by

$$(x+1)(x-2)^2(x^4+1)^2 = (x+1)(x-2)^2(x^2-\sqrt{2}x+1)^2(x^2+\sqrt{2}x+1)^2.$$

The factor $x+1$ appears with a highest power of 1, the factor $x-2$ appears with a highest power of 2 and each Type II factor $x^2+\sqrt{2}x+1$, $x^2-\sqrt{2}x+1$ appears with a highest power of 2. So, the partial fraction decomposition looks like,

$$\begin{aligned}\frac{x^5+1}{(x+1)(x-2)^2(x^4+1)^2} \quad &= \quad \frac{A_1}{x+1} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2} + \\ &\quad + \frac{B_1x+C_1}{x^2-\sqrt{2}x+1} + \frac{B_2x+C_2}{(x^2-\sqrt{2}x+1)^2} + \\ &\quad + \frac{B_3x+C_3}{x^2+\sqrt{2}x+1} + \frac{B_4x+C_4}{(x^2+\sqrt{2}x+1)^2},\end{aligned}$$

where the 11 constants $A_i, i = 1, 2, 3$, $B_i, i = 1, 2, 3, 4$ and $C_i, i = 1, 2, 3, 4$ are all to be found !! Don't worry, this doesn't happen much in practice. This example was included to reinforce the actual finding of the FORM of the decomposition, not the actual constants.

Okay, now that we know how to find the FORM of a partial fraction decomposition, we can proceed further to find the constants that appear in it, and finally integrate some rational functions. The next examples show how this is done.

Example 325. Evaluate $\displaystyle\int \frac{5x-7}{x^2-3x+2}\,dx$

Solution **Step 1. Find the form of the partial fraction decomposition**. We are in the case where $n = 1$ and $m = 2$, so $n < m$. In this case, we factor the denominator completely and find $x^2 - 3x + 2 = (x-1)(x-2)$. Since each one of its factors $(x-1)$, $(x-2)$ is a linear factor, the partial fraction decomposition of this function looks like,

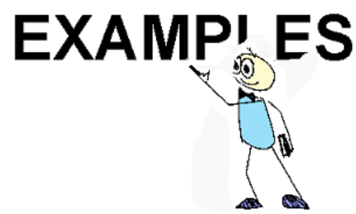
$$\begin{aligned}\frac{5x-7}{x^2-3x+2} &= \frac{5x-7}{(x-2)(x-1)}\\ &= \frac{A}{x-2} + \frac{B}{x-1},\end{aligned}$$

where A, B are to be found!

Step 2. Find the constants $A, B, \ldots$. Multiplying both sides by the denominator, $(x-1)(x-2)$, we find that

$$5x - 7 = A(x-1) + B(x-2),$$

must hold for all x. At this point, one may proceed in many different ways. All we need to do is to find the value of A, B, right? The basic idea is to plug in certain values of x and then obtain a system of two equations in the two unknowns A, B, which we can solve. For example,



Method 1: The "Plug-in" Method. If we set $x = 1$, then $5x - 7 = A(x-1) + B(x-2)$ means that $5 \cdot 1 - 7 = -2 = 0 + B(-1)$, and so $B = 2$.

On the other hand, if we set $x = 2$, then $5x - 7 = A(x-1) + B(x-2)$ means that $5 \cdot 2 - 7 = 3 = A(1) + 0 = A$, and so $A = 3$. So, $A = 3, B = 2$.

Method 2: Comparing Coefficients. Since $5x - 7 = A(x-1) + B(x-2)$ must hold for every value of x the two polynomials (represented by the left-side and by the right-side) are equal and so their coefficients must be equal, as well. So,

$$\begin{aligned}5x-7 &= A(x-1) + B(x-2),\\ &= (A+B)x - A - 2B,\\ &= (A+B)x - (A+2B).\end{aligned}$$

Comparing the coefficients we find

$$A + B = 5 \quad \textit{and} \quad A + 2B = 7,$$

which represents a simple system of two equations in the two unknowns A, B whose solution is given by $A = 3, B = 2$.

Method 3: The Derivative Method. Since $5x - 7 = A(x-1) + B(x-2)$ must hold for every value of x the same must be true of its derivative(s) and so,

$$\begin{aligned}5x-7 &= A(x-1) + B(x-2),\\ \frac{d}{dx}(5x-7) &= \frac{d}{dx}(A(x-1) + B(x-2)),\\ 5 &= A + B,\end{aligned}$$

and so we can write A, say, in terms of B. In this case, we get $A = 5 - B$. We substitute this back into the relation $5x - 7 = A(x-1) + B(x-2)$ to find

$5x-7=(5-B)(x-1)+B(x-2)=5x-B-5$, and so $B=2$. Since $A+B=5$ it follows that $A=3$.

Regardless of which method we use to find A, B (each one has its advantages and disadvantages), we will obtain the partial fraction decomposition

$$\frac{5x-7}{x^2-3x+2}=\frac{3}{x-2}+\frac{2}{x-1}.$$

The right-hand side is easily integrated in terms of natural logarithms and so

$$\begin{aligned}\int \frac{5x-7}{x^2-3x+2}\,dx &= \int \frac{3}{x-2}\,dx+\int \frac{2}{x-1}\,dx,\\ &= 3\ln|x-2|+2\ln|x-1|+C,\\ &= \ln\left|(x-2)^3(x-1)^2\right|+C,\end{aligned}$$

after using the basic properties of the logarithm and the absolute value, namely, $\triangle \ln \square = \ln\left(\square^{\triangle}\right)$ and $|\square\triangle| = |\square||\triangle|$.

NOTE: Normally, Method 1 outlined in Example 325 is the most efficient method in finding the values of $A, B, \ldots$.

Example 326. Evaluate $\displaystyle\int \frac{x}{(x-1)(x-2)(x+3)}\,dx$

Solution We can use Table 7.6 since the factors of the denominator are all linear and simple (there are no powers). We write

$$\frac{x}{(x-1)(x-2)(x+3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x+3}.$$

We can also get the value of B using the same method (even though it is not a simple factor!), but we can't get the value of A. The point is that the method of Table 7.6 even works for *powers* of linear factors **but only for the highest such power!**. For example, we can get B in Example 327 (because it corresponds to the highest power of the linear factor $(x-1)$, which is 2), but we can't get the value of A using this approach. So, if you cover the $(x-1)^2$-term and plug in the value $x=1$ in "what's left over", you'll see that $B=\frac{2}{3}$.

Then, using the Shortcut in Table 7.6, since $(x-\boxed{1})$ is the factor which is associated with A, we find the value of A as

$$A=\frac{\boxed{1}}{(covered)(\boxed{1}-2)(\boxed{1}+3)}=\frac{1}{(-1)(4)}=-\frac{1}{4},$$

and, since $(x-\boxed{2})$ is the factor which is associated with B, its value is given by

$$B=\frac{\boxed{2}}{(\boxed{2}-1)(covered)(\boxed{2}+3)}=\frac{2}{(1)(5)}=\frac{2}{5},$$

and finally, since $(x-(-3))=(x+3)$ is the factor which is associated with C, the value of C is,

$$C=\frac{\boxed{(-3)}}{(\boxed{-3}-1)(\boxed{-3}-2)(covered)}=\frac{(-3)}{(-4)(-5)}=-\frac{3}{20}.$$

Okay, now we can integrate the function readily since

$$\begin{aligned}\int \frac{x}{(x-1)(x-2)(x+3)}\,dx &= \int\left(\frac{-\frac{1}{4}}{x-1}+\frac{\frac{2}{5}}{x-2}+\frac{-\frac{3}{20}}{x+3}\right)dx,\\ &= -\frac{1}{4}\int\frac{dx}{x-1}+\frac{2}{5}\int\frac{dx}{x-2}-\frac{3}{20}\int\frac{dx}{x+3},\\ &= -\frac{1}{4}\ln|x-1|+\frac{2}{5}\ln|x-2|-\frac{3}{20}\ln|x+3|+C.\end{aligned}$$

Shortcut: The Cover-up Method

If the **denominator of the rational function f has simple linear factors** (Type I factors, see Chapter 5), then we can find its partial fraction decomposition fairly rapidly as follows.

For example, if

$$f(x) = \frac{A}{x - x_1} + \frac{B}{x - x_2} + \frac{C}{x - x_3} + \ldots,$$

then

$$\lim_{x \to x_1} (x - x_1) f(x) = A,$$
$$\lim_{x \to x_2} (x - x_2) f(x) = B,$$
$$\lim_{x \to x_3} (x - x_3) f(x) = C,$$

and so on.

In practice, if we have the three unknowns A, B, C, we find their values like this:

Write $f(x)$ as

$$f(x) = \frac{p(x)}{(x - x_1)(x - x_2)(x - x_3)}.$$

1. With your finger cover the factor $(x - x_1)$ only!
2. Plug in the value $x = x_1$ in "what's left over" (*i.e.,* the part that's not covered).
3. The number you get in (2) is the value of A.

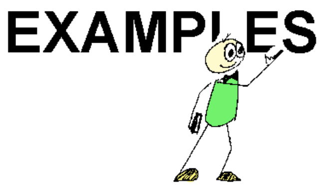
The other values, B, C, are obtained in the same way, except that we cover the factor $(x - x_2)$ only (in the case of B) and $(x - x_3)$ only (in the case of C), and continue as above.

Table 7.6: Finding a Partial Fraction Decomposition in the Case of Simple Linear Factors

Example 327. Evaluate $\int \frac{x+1}{(x-1)^2(x+2)}\,dx.$

Solution Now the denominator has only one simple linear factor, namely, $(x+2) = (x-(-2))$. Notice that the root is "-2" and NOT "2". The other factor is NOT SIMPLE because there is a term of the form $(x-1)^2$. The partial fraction decomposition of this rational function now looks like,

$$\frac{x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}.$$



Since $x+2$ is a simple linear factor we can use Table 7.6 to find the value of C corresponding to that factor. We get,

$$C = \frac{\boxed{-2}+1}{(\boxed{-2}-1)^2(covered)} = -\frac{1}{9}.$$

We're missing A, B, right? The easiest way to get these values is simply to **substitute two arbitrary values of x**, as in the **Plug-in Method** of Example 325. For example, we know that

$$\frac{x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} - \frac{\frac{1}{9}}{x+2}, \tag{7.50}$$

so, setting $x = 2$ say, we must have,

$$\begin{aligned} \frac{2+1}{(2-1)^2(2+2)} &= \frac{A}{2-1} + \frac{B}{(2-1)^2} - \frac{\frac{1}{9}}{2+2}, \\ \frac{3}{4} &= A + B - \frac{1}{36}, \\ \frac{7}{9} &= A + B. \end{aligned}$$

Our first equation for the unknowns A, B is then

$$A + B = \frac{7}{9}. \tag{7.51}$$

We only need another such equation. Let's put $x = -1$ into (7.50), above (because it makes its left-hand side equal to zero!) Then,

$$\begin{aligned} 0 &= \frac{A}{(-1-1)} + \frac{B}{(-1-1)^2} - \frac{\frac{1}{9}}{(-1+2)}, \\ 0 &= -\frac{A}{2} + \frac{B}{4} - \frac{1}{9}, \\ \frac{1}{9} &= -\frac{A}{2} + \frac{B}{4}. \end{aligned}$$

Our second and final equation for the unknowns A, B is now

$$-\frac{A}{2} + \frac{B}{4} = \frac{1}{9}. \tag{7.52}$$

Solving (7.53),(7.52) simultaneously, we get the values

$$A = \frac{1}{9}, \quad B = \frac{2}{3}.$$

Feeding these values back into equation 7.50 we get the decomposition

$$\frac{x+1}{(x-1)^2(x+2)} = \frac{\frac{1}{9}}{x-1} + \frac{\frac{2}{3}}{(x-1)^2} - \frac{\frac{1}{9}}{x+2}.$$

It follows that

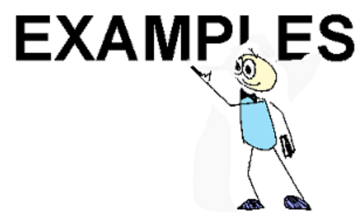
$$\begin{aligned}
\int \frac{x+1}{(x-1)^2(x+2)}\,dx &= \int \left(\frac{\frac{1}{9}}{x-1} + \frac{\frac{2}{3}}{(x-1)^2} - \frac{\frac{1}{9}}{x+2}\right)\,dx,\\
&= \frac{1}{9}\int \frac{1}{x-1}\,dx + \frac{2}{3}\int \frac{1}{(x-1)^2}\,dx - \frac{1}{9}\int \frac{1}{x+2}\,dx,\\
&= \frac{1}{9}\ln|x-1| - \frac{2}{3}\frac{1}{x-1} - \frac{1}{9}\ln|x+2| + C,\\
&= \frac{\ln|x-1|}{9} - \frac{2}{3(x-1)} - \frac{\ln|x+2|}{9} + C,
\end{aligned}$$

where we used the *Power Rule for Integrals* in the evaluation of the second integral, (*i.e.,* the first of Table 6.5).

The moral is this:

Try to use Table 7.6 as much as possible in your evaluation of the constants $A, B, C, \ldots$ in the partial fraction decomposition of the given rational function.

If there are powers of linear factors present (as in Example 327), you can still use the same method provided you are looking for that constant which corresponds to the linear factor with the highest power!



In the following example we introduce a Type II factor into the denominator and we evaluate an integral involving such a factor.

Example 328. Evaluate

$$\int \frac{x}{x^4-1}\,dx.$$

Solution Since $x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$ the partial fraction decomposition of this integrand looks like

$$\frac{x}{x^4-1} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{B_1x+C_1}{x^2+1},$$

since the factors of the denominator are simple and each one occurs with highest power 1, (*i.e.,* $p=1,\ q=1$). Notice that we are using the symbols A_1, A_2, B_1, C_1 instead of A, B, C, D. This doesn't change anything. Now, we can proceed as in Table 7.6 and find A_1, A_2 using the method outlined there. We find,

$$\frac{x}{(x-1)(x+1)(x^2+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{B_1x+C_1}{x^2+1},$$

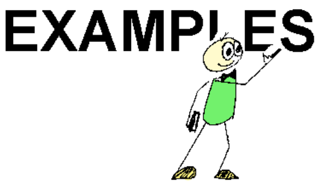
so covering the $(x-1)$ term on the left and setting $x=1$ in *what's left over* we find $A_1 = \frac{1}{4}$. Next, covering the $(x+1) = (x-(-1))$ term on the left and setting $x=-1$ there, gives us $A_2 = \frac{1}{4}$, as well. The constants B_1, C_1 are found using the *Plug-in Method.* So we plug-in some two other values of x, say, $x = 0, 2$ in order to **get a system of equations** (two of them) in the two given unknowns, B_1, C_1 (because WE KNOW the values of A_1, A_2). Solving this system, we get $B_1 = -\frac{1}{2}, C_1 = 0$ so that the partial fraction decomposition takes the form

$$\frac{x}{x^4-1} = \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{x+1} - \frac{\frac{1}{2}x}{x^2+1}.$$

It follows that

$$\begin{aligned}
\int \frac{x}{x^4-1\,dx} &= \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{x\,dx}{x^2+1},\\
&= \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + C,
\end{aligned}$$

where the last integral on the right was evaluated using the substitution $u = x^2 + 1$, $du = 2x\ dx$, etc.



Example 329. Evaluate $\displaystyle\int \frac{3\ dx}{x^2(x^2+9)}$.

Solution The partial fraction decomposition looks like

$$\frac{3}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9} \swarrow \textit{Type II factor}.$$

non-simple root, highest power=2

We shall use a combination of the *Plug-in Method* and the method of comparing coefficients, see Example 325.

We multiply both sides by the denominator to find,

$$3 = Ax(x^2+9) + B(x^2+9) + (Cx+D)(x^2). \tag{7.53}$$

Next, we set $x = 0 \Rightarrow 3 = 0 + 9B + 0 = 9B \Rightarrow B = \frac{1}{3}$ (by the Plug-in Method).

Now, we expand the right-hand side of equation (7.53) completely and collect like terms. This gives the equivalent equation

$$3 = (A+C)x^3 + (B+D)x^2 + (9A)x + 3,$$

(since we already know that $B = \frac{1}{3}$), or,

$$(A+C)x^3 + (1/3+D)x^2 + (9A)x = 0.$$

Since this last relation must be true for every value of x, we may compare the coefficients and obtain (recall that we already know B),

$$\begin{cases} A + C = 0 & \text{(coefficient of } x^3 = 0) \\ 1/3 + D = 0 & \text{(coefficient of } x^2 = 0) \\ 9A = 0 & \text{(coefficient of } x = 0) \end{cases}$$

Solving this last system of three equations in the three unknowns A, C, D, simultaneously gives $A = 0$, $C = 0$ and $D = -B = -\frac{1}{3} \Rightarrow D = -\frac{1}{3}$, *i.e.*,

$$\begin{aligned} \frac{3}{x^2(x^2+9)} &= \frac{1/3}{x^2} + \frac{-1/3}{x^2+9}, \\ &= \frac{1}{3}\left(\frac{1}{x^2} - \frac{1}{x^2+9}\right). \end{aligned}$$

It follows that

$$\begin{aligned}
\int \frac{3}{x^2(x^2+9)}\,dx &= \frac{1}{3}\int \left(\frac{1}{x^2} - \frac{1}{x^2+9}\right)\,dx \\
&= \frac{1}{3}\int \frac{dx}{x^2} - \frac{1}{3}\int \frac{dx}{x^2+9} \\
&= -\frac{1}{3x} - \frac{1}{3}\int \frac{dx}{9\left(\frac{x^2}{9}+1\right)} \\
&= -\frac{1}{3x} - \frac{1}{27}\int \frac{dx}{\left(\frac{x}{3}\right)^2+1} \\
&= -\frac{1}{3x} - \frac{1}{27}\int \frac{3\,du}{u^2+1} \qquad \left\{ \begin{array}{l} \textit{Set } u = \frac{x}{3},\, du = \frac{dx}{3}, \\ \textit{so, } dx = 3\,du, \end{array} \right. \\
&= -\frac{1}{3x} - \frac{1}{9}\mathrm{Arctan}\, u + C \\
&= -\frac{1}{3x} - \frac{1}{9}\mathrm{Arctan}\left(\frac{x}{3}\right) + C.
\end{aligned}$$

$\boxed{\textbf{Example 330.}}$ Evaluate $\displaystyle\int \frac{x^3}{x^2-2x+1}\,dx$.

Solution The numerator has degree $n = 3$ while the denominator has degree $m = 2$. **Since $n > m$ we must divide the numerator by the denominator** using *long division.*

$$\begin{array}{rl}
 & \underline{\;x+2\qquad\quad} \\
x^2-2x+1 & |x^3 \\
 & \underline{x^3-2x^2+x} \\
 & \qquad 2x^2-x \\
 & \qquad \underline{2x^2-4x+2} \\
\text{Remainder}\longrightarrow & \qquad \boxed{3x-2}
\end{array}$$

The result is

$$\frac{x^3}{x^2-2x+1} = x+2+\frac{3x-2}{x^2-2x+1}$$

Use partial fractions here!

Since $n < m$ for the rational function on the right, its partial fraction decomposition is

$$\frac{3x-2}{x^2-2x+1} = \frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

We use the Plug-in Method: Remember to multiply both sides of the equation by the denominator so that

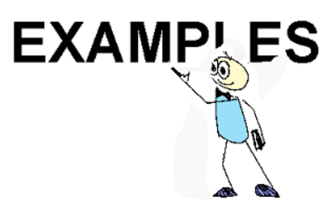
$$3x-2 = A(x-1)+B.$$

Set $x = 1 \Rightarrow 1 = B$.

Set $x = 0 \Rightarrow -2 = -A+B = -A+1$ which means that $A = 3$.

So, the decomposition looks like

$$\frac{3x-2}{(x-1)^2} = \frac{3}{x-1} + \frac{1}{(x-1)^2},$$



where, as a summary, we obtained

$$\frac{x^3}{x^2-2x+1} = \underbrace{x+2}_{\text{found by division}} + \underbrace{\frac{3}{x-1} + \frac{1}{(x-1)^2}}_{\text{found by partial fractions}}$$

Now the final integration is easy, that is,

$$\begin{aligned}
\int \frac{x^3}{x^2-2x+1}\,dx &= \int \left(x+2+\frac{3x-2}{x^2-2x+1}\right)\,dx \\
&= \frac{x^2}{2}+2x+\int \frac{3x-2}{x^2-2x+1}\,dx \\
&= \frac{x^2}{2}+2x+\int \left(\frac{3}{x-1}+\frac{1}{(x-1)^2}\right)\,dx \\
&= \frac{x^2}{2}+2x+\int \frac{3\,dx}{x-1}+\int \frac{1\,dx}{(x-1)^2} \\
&= \frac{x^2}{2}+2x+3\ln|x-1|-\frac{1}{x-1}+C.
\end{aligned}$$

Example 331. Evaluate $I = \displaystyle\int_0^1 \frac{3x^2}{x^2+2x+1}\,dx$

Solution In this example we see that $n = m$ or that the degree of the nominator equals the degree of the denominator. So we must divide the two polynomials using the method of *long division*. Then,

$$\begin{array}{r|l}
 & \;3 \\ \hline
x^2+2x+1 & 3x^3 \\
 & \underline{3x^2+6x+3} \\
 & \quad -6x-3,
\end{array}$$

after which we find

$$\begin{aligned}
\frac{3x^2}{x^2+2x+1} &= 3+\frac{(-6x-3)}{x^2+2x+1} \\
&= 3-\frac{(6x+3)}{x^2+2x+1}.
\end{aligned}$$

We now see that

$$\begin{aligned}
I &= \int_0^1 \frac{3x^2}{x^2+2x+1}\,dx = \int_0^1 \left(3-\frac{6x+3}{x^2+2x+1}\right)\,dx \\
&= \int_0^1 3\,dx - \underbrace{\int_0^1 \frac{6x+3}{x^2+2x+1}\,dx}_{\uparrow\ \text{Use partial fractions here!}}
\end{aligned}$$

The partial fraction decomposition of the rational function on the right has the form

$$\frac{6x+3}{x^2+2x+1} = \frac{6x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

where we can use the method of *comparing coefficients* (remember to multiply both sides by the denominator),

$$\begin{aligned}
6x+3 &= A(x+1)+B \\
&= Ax+(A+B)
\end{aligned}$$

So, $A = 6$ and $A + B = 3 \Rightarrow B = -3$. It follows that

$$\begin{aligned}
I &= \int_0^1 \frac{3x^2}{x^2+2x+1}\,dx \\
&= \int_0^1 3\,dx - \int_0^1 \frac{6x+3}{x^2+2x+1}\,dx \\
&= 3 - \int_0^1 \left(\frac{6}{x+1} - \frac{3}{(x+1)^2}\right)\,dx \\
&= 3 - 6\ln(x+1)\Big|_0^1 + 3\int_0^1 (x+1)^{-2}\,dx \\
&= 3 - 6\ln 2 + 3\left(\frac{(x+1)^{-1}}{(-1)}\right)\Big|_0^1 \\
&= 3 - 6\ln 2 + 3\left(-\frac{1}{2}+1\right) \\
&= 3 - 6\ln 2 + \frac{3}{2} \\
&= \frac{9}{2} - 6\ln 2.
\end{aligned}$$

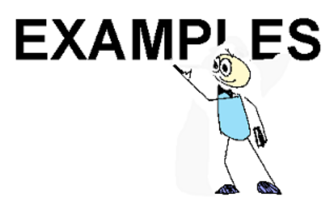
Example 332. Evaluate $\displaystyle\int \frac{x^5+2x-2}{x^4-1}\,dx.$

Solution In this example we have $n > m$ since the numerator has the higher degree. Using long division we find,

$$\begin{array}{rl}
 & \underline{\;x\qquad\qquad\quad} \\
x^4-1 & |x^5+2x-2 \\
 & \underline{x^5-x} \\
 & \qquad 3x-2.
\end{array}$$

From this we see that,

$$\begin{aligned}
\frac{x^5+2x-2}{x^4-1} &= x + \frac{3x-2}{x^4-1} \\
&= x + \frac{3x-2}{(x-1)(x+1)(x^2+1)}.
\end{aligned}$$



Now, the partial fraction decomposition of the rational function above, on the right, looks like

$$\frac{3x-2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \tag{7.54}$$

where A, B may be found using the method of Table 7.6 while C, D may be found using the Plug-in Method. Covering the factors $(x-1)$, $(x+1)$ respectively we obtain $A = \frac{1}{4}$ and $B = \frac{5}{4}$. Okay, this means that (7.54) is the same as

$$\frac{3x-2}{(x-1)(x+1)(x^2+1)} = \frac{\frac{1}{4}}{x-1} + \frac{\frac{5}{4}}{x+1} + \frac{Cx+D}{x^2+1} \tag{7.55}$$

where all we need to find now is C, D. So, let's use the Plug-in Method.

Setting $x = 0$ in (7.55) gives

$$\begin{aligned}
2 &= -\frac{1}{4} + \frac{5}{4} + \frac{D}{1}, \quad \text{or, solving for } D, \\
D &= 2 + \frac{1}{4} - \frac{5}{4} \\
&= 1.
\end{aligned}$$

So, (7.54) now looks like,

$$\frac{3x-2}{(x-1)(x+1)(x^2+1)} = \frac{\frac{1}{4}}{x-1}+\frac{\frac{5}{4}}{x+1}+\frac{Cx+1}{x^2+1}, \tag{7.56}$$

and all that's missing is the value of C, right? For this last value, we can substitute *any* value of x other than one of the roots of the denominator. For example, we can set $x=2$. Then,

$$\begin{aligned}\frac{4}{15} &= \frac{1}{4}+\frac{5}{12}+\frac{2C+1}{5}, \quad \text{or, solving for } C,\\ C &= -\frac{3}{2}.\end{aligned}$$

So,

$$\frac{3x-2}{(x-1)(x+1)(x^2+1)} = \frac{\frac{1}{4}}{x-1}+\frac{\frac{5}{4}}{x+1}+\frac{-\frac{3}{2}x+1}{x^2+1},$$

is the partial fraction decomposition of the rational function on the left. We now proceed to the integral.

$$\begin{aligned}\int \frac{x^5+2x-2}{x^4-1}\,dx &= \int \left(x+\frac{3x-2}{(x-1)(x+1)(x^2+1)}\right)\,dx\\
&= \frac{x^2}{2}+\int\left(\frac{\frac{1}{4}}{x-1}+\frac{\frac{5}{4}}{x+1}+\frac{-\frac{3}{2}x+1}{x^2+1}\right)\,dx\\
&= \frac{x^2}{2}+\frac{1}{4}\int\frac{dx}{x-1}+\frac{5}{4}\int\frac{dx}{x+1}-\frac{3}{2}\int\frac{x\,dx}{x^2+1}+\\
&\quad+\int\frac{dx}{x^2+1},\\
&= \frac{x^2}{2}+\frac{1}{4}\ln|x-1|+\frac{5}{4}\ln|x+1|-\frac{3}{4}\int\frac{du}{u}+\\
&\quad+\text{Arctan}\,x\\
&\quad\left(\text{where we used the substitution } u=x^2+1, du=2xdx\right)\\
&= \frac{x^2}{2}+\frac{1}{4}\ln|x-1|+\frac{5}{4}\ln|x+1|-\frac{3}{4}\ln\left(x^2+1\right)+\\
&\quad+\text{Arctan}\,x+C.\end{aligned}$$

SNAPSHOTS

Example 333. Evaluate $I=\displaystyle\int\frac{x+4}{x^2+5x-6}\,dx$

Solution The partial fraction decomposition has the form,

$$\frac{x+4}{x^2+5x-6}=\frac{x+4}{(x-1)(x+6)}=\frac{A}{x-1}+\frac{B}{x+6},$$

and so, either the Plug-in Method or the method of Table 7.6, gives the values of A,B. In this case, $x+4=A(x+6)+B(x-1)$ which means that if we set

$$x=1 \Rightarrow 5=7A \Rightarrow A=\tfrac{5}{7},$$

$$\text{or} \quad x=-6 \Rightarrow -2=-7B \Rightarrow B=\tfrac{2}{7}.$$

Finally,

$$\begin{aligned}I &= \int\left(\frac{5/7}{x-1}+\frac{2/7}{x+6}\right)dx\\ &= \frac{5}{7}\ln|x-1|+\frac{2}{7}\ln|x+6|+C.\end{aligned}$$

Example 334. Evaluate $I = \int \frac{dx}{1+\sqrt{x}}$

Solution We start with a substitution, $u = \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$ or $dx = 2u\ du$. Then,

$$\begin{aligned}
I &= \int \frac{dx}{1+\sqrt{x}} \\
&= \int \frac{2u\ du}{1+u} \\
&= 2\int \frac{u}{1+u}\ du \quad \text{(rational function in } u,\ n=m=1\text{)}, \\
&= 2\int \left(1 - \frac{1}{1+u}\right)\ du \quad \text{(after using long division)}, \\
&= 2u - 2\int \frac{du}{1+u} \\
&= 2u - 2\ln|1+u| + C \\
&= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + C.
\end{aligned}$$

Example 335. Evaluate $I = \int_0^1 \ln(x^2+1)\ dx$

Solution We use Integration by Parts.

$$\text{Set} \left\{ \begin{array}{ll} u = \ln(x^2+1) & dv = 1dx \\ du = \dfrac{2x\ dx}{x^2+1} & v = x. \end{array} \right.$$

Then,

$$\begin{aligned}
I &= uv - \int v\ du \\
&= x\ln(x^2+1)\Big|_0^1 - \int_0^1 x\left(\frac{2x}{x^2+1}\right)\ dx \\
&= \ln 2 - 0 - 2\int_0^1 \frac{x^2}{1+x^2}\ dx \text{ (rational function, } n=m=2\text{)}, \\
&= \ln 2 - 2\int_0^1 \left(1 - \frac{1}{1+x^2}\right)\ dx \text{ (after long division)}, \\
&= \ln 2 - 2 + 2\int_0^1 \frac{dx}{1+x^2} \\
&= \ln 2 - 2 + 2\tan^{-1} x\Big|_0^1 \quad \text{(by Table 6.7)}, \\
&= \ln 2 - 2 + 2(\tan^{-1}(1) - 0) \\
&= \ln 2 - 2 + 2\cdot\frac{\pi}{4} \\
&= \ln 2 - 2 + \frac{\pi}{2}.
\end{aligned}$$

NOTES:

Exercise Set 34.

1. $\displaystyle\int \frac{x}{x-1}\, dx$
2. $\displaystyle\int \frac{x+1}{x}\, dx$
3. $\displaystyle\int \frac{x^2}{x+2}\, dx$
4. $\displaystyle\int \frac{x^2}{x^2+1}\, dx$
5. $\displaystyle\int \frac{x^2}{(x-1)(x+1)}\, dx$
6. $\displaystyle\int \frac{2x}{(x-1)(x-3)}\, dx$
7. $\displaystyle\int \frac{3x^2}{(x-1)(x-2)(x-3)}\, dx$
8. $\displaystyle\int_0^1 \frac{x^3-1}{x+1}\, dx$
9. $\displaystyle\int \frac{3x}{(x-1)^2}\, dx$
10. $\displaystyle\int \frac{2x-1}{(x-2)^2(x+1)}\, dx$
11. $\displaystyle\int \frac{x^4+1}{x^2+1}\, dx$
12. $\displaystyle\int \frac{1}{(x^2+1)(x^2+4)}\, dx$
13. $\displaystyle\int \frac{1}{x^2(x-1)(x+2)}\, dx$
14. $\displaystyle\int \frac{x^5+1}{(x^2-2x)(x^4-1)}\, dx$
15. $\displaystyle\int \frac{2}{x(x-1)^2(x^2+1)}\, dx$

Suggested Homework Set 26. *Do problems 1, 3, 5, 7, 10, 14*

NOTES:

7.5 Products of Trigonometric Functions

7.5.1 Products of Sines and Cosines

In this section we provide some insight into the integration of products and powers of trigonometric functions. For example, we will be considering integrals of the form

$$\int \cos^2 x \sin^3 x \ dx, \quad \int \sin^5 x \ dx, \quad \int \cos^4 x \ dx.$$

In almost all the cases under consideration extensive use will be made of some fundamental trigonometric identities, such as those presented in the display below. Furthermore you should recall you angle-sum/angle-difference identities (see Section 1.3) or the text below.

A **trigonometric integral** is an integral whose integrand contains only trigonometric functions and their powers. These are best handled with the repeated use of *trigonometric identities*. Among those which are most commonly used we find (here u or x is in radians):

$$\cos^2 u = \frac{1+\cos 2u}{2}, \qquad \sin^2 u = \frac{1-\cos 2u}{2}, \tag{7.57}$$

$$\begin{aligned} \cos^2 u + \sin^2 u &= 1, & (7.58)\\ \sec^2 u - \tan^2 u &= 1, & (7.59)\\ \csc^2 u - \cot^2 u &= 1, & (7.60)\end{aligned}$$

$$\sin 2u = 2 \sin u \ \cos u. \tag{7.61}$$

Of these identities, **(7.57) is used to reduce the power in a trigonometric integral by 1**. As you can see, the left-side is a square while the right-side is not.

Example 336. Evaluate $\int \cos^4 x \ dx$.

Solution One always starts a problem like this by applying some *trigonometric identities*. Before you integrate such functions you should be using trigonometric identities to simplify the form of the integrand, first!

This problem is tackled by **writing $\cos^4 x$ as a product of two squares** and then using (7.57) repeatedly as follows:

$$\begin{aligned} \cos^4 x &= \cos^2 x \ \cos^2 x \\ &= \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) \\ &= \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right) \\ &= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right), \end{aligned}$$

since we can use (7.57) with $u = 2x$ to find that

$$\cos^2 2x = \frac{1+\cos 4x}{2}.$$

Combining the last two identities we find that

$$\begin{aligned}\cos^4 x &= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\right)\\ &= \frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}.\end{aligned}$$

Okay, now the last equation is in a form we can integrate just by using simple substitutions (because there are *no powers* left). For example, since we can let $u = 2x, du = 2dx$ in the second integral and $u = 4x, du = 4dx$ in the third integral we see that

$$\begin{aligned}\int \cos^4 x\ dx &= \frac{3}{8}\int 1\ dx + \frac{1}{2}\int \cos 2x\ dx + \frac{1}{8}\int \cos 4x\ dx\\ &= \frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.\end{aligned}$$

The same exact idea may be used to evaluate the integral of its counterpart, $\sin^4 x$.

Example 337. Evaluate $\int \sin^4 x\ dx$.

Solution We write $\sin^4 x$ as a product of two squares and then use (7.57) repeatedly just as before.

$$\begin{aligned}\sin^4 x &= \sin^2 x\ \sin^2 x\\ &= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)\\ &= \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right)\\ &= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right),\end{aligned}$$

just as before. So,

$$\sin^4 x = \frac{3}{8} - \frac{\cos 2x}{2} + \frac{\cos 4x}{8},$$

and it follows that

$$\begin{aligned}\int \sin^4 x\ dx &= \frac{3}{8}\int 1\ dx - \frac{1}{2}\int \cos 2x\ dx + \frac{1}{8}\int \cos 4x\ dx\\ &= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.\end{aligned}$$

In addition to (7.57)-(7.61) there are a few other identities which may be useful as they help to *untangle* the products. For any two angles, A, B, these are

$$\sin A\ \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}, \qquad (7.63)$$

$$\sin A\ \cos B = \frac{\sin(A-B) + \sin(A+B)}{2}, \qquad (7.64)$$

$$\cos A\ \cos B = \frac{\cos(A-B) + \cos(A+B)}{2}. \qquad (7.65)$$

We present the details for evaluating integrals of the form

$$\int \cos^m x \ \sin^n x \ dx \tag{7.62}$$

where m, n are positive integers.

- ***m is odd, n is even***

Solve (7.58) for $\cos^2 x$. In the integrand, substitute every term of the form $\cos^2 x$ by $1-\sin^2 x$. Since m is odd, there is always one extra cosine term. Collecting terms we see that we have an integrand involving only sine functions and their powers and only ONE simple cosine factor. Follow this with a substitution of variable, namely, $u = \sin x,\ du = \cos x\ dx$, which now reduces the integrand to a polynomial in u and this is easily integrated.

- ***m is odd, n is odd***

Factor out a copy of each of $\sin x,\ \cos x$, leaving behind *even powers* of both $\sin x,\ \cos x$. Convert either one of these even powers in terms of the other using (7.58), and then perform a simple substitution, as before.

- ***m is even, n is odd***

Proceed as in the case where m is odd and n is even with the words *sine* and *cosine* interchanged. So, we solve (7.58) for $\sin^2 x$. In the integrand, substitute every term of the form $\sin^2 x$ by $1 - \cos^2 x$. Since n is odd, there is always one extra sine term. Collecting terms we see that we have an integrand involving only cosine functions and their powers and only ONE simple sine term. Follow this with a simple substitution of variable, namely, $u = \cos x,\ du = -\sin x\ dx$, which now reduces the integrand to a polynomial in u which is easily integrated.

- ***m is even, n is even***

This is generally the most tedious case. Remove all even powers of the sine and cosine by applying (7.57) repeatedly and reduce the integrand to a more recognizable form after a change of variable. You may then have to apply anyone or more of the three cases, above (see Examples 336, 337).

Table 7.7: Powers and Products of Sine and Cosine Integrals

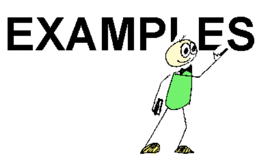
Example 338. Evaluate $\displaystyle\int \cos^3 x\ dx$.

Solution In this case $m = 3, n = 0$, and so m is odd and n is even (remember that we consider 0 to be an *even* number). Now we solve (7.58) for $\cos^2 x$ and substitute the remaining term into the cosine expression above leaving one cosine term as a factor. Thus,

$$\begin{aligned}\cos^3 x &= \cos^2 x \cos x \\ &= (1-\sin^2 x)\cos x.\end{aligned}$$

Finally,

$$\begin{aligned}\int \cos^3 x\ dx &= \int (1-\sin^2 x)\cos x\ dx \\ &= \int (1-u^2)\ du \text{ (since we set } u = \sin x, \quad du = \cos x\ dx), \\ &= u - \frac{u^3}{3} + C, \\ &= \sin x - \frac{\sin^3 x}{3} + C, \quad \text{(back-substitution).}\end{aligned}$$



Example 339. Evaluate $\displaystyle\int \sin^3 x\ dx$.

Solution In this case $m = 0, n = 3$, and so m is even and n is odd. So we solve (7.58) for $\sin^2 x$ to find $\sin^2 x = 1 - \cos^2 x$ and substitute this last term into the expression leaving one "sine" term as a factor. Thus,

$$\begin{aligned}\sin^3 x &= \sin^2 x \sin x \\ &= (1-\cos^2 x)\sin x.\end{aligned}$$

Just as before we will find,

$$\begin{aligned}\int \sin^3 x\ dx &= \int (1-\cos^2 x)\sin x\ dx \\ &= \int -(1-u^2)\ du \text{ (but now we let } u = \cos x,\ du = -\sin x\ dx), \\ &= -u + \frac{u^3}{3} + C, \\ &= -\cos x + \frac{\cos^3 x}{3} + C, \quad \text{(back-substitution).}\end{aligned}$$

Example 340. Evaluate $\displaystyle\int \sin^4 x\ \cos^2 x\ dx$.

Solution Here $m = 2, n = 4$, right? Since both m, n are even we remove all even powers of the sine and cosine by applying (7.57) over and over again thereby reducing the powers more and more! Thus,

$$\begin{aligned}\sin^4 x\ \cos^2 x &= \sin^2 x \sin^2 x \cos^2 x \\ &= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right) \\ &= \frac{1}{8}\left((1-\cos^2 2x)\ (1-\cos 2x)\right) \\ &= \frac{1}{8}\left(1-\cos 2x - \cos^2 2x + \cos^3 2x\right).\end{aligned}$$

where the first three terms may be integrated without much difficulty. The last term above may be converted to the case of Example 338, with the substitution $u = 2x,\ du = 2dx$, for example. So,

$$\begin{aligned}
\int \sin^4 x\ \cos^2 x\ dx &= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right)\ dx \\
&= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{1}{8}\int \left(\frac{1+\cos 4x}{2}\right)\ dx + \frac{1}{8}\int \cos^3 2x\ dx \\
&= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{x}{16} - \frac{\sin 4x}{64} + \frac{1}{16}\int \cos^3 u\ du \\
&= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{x}{16} - \frac{\sin 4x}{64} + \frac{1}{16}\left(\sin u - \frac{\sin^3 u}{3}\right) + C \\
&\quad \text{(by the result of Example 338)} \\
&= \frac{x}{8} - \frac{\sin 2x}{16} - \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin 2x}{16} - \frac{\sin^3 2x}{48} + C, \\
&\quad \text{(since } u = 2x,\text{)} \\
&= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{48} + C.
\end{aligned}$$

Example 341. Evaluate $\int \sin^3 x \cos^3 x\ dx$.

Solution Now, $m = 3, n = 3$ and so both exponents are odd. In accordance with Table 7.7 we **factor out a copy of each of the sine and cosine term** leaving only even powers of the remaining product. Thus,

$$\sin^3 x \cos^3 x = \left(\sin^2 x \cos^2 x\right) \sin x\ \cos x.$$

Now we use (7.58) in order to convert either one of the even powers to powers involving the other. For example,

$$\begin{aligned}
\left(\sin^2 x \cos^2 x\right) \sin x\ \cos x &= \sin^2 x \left(1 - \sin^2 x\right) \sin x\ \cos x \\
&= \left(\sin^2 x - \sin^4 x\right) \sin x \cos x \\
&= \left(\sin^3 x - \sin^5 x\right) \cos x,
\end{aligned}$$

and this is where we STOP. Now we use the substitution $u = \sin x,\ du = \cos x\ dx$ to transform the integral into a polynomial in u which is easily integrated. The details are,

$$\begin{aligned}
\int \sin^3 x \cos^3 x\ dx &= \int \left(\sin^3 x - \sin^5 x\right) \cos x\ dx \\
&= \int (u^3 - u^5)\ du \\
&= \frac{u^4}{4} - \frac{u^6}{6} + C, \\
&= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C.
\end{aligned}$$

Example 342. Evaluate $\int \sin^3 x \cos^2 x\ dx$.

Solution Use Table 7.7 with $m = 2, n = 3$ so that m is even and n is odd. Thus, we solve (7.58) for $\sin^2 x$ to find $\sin^2 x = 1 - \cos^2 x$ and substitute this last term into the expression $\sin^3 x \cos^2 x$ leaving one "sine" term as a factor. Thus,

$$\begin{aligned}
\sin^3 x \cos^2 x &= \sin^2 x \cos^2 x \sin x \\
&= (1 - \cos^2 x) \cos^2 x\ \sin x, \\
&= \left(\cos^2 x - \cos^4 x\right) \sin x.
\end{aligned}$$

Finally,

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \left(\cos^2 x - \cos^4 x\right) \sin x \, dx \\
&= \int -(u^2 - u^4) \, du, \quad (\text{we let } u = \cos x, \ du = -\sin x \, dx), \\
&= -\frac{u^3}{3} + \frac{u^5}{5} + C, \\
&= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C
\end{aligned}$$

Example 343. Evaluate $\int \sin^5 x \cos^3 x \, dx$.

Solution Now $m = 3, n = 5$ so both m, n are odd. Using Table 7.7 we remember to factor out a copy of each of $\sin x$ and $\cos x$ thereby leaving behind only EVEN exponents. Symbolically,

$$\begin{aligned}
\sin^5 x \cos^3 x &= \left(\sin^4 x \cos^2 x\right) \sin x \ \cos x \\
&= \sin^4 x \left(1 - \sin^2 x\right) \sin x \ \cos x \\
&= \left(\sin^4 x - \sin^6 x\right) \sin x \cos x \\
&= \left(\sin^5 x - \sin^7 x\right) \cos x,
\end{aligned}$$

and, once again, this is where we STOP. Use the substitution $u = \sin x$, $du = \cos x \, dx$ to transform the integral into a polynomial in u so that

$$\begin{aligned}
\int \sin^5 x \cos^3 x \, dx &= \int \left(\sin^5 x - \sin^7 x\right) \cos x \, dx \\
&= \int (u^5 - u^7) \, du \\
&= \frac{u^6}{6} - \frac{u^8}{8} + C, \\
&= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C.
\end{aligned}$$

Example 344. Evaluate $\int \cos^3 x \sin^4 x \, dx$.

Solution In this final problem before the *Snapshots* we note that $m = 3, n = 4$ so that m is odd and n is even. We use Table 7.7 again. We factor out a cosine term out of the integrand leaving us with only even exponents, that is,

$$\cos^3 x \sin^4 x = \left(\cos^2 x \ \sin^4 x\right) \cos x.$$

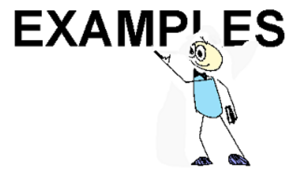
We replace every term in the integrand of the form $\cos^2 x$ by $1 - \sin^2 x$. So,

$$\begin{aligned}
\cos^3 x \sin^4 x &= \left(\cos^2 x \ \sin^4 x\right) \cos x \\
&= \left(1 - \sin^2 x\right) \sin^4 x \ \cos x \\
&= \left(\sin^4 x - \sin^6 x\right) \ \cos x.
\end{aligned}$$

This last expression is easily integrated after a simple substitution, *i.e.,*

$$\begin{aligned}
\int \cos^3 x \ \sin^4 x \, dx &= \int \left(\sin^4 x - \sin^6 x\right) \ \cos x \, dx \\
&= \int (u^4 - u^6) \, du \ (\text{set } u = \sin x, \ du = \cos x \, dx), \\
&= \frac{u^5}{5} - \frac{u^7}{7} + C, \\
&= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.
\end{aligned}$$

SNAPSHOTS



Example 345. Evaluate $\int \cos^4 2x \, dx$.

Solution We bring the integrand into a *standard form* so that the same symbol appears everywhere in the integral. So we must perform a substitution $u = 2x$, $du = 2dx$. This transforms the integral to

$$\int \cos^4 2x \, dx = \frac{1}{2} \int \cos^4 u \, du.$$

Now $m = 4, n = 0$, so

$$\begin{aligned}
\int \cos^4 2x \, dx &= \frac{1}{2} \int \cos^4 u \, du \\
&= \frac{1}{2} \int \cos^2 u \; \cos^2 u \, du \\
&= \frac{1}{2} \int \left(\frac{1+\cos 2u}{2}\right)\left(\frac{1+\cos 2u}{2}\right) \, du \\
&= \frac{1}{8} \int (1 + 2\cos 2u + \cos^2 2u) \, du \\
&= \frac{u}{8} + \frac{\sin 2u}{8} + \frac{1}{16} \int \cos^2 v \, dv \quad (\text{where } v = 2u), \\
&= \frac{u}{8} + \frac{\sin 2u}{8} + \frac{1}{16} \int \left(\frac{1+\cos 2v}{2}\right) \, dv \\
&= \frac{u}{8} + \frac{\sin 2u}{8} + \frac{v}{32} + \frac{1}{32} \int \cos 2v \, dv \\
&= \frac{u}{8} + \frac{\sin 2u}{8} + \frac{v}{32} + \frac{1}{32} \frac{\sin 2v}{2} + C \\
&= \frac{u}{8} + \frac{\sin 2u}{8} + \frac{u}{16} + \frac{\sin 4u}{64} + C \\
&= \frac{x}{4} + \frac{\sin 4x}{8} + \frac{x}{8} + \frac{\sin 8x}{64} + C \\
&= \frac{3x}{8} + \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + C.
\end{aligned}$$

NOTES:

Example 346. Evaluate $\int x\cos^2(3x^2+1)\ dx$.

Solution Let $u = 3x^2+1,\ du = 6x\ dx$. Then

$$\begin{aligned}\int x\cos^2(3x^2+1)\ dx &= \frac{1}{6}\int \cos^2 u\ du\\ &= \frac{1}{6}\int \left(\frac{1+\cos 2u}{2}\right)\ du\\ &= \frac{u}{12}+\frac{1}{24}\sin 2u + C\\ &= \frac{3x^2+1}{12}+\frac{1}{24}\sin\big(2(3x^2+1)\big)+C\\ &= \frac{1}{12}+\frac{x^2}{4}+\frac{1}{24}\sin(6x^2+2)+C.\end{aligned}$$

Example 347. Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x\ \cos^2 x\ dx$.

Solution Use the identity $\sin 2x = 2\sin x\ \cos x$ to find

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x\ \cos^2 x\ dx &= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2x\ dx\\ &= \frac{1}{8}\int_0^{\pi} \sin^2 u\ du\\ &= \frac{1}{8}\int_0^{\pi}\left(\frac{1-\cos 2u}{2}\right)\ du\\ &= \frac{\pi}{16}-\frac{1}{16}\int_0^{\pi}\cos 2u\ du\\ &= \frac{\pi}{16}-\frac{\sin 2u}{32}\Big|_0^{\pi}\\ &= \frac{\pi}{16}.\end{aligned}$$

Example 348. Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x\ \cos^4 x\ dx$.

Solution Use Table 7.7 with $m=4, n=2$, so that both m, n are even. Now,

$$\begin{aligned}\sin^2 x\ \cos^4 x &= \sin^2 x\ \cos^2 x\ \cos^2 x\\ &= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)\\ &= \frac{1}{8}\left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right).\end{aligned}$$

We let $u=2x,\ du=2dx$, so that when $x=0,\ u=0$ and when $x=\frac{\pi}{2},\ u=\pi$. Now the integral becomes

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x\ \cos^4 x\ dx &= \frac{1}{16}\int_0^{\pi}\left(1+\cos u-\cos^2 u-\cos^3 u\right)\ du\\ &= \frac{\pi}{16}+0-\frac{1}{32}\int_0^{\pi}(1+\cos 2u)\ du-\frac{1}{16}\int_0^{\pi}\cos^3 u\ du\\ &= \frac{\pi}{16}-\frac{\pi}{32}-0-\frac{1}{16}\left(\sin u-\frac{\sin^3 u}{3}\right)\Big|_0^{\pi},\quad \text{(by Example 338)},\\ &= \frac{\pi}{32}.\end{aligned}$$

Exercise Set 35.

Evaluate the following integrals.

1. $\int \sin^3 3x \ dx$
2. $\int \cos^3(2x-1) \ dx$
3. $\int_0^{\frac{\pi}{2}} \sin^2 x \ \cos^3 x \ dx$
4. $\int \cos^2(x-2) \ \sin^3(x-2) \ dx$
5. $\int_{\frac{\pi}{2}}^{\pi} \sin^3 x \ \cos x \ dx$
6. $\int x \ \sin^2(x^2) \ \cos^2(x^2) \ dx$
7. $\int \sin^4 x \ \cos^4 x \ dx$
8. $\int \sin^4 x \ \cos^5 x \ dx$
9. $\int \cos^4 2x \ dx$
10. $\int \sin^5 x \ \cos^3 x \ dx$
11. $\int \sin^5 x \ \cos^4 x \ dx$
12. $\int \sin^6 x \ \ dx$
13. $\int \cos^7 x \ dx$

Suggested Homework Set 27. *Do problems 1, 3, 4, 6,*

NOTES:

7.5.2 Fourier Coefficients

The motivation behind this topic dates back to work by **Joseph-Louis Fourier**, (1768-1830), a French engineer (and mathematician) who discussed heat flow through a bar. This gives rise to the so-called **Heat Diffusion Problem**, given by a *partial differential equation* involving an unknown function u of the two variables x, t denoted by $u = u(x,t)$ (see Chapter **??**) for functions of many variables)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}$$

where $u(0,t) = 0 = u(L,t)$, $u(x,0) = f(x)$, and f is given and K is a constant. Think of f as describing the initial state of the bar at time $t = 0$, and $u(x,t)$ as being the temperature distribution along the bar at the point x in time t. The **boundary conditions** or conditions at the end-points are given in such a way that the bar's "ends" are kept at a fixed temperature, say 0 degrees (in whatever units).

Some advanced methods are needed to solve this particular kind of differential equation but, in order to motivate the type of integrals we will be studying here, we will assume that we already know the solution to our problem. That is, if the function $f(x)$ is simple enough Fourier obtained the expression for this function $u(x,t)$ of two variables,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{N} b_n u_n(x,t) \\ &= \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 Kt/L^2}, \end{aligned}$$

where the numbers b_n are constants that need to be determined, and that generally depend on f. The other quantities are physical constants (*e.g.,* K is the conductivity of the bar, L its length, etc.). The b_n appearing here are called **Fourier coefficients** of the function f. What are they in terms of f? We have enough information now to actually derive their form:

First, we assume that $t = 0$ so that we are looking at the initial temperature of the bar, $u(x,0)$. Since $u(x,0) = f(x)$ we get

$$f(x) = \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{L}.$$

To find the form of the b_n we fix an integer m, where $1 \le m \le N$, multiply both sides of the last display by $\sin\frac{m\pi x}{L}$ and then integrate both sides over the interval $[0, L]$ so that,

$$\begin{aligned} f(x)\sin\frac{m\pi x}{L} &= \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L} \\ \int_0^L f(x)\sin\frac{m\pi x}{L}\,dx &= \int_0^L \left(\sum_{n=1}^{N} b_n \sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L}\right)\,dx \\ &= \sum_{n=1}^{N}\int_0^L b_n \sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L}\,dx \\ &= \sum_{n=1}^{N} b_n \int_0^L \sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L}\,dx. \end{aligned}$$

But, by the methods of Section 7.5,

$$\int_0^L \sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L}\,dx = \begin{cases} 0, & \text{if } m \ne n, \\ \frac{L}{2}, & \text{if } m = n. \end{cases}$$

This relation means that whenever $n \neq m$, our fixed integer, the corresponding integral in the last displayed sum above is zero. In other words,

$$\begin{aligned}
\int_0^L f(x) \sin \frac{m\pi x}{L}\ dx &= \sum_{n=1}^{N} b_n \int_0^L \sin \frac{n\pi x}{L}\ \sin \frac{m\pi x}{L}\ dx \\
&= \sum_{n=1, n\neq m}^{N} b_n \int_0^L \sin \frac{n\pi x}{L}\ \sin \frac{m\pi x}{L}\ dx + \\
&\quad + b_m \int_0^L \sin \frac{m\pi x}{L}\ \sin \frac{m\pi x}{L}\ dx \\
&= 0 + b_m\ \frac{L}{2} \\
&= b_m\ \frac{L}{2}.
\end{aligned}$$

Solving for b_m we get

$$b_m = \frac{2}{L}\ \int_0^L f(x) \sin \frac{m\pi x}{L}\ dx.$$

If f is an *odd* function, that is, if $f(-x) = -f(x)$, then we know that the product of two odd functions is an *even* function (see Chapter 5) and so

$$\frac{2}{L}\ \int_0^L f(x) \sin \frac{m\pi x}{L}\ dx = \frac{1}{L}\ \int_{-L}^L f(x) \sin \frac{m\pi x}{L}\ dx.$$

It follows that

$$b_m = \frac{1}{L}\ \int_{-L}^L f(x) \sin \frac{m\pi x}{L}\ dx.$$

This is an example of a *Fourier Sine Coefficient* of f.

NOTE: As you can see, the methods of integration in this Chapter come in really handy when one needs to evaluate integrals called ***Fourier coefficients***, that is, integrals that look like the, so-called, *Fourier cosine coefficient*,

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L}\ dx,$$

where $n = 0, 1, 2, 3, ...$ or, the *Fourier sine coefficient*,

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L}\ dx,$$

where $L > 0$ is a fixed number, and $n = 1, 2, 3, ...$ Integrals of this type appear in the ***Theory of Fourier Series*** and were originally studied in order to solve the one-dimensional heat equation. In most cases, the functions f are usually given as polynomials or piecewise constant or linear functions. In either case, the methods of this Chapter allow for a very speedy finding of these coefficients as we will see.

Example 349. *Evaluate the Fourier cosine and the Fourier sine coefficients of the function f defined by setting $L = 1$ and*

$$f(x) = \begin{cases} x - 1, & \textit{if} \quad 0 \leq x \leq 1, \\ x + 1, & \textit{if} \quad -1 \leq x < 0. \end{cases}$$

Solution By definition, we know that (since $L = 1$),

$$
\begin{aligned}
a_n &= \int_{-1}^{1} f(x)\cos n\pi x \; dx, \\
&= \int_{-1}^{0} (x+1)\cos n\pi x \; dx + \int_{0}^{1} (x-1)\cos n\pi x \; dx, \\
&= \frac{1-\cos n\pi}{n^2\pi^2} + \frac{\cos n\pi - 1}{n^2\pi^2}, \\
&= 0,
\end{aligned}
$$

since the table corresponding to the first integration looks like,

$(x+1)$ $+$	$\cos n\pi x$
1 $-$	$\dfrac{\sin n\pi x}{n\pi}$
0 $+$	$-\dfrac{\cos n\pi x}{n^2\pi^2}$
$\ldots$	$\ldots$

and this means that the corresponding definite integral is given by

$$
\begin{aligned}
\int_{-1}^{0} (x+1)\cos n\pi x \; dx &= \left\{ \frac{(x+1)\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2} \right\}\Bigg|_{-1}^{0}, \\
&= \frac{1}{n^2\pi^2} - \frac{\cos(-n\pi)}{n^2\pi^2}, \\
&= \frac{1-\cos n\pi}{n^2\pi^2}.
\end{aligned}
$$

The other integral is done similarly. For instance, the table corresponding to the second integral is given by

$(x-1)$ $+$	$\cos n\pi x$
1 $-$	$\dfrac{\sin n\pi x}{n\pi}$
0 $+$	$-\dfrac{\cos n\pi x}{n^2\pi^2}$
$\ldots$	$\ldots$

This gives the defnite integral,

$$
\begin{aligned}
\int_{0}^{1} (x-1)\cos n\pi x \; dx &= \left\{ \frac{(x-1)\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2} \right\}\Bigg|_{0}^{1}, \\
&= \frac{\cos(n\pi)}{n^2\pi^2} - \frac{1}{n^2\pi^2}, \\
&= \frac{\cos n\pi - 1}{n^2\pi^2}.
\end{aligned}
$$

Adding the two contributions, we get that $a_n = 0$ as we claimed.

The *Fourier sine coefficients* of this function are found using the same methods. For example,

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\sin n\pi x\ dx, \\ &= \int_{-1}^{0} (x+1)\sin n\pi x\ dx + \int_{0}^{1} (x-1)\sin n\pi x\ dx, \\ &= -\frac{1}{n\pi} - \frac{1}{n\pi}, \\ &= -\frac{2}{n\pi}, \end{aligned}$$

where $n = 1, 2, 3, \ldots$, since the table corresponding to the first integration looks like,

$(x+1)$	$+$	$\sin n\pi x$
1	$-$	$-\dfrac{\cos n\pi x}{n\pi}$
0	$+$	$-\dfrac{\sin n\pi x}{n^2\pi^2}$
$\ldots$		$\ldots$

and this means that the corresponding definite integral is given by

$$\begin{aligned} \int_{-1}^{0} (x+1)\sin n\pi x\ dx &= \left\{ -\frac{(x+1)\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2} \right\}\Bigg|_{-1}^{0}, \\ &= -\frac{1}{n\pi} - 0, \\ &= -\frac{1}{n\pi}, \end{aligned}$$

while the table corresponding to the second integration looks like,

$(x-1)$	$+$	$\sin n\pi x$
1	$-$	$-\dfrac{\cos n\pi x}{n\pi}$
0	$+$	$-\dfrac{\sin n\pi x}{n^2\pi^2}$
$\ldots$		$\ldots$

and this means that the corresponding definite integral is given by

$$\begin{aligned} \int_{0}^{1} (x-1)\sin n\pi x\ dx &= \left\{ -\frac{(x-1)\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2} \right\}\Bigg|_{0}^{1}, \\ &= 0 - \frac{1}{n\pi}, \\ &= -\frac{1}{n\pi}. \end{aligned}$$

It follows that $b_n = -\dfrac{2}{n\pi}$, for $n = 1, 2, 3, \ldots$.

7.5.3 Products of Secants and Tangents

In the previous section we looked at the problem of integrating products of sine and cosine functions and their powers. In this section we provide some additional insight into the integration of products and powers of secant and tangent functions. So, we'll be looking at how to evaluate integrals of the form

$$\int \sec x \tan^2 x \; dx, \quad \int \sec^3 x \; dx, \quad \int \sec^2 x \tan^2 x \; dx.$$

Cases involving the functions Cosecant and Cotangent are handled similarly. In almost all the cases under consideration we will be appealing to the fundamental trigonometric identities of the previous section that is, (7.59) and (7.60). We recall the fundamental identity (7.59) here:

$$\sec^2 u - \tan^2 u = 1,$$

valid for any real number u.

Example 350. Evaluate $I = \int \sec x \; dx$.

Solution This one requires a very clever use of an identity.

$$\begin{aligned}
\int \sec x \; dx &= \int \frac{\sec x \; (\sec x + \tan x)}{\sec x + \tan x} \; dx \quad \text{(this is the idea!)}, \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx \\
&= \int \frac{dv}{v} \quad \text{(using the substitution } v = \tan x + \sec x\text{)}, \\
&= \ln|v| + C \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

Example 351. Evaluate $I = \int \sec^3 x \; dx$.

Solution We rewrite the integrand as $\sec^3 x = \sec^2 x \sec x$ and then use (7.59) in the form $\sec^2 x = \tan^2 x + 1$. This gives us,

$$\begin{aligned}
\int \sec^3 x \; dx &= \int \sec^2 x \sec x \; dx \\
&= \int (\tan^2 x + 1) \sec x \; dx \\
&= \int \tan^2 x \sec x \; dx + \int \sec x \; dx \\
&= \int \tan^2 x \sec x \; dx + \ln|\sec x + \tan x|, \quad \text{(by Example 350)}.
\end{aligned}$$

In order to evaluate the first integral on the right, above, we use Integration by Parts. Thus, we write $u = \tan x, \;\; du = \sec^2 x \; dx$ and $dv = \sec x \tan x \; dx, \;\; v = \sec x$. Since

$$\int u \; dv = uv - \int v \; du,$$

we get

$$\begin{aligned}
\int \tan^2 x \;\; \sec x \; dx &= \int \tan x \cdot \sec x \tan x \; dx \\
&= \tan x \sec x - \int \sec^3 x \; dx.
\end{aligned}$$

Combining these results with the previous display we find,

$$\begin{aligned}
\int \sec^3 x\ dx &= \int \tan^2 x \sec x\ dx + \ln|\sec x + \tan x|, \\
&= \tan x \sec x - \int \sec^3 x\ dx + \ln|\sec x + \tan x|, \\
2\int \sec^3 x\ dx &= \tan x \sec x + \ln|\sec x + \tan x|, \\
\int \sec^3 x\ dx &= \frac{1}{2}(\tan x \sec x + \ln|\sec x + \tan x|) + C.
\end{aligned}$$

Example 352. Evaluate $I = \int \sec x \tan^2 x\ dx$.

Solution This is a by-product of Example 351. In the first part of the proof we showed that

$$\begin{aligned}
\int \tan^2 x \sec x\ dx &= \int \sec^3 x\ dx - \ln|\sec x + \tan x| \\
&= \frac{\tan x \sec x + \ln|\sec x + \tan x|}{2} - \ln|\sec x + \tan x| + C, \\
&= \frac{\tan x \sec x}{2} - \frac{\ln|\sec x + \tan x|}{2} + C.
\end{aligned}$$

Example 353. Evaluate $I = \int \sec^4 x\ dx$.

Solution We rewrite the integrand as $\sec^4 x = \sec^2 x \sec^2 x$, and we solve for $\sec^2 x$ in 7.58. Now,

$$\begin{aligned}
\int \sec^4 x\ dx &= \int \sec^2 x\ \sec^2 x\ dx \\
&= \int \sec^2 x\left(1 + \tan^2 x\right)\ dx \\
&= \int \sec^2 x\ dx + \int \sec^2 x \tan^2 x\ dx \\
&\qquad \text{(and let } u = \tan x,\ du = \sec^2 x\ dx \text{ in the second integral),} \\
&= \tan x + \int u^2\ du \\
&= \tan x + \frac{u^3}{3} + C \\
&= \tan x + \frac{\tan^3 x}{3} + C.
\end{aligned}$$

NOTES:

Example 354. Evaluate $I = \int \sec^2 x \tan^3 x \; dx$.

Solution We note that $D(\tan x) = \sec^2 x$ and so the substitution $u = \tan x$, $du = \sec^2 x \; dx$ reduces the integral to

$$\begin{aligned} \int \tan^3 x \; \sec^2 x \; dx &= \int u^3 \; du \\ &= \frac{u^4}{4} + C \\ &= \frac{\tan^4 x}{4} + C. \end{aligned}$$

Example 355. For k an EVEN integer, $k \geq 2$, evaluate $I = \int \sec^k x \; dx$, and show that it can be written as

$$\int \sec^k x \; dx = \int \left(1+u^2\right)^{\frac{k}{2}-1} \; du,$$

where $u = \tan x$.

Solution Since k is even we can write it in the form $k = 2p$ where p is some integer. So, replace k by $2p$. We factor out ONE term of the form $\sec^2 x$. Next, we use the fact that $\sec^2 x = 1 + \tan^2 x$ in order to remove all the other terms of the form $\sec^2 x$ from the integrand. This leaves us with only terms of the form $\tan^2 x$ and only one term of the form $\sec^2 x$. Thus,

$$\begin{aligned} \int \sec^{2p} x \; dx &= \int \sec^{2p-2} x \; \sec^2 x \; dx \\ &= \int \left(\sec^2 x\right)^{p-1} \; \sec^2 x \; dx \\ &= \int \left(1+\tan^2 x\right)^{p-1} \; \sec^2 x \; dx \text{ (we set } u = \tan x, \; du = \sec^2 x \; dx), \\ &= \int \left(1+u^2\right)^{p-1} \; du, \end{aligned}$$

and this last integral is just a polynomial in u of degree $2p - 2$. As a result, it can be integrated easily using term-by-term integration.

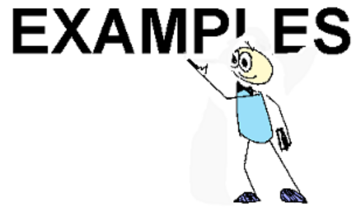
NOTE: If we set $p = 2$ so that $k = 4$, then

$$\begin{aligned} \int \sec^4 x \; dx &= \int \left(1+u^2\right)^{2-1} \; du, \\ &= \int (1+u^2) \; du, \\ &= u + \frac{u^3}{3} + C, \\ &= \tan x + \frac{\tan^3 x}{3} + C, \end{aligned}$$

which agrees with the result we obtained in Example 352, above.

Example 356. For k an ODD integer, $k \geq 3$, evaluate $I = \int \sec^k x\ dx$. Show that

$$\int \sec^k x\ dx = \frac{\sec^{k-2} x\ \tan x}{k-1} + \frac{k-2}{k-1} \int \sec^{k-2} x\ dx, \tag{7.66}$$



Solution We have already covered the cases $k = 1$ and $k = 3$ in Examples 350, and 351. Since k is an odd number we can write it in the form $k = 2p + 1$. This is because every odd number can be written as 1 plus an even number. Let's assume this is done, so $k = 2p + 1$. Now we write the integral as

$$\begin{aligned} \int \sec^k x\ dx &= \int \sec^{2p+1} x\ dx \\ &= \int \sec^{2p-1} x\ \sec^2 x\ dx. \end{aligned}$$

Now we use Integration by Parts on the preceding integral with the following substitutions: Let

$$u = \sec^{2p-1} x, \quad du = (2p-1) \sec^{2p-2} x \cdot (\sec x \tan x)\ dx,$$

and

$$dv = \sec^2 x\ dx, \ \ v = \tan x.$$

Then

$$\begin{aligned} I &= \int \sec^k x\ dx \\ &= \int \sec^{2p+1} x\ dx \\ &= \int \sec^{2p-1} x\ \sec^2 x\ dx \\ &= \sec^{2p-1} x\ \tan x - \int (\tan x) \cdot (2p-1) \cdot \sec^{2p-2} x \cdot (\sec x \tan x)\ dx \\ &= \sec^{2p-1} x\ \tan x - (2p-1) \int \tan x \cdot \sec^{2p-2} x \cdot (\sec x \tan x)\ dx \\ &= \sec^{2p-1} x\ \tan x - (2p-1) \int \sec^{2p-1} x\ \tan^2 x\ dx \\ &= \sec^{2p-1} x\ \tan x - (2p-1) \int \sec^{2p-1} x\ \left(\sec^2 x - 1\right)\ dx \\ &= \sec^{2p-1} x\ \tan x - (2p-1) \int \sec^{2p+1} x\ dx \\ & \quad +(2p-1) \int \sec^{2p-1} x\ dx. \\ &= \sec^{2p-1} x\ \tan x - (2p-1)\ I + (2p-1) \int \sec^{2p-1} x\ dx. \end{aligned}$$

Collecting the terms involving I in the last integral we see that

$$I\,(1 + (2p-1)) = \sec^{2p-1} x\ \tan x + (2p-1) \int \sec^{2p-1} x\ dx,$$

or

$$\int \sec^{2p+1} x\ dx = \frac{\sec^{2p-1} x\ \tan x}{2p} + \frac{2p-1}{2p} \int \sec^{2p-1} x\ dx, \tag{7.67}$$

which is of the SAME FORM as the original one in (7.66) but with $k = 2p+1$. This last integral is the same as (7.66) once we replace $2p$ by $k-1$ as required.

We present the details for evaluating integrals of the form

$$\int \sec^m x \ \tan^n x \ dx \tag{7.68}$$

where m, n are positive integers.

- ***m is even, (n is even or odd)***

Assume $m > 2$, otherwise the integral is easy. Solve (7.59) for $\sec^2 x$. In the integrand, substitute every term of the form $\sec^2 x$ by $1+\tan^2 x$. Since m is even and $m \geq 2$ by hypothesis, there is always one extra term of the form $\sec^2 x$ which we can factor out. Collecting terms we see that we have an integrand involving only tangent functions and their powers and only ONE $\sec^2 x-$term. Follow this with a simple substitution of variable, namely, $u = \tan x,\ du = \sec^2 x \ dx$, which now reduces the integral to an integral with a polynomial in u and this is easily integrated.

- ***m is odd, n is odd***

Factor out one term of the form $\sec x \ \tan x$ out of the integrand. This leaves only even powers of each of $\tan x$ and $\sec x$. We solve (7.59) for $\tan^2 x$. In the integrand, substitute every term of the form $\tan^2 x$ by $\sec^2 x - 1$. Now the integrand has been rewritten as a product of secant functions along with an additional factor of $\sec x \ \tan x$. Next, we follow this with a simple substitution of variable, namely, $u = \sec x,\ du = \sec x \ \tan x \ dx$, which now reduces the integral to an integral with a polynomial in u and this is easily integrated.

- ***m is odd, n is even***

This is by far the most tedious, but not impossible, case. Since n is even every term of the form $\tan^2 x$ may be replaced by an equivalent term of the form $\sec^2 x - 1$. The integral now takes the form

$$\int \sec^m x \ \tan^n x \ dx = \int \sec^m x \ \left(\sec^2 x - 1\right)^{\frac{n}{2}} \ dx,$$

where $n/2$ is an integer (since n is even). The last integral, when expanded completely, gives a sum of integrals of the form

$$\int sec^k x \ dx,$$

where k is an integer. Depending on whether k is even or odd, as the case may be, we can apply the results of Examples 355, 356 respectively, in order to evaluate it.

Table 7.8: Powers and Products of Secant and Tangent Integrals

NOTE: If we set $k = 2p + 1 = 3$, so that $p = 1$, then equation 7.66 becomes

$$\begin{aligned}\int \sec^3 x\, dx &= \frac{\sec x\ \tan x}{2} + \frac{1}{2}\int \sec x\, dx, \text{ (and by Example 350)},\\ &= \frac{\sec x\ \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C,\end{aligned}$$

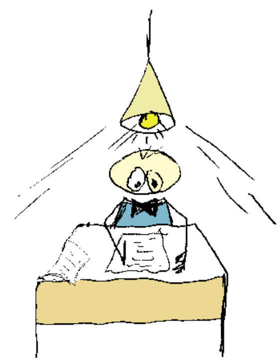
which agrees with the result we obtained in Example 351, below.

The general rules for evaluating such integrals can now be summarized in Table 7.8, above.

Example 357. Evaluate $\displaystyle\int \sec^2 x \tan^4 x\, dx$.

Solution We use Table 7.8. In this example, $m = 2$, and $n = 4$ so m is even and n is even. We *factor* out the only term of the form $\sec^2 x$, and then substitute $u = \tan x$ in the remaining expression consisting only of "tangents". We see that,

$$\begin{aligned}\int \sec^2 x \tan^4 x\, dx &= \int u^4\, du\\ &= \frac{u^5}{5} + C\\ &= \frac{\tan^5 x}{5} + C.\end{aligned}$$



Example 358. Evaluate $\displaystyle\int \sec^3 x \tan^5 x\, dx$.

Solution Now $m = 3, n = 5$ so both m, n are odd. In this case we factor a term of the form $\sec x \tan x$ out of the integrand. Then we must replace every term of the form $\tan^2 x$ by $\sec^2 x - 1$. Finally, we substitute $u = \sec x$ in the remaining integral. So,

$$\begin{aligned}\int \sec^3 x \tan^5 x\, dx &= \int \sec^2 x \tan^4 x \cdot (\sec x \tan x)\, dx\\ &= \int \sec^2 x \cdot \tan^2 x \cdot \tan^2 x \cdot (\sec x \tan x)\, dx\\ &= \int \sec^2 x \left(\sec^2 x - 1\right)^2 \cdot (\sec x \tan x)\, dx \quad \text{(and let } u = \sec x\text{)},\\ &= \int u^2 \left(u^2 - 1\right)^2\, du\\ &= \int \left(u^6 - 2u^4 + u^2\right)\, du\\ &= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C\\ &= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C.\end{aligned}$$

Example 359. Evaluate $I = \displaystyle\int \sec^5 x\, dx$.

Solution We use Example 356 with $k = 5$. We know that

$$\int \sec^k x\, dx = \frac{\sec^{k-2} x\ \tan x}{k-1} + \frac{k-2}{k-1}\int \sec^{k-2} x\, dx,$$

Fundamental Results

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C,$$

and

$$\int \csc\theta \, d\theta = \ln|\csc\theta - \cot\theta| + C.$$

Table 7.9: The Antiderivative of the Secant and Cosecant Functions

in general so, in our case,

$$\int \sec^5 x \, dx = \frac{\sec^3 x \, \tan x}{4} + \frac{3}{4}\int \sec^3 x \, dx.$$

But we know from Example 351 that

$$\int \sec^3 x \, dx = \frac{1}{2}\left(\tan x \sec x + \ln|\sec x + \tan x|\right) + C.$$

This means that

$$\begin{aligned}\int \sec^5 x \, dx &= \frac{\sec^3 x \, \tan x}{4} + \frac{3}{4}\int \sec^3 x \, dx, \\ &= \frac{\sec^3 x \, \tan x}{4} + \frac{3}{8}\left(\tan x \sec x + \ln|\sec x + \tan x|\right) + C.\end{aligned}$$

NOTES:

Exercise Set 36.

Evaluate the following integrals using any method.

1. $\int \tan x \ dx$
2. $\int \sec^2(3x+1) \ dx$
3. $\int \sec x \tan x \ dx$
4. $\int \sec^2 x \tan x \ dx$
5. $\int \sec^2 x \tan^2 x \ dx$
6. $\int \sec^2 x \tan^5 x \ dx$
7. $\int \sec x \tan^3 x \ dx$
8. $\int \sec^4 x \tan^4 x \ dx$
9. $\int \sec^3 x \tan^3 x \ dx$
10. $\int \sec^3(2x) \tan^5(2x) \ dx$
11. $\int \sec^2 2x \tan^5 2x \ dx$
12. $\int \tan^3 x \ dx$
13. $\int \sec^7 x \ dx$
14. $\int \sec x \tan^2 x \ dx$
15. $\int \sec^3 x \tan^2 x \ dx$

Suggested Homework Set 28. *Do problems 4, 5, 7, 11, 13*

NOTES:

7.6 Trigonometric Substitutions

7.6.1 Completing the Square in a Quadratic (Review)

In this section we review briefly the classical method of **completing the square** in a quadratic polynomial. This idea is very useful in evaluating the integral of rational functions whose denominators are quadratics and where, at first sight, the Substitution Rule fails to give any simplif24 2015ication! For example, using this method we can show that

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1}.$$

Now, the form of the second integral makes us think of the substitution $u = x+1$ which converts that integral to the form

$$\int \frac{dx}{(x+1)^2+1} = \int \frac{du}{u^2+1} = \text{Arctan}\, u + C,$$

and back-substitution gives us the final answer,

$$\int \frac{dx}{(x+1)^2+1} = \text{Arctan}\,(x+1) + C.$$

The method in this section is particularly useful when we need to integrate the reciprocal of a quadratic irreducible polynomial (or Type II factor, cf., Chapter 5), that is a polynomial of the form ax^2+bx+c where $b^2-4ac<0$ (when $b^2-4ac \geq 0$, there is no problem since the polynomial has only Type I or linear factors, so one can use the method of Partial Fractions).

Integrands involving quadratics, ax^2+bx+c

If you see the expression ax^2+bx+c or even $\sqrt{ax^2+bx+c}$ in an integral, proceed as follows: Write

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right), \\ &= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right], \end{aligned}$$

so that the quadratic may be expressed in the form of a **"square"** (the $x+b/(2a)$ part) **plus some extra stuff** (the remaining expression $(c/a)-(b^2/(4a^2))$), or

$$ax^2+bx+c = a\left[\underbrace{\left(x+\frac{b}{2a}\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)}_{\textbf{“completing the square”}}\right].$$

You have just "completed the square". Next, you use the substitution

$$u = x+\frac{b}{2a}$$

to get the right-hand side into the form $a(u^2 \pm A^2)$, where the sign of A^2 is positive or negative depending on whether $(4ac-b^2)/a^2$ is positive or negative, respectively.

If a, b, c are any three numbers with $a \neq 0$, then

$$ax^2 + bx + c = a\left[\underbrace{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)}_{\text{“Completing the square”}}\right].$$

Table 7.10: Completing the Square in a Quadratic Polynomial

This will simplify your integral to the point where you can write down the answer almost immediately! We summarize this procedure in Table 7.10.

Example 360. Rewrite the following polynomials by “completing the square”.

1. $x^2 - 2x + 2.$
2. $x^2 + 2x + 5.$
3. $x^2 - 4x + 3.$
4. $x^2 - x + 1.$
5. $2x^2 - 4x + 4.$

Solution **1)** Use Table 7.10 with $a = 1$, $b = -2$ and $c = 2$. Then,

$$\begin{aligned} x^2 - 2x + 2 &= \left(x - \frac{2}{2 \cdot 1}\right)^2 + \left(\frac{4 \cdot 2 - 4}{4 \cdot 1}\right), \\ &= (x-1)^2 + 1. \end{aligned}$$

2) We use Table 7.10 with $a = 1$, $b = 2$ and $c = 5$. Then,

$$\begin{aligned} x^2 + 2x + 5 &= \left(x + \frac{2}{2 \cdot 1}\right)^2 + \left(\frac{4 \cdot 5 - 4}{4 \cdot 1}\right), \\ &= (x+1)^2 + 4. \end{aligned}$$

3) Use Table 7.10 with $a = 1$, $b = -4$ and $c = 3$. Then,

$$\begin{aligned} x^2 - 4x + 3 &= \left(x - \frac{4}{2 \cdot 1}\right)^2 + \left(\frac{4 \cdot 3 - 16}{4 \cdot 1}\right), \\ &= (x-2)^2 - 1. \end{aligned}$$

4) We use Table 7.10 with $a = 1$, $b = -1$ and $c = 1$. Then,

$$\begin{aligned} x^2 - x + 1 &= \left(x - \frac{1}{2}\right)^2 + \left(\frac{4 - 1}{4}\right), \\ &= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}. \end{aligned}$$

5) Use Table 7.10 with $a = 2$, $b = -4$ and $c = 4$. Then,

$$\begin{aligned} 2x^2 - 4x + 4 &= 2 \cdot \left(x - \frac{4}{4}\right)^2 + \left(\frac{32 - 16}{16}\right), \\ &= 2 \cdot \left((x-1)^2 + 1\right), \\ &= 2(x-1)^2 + 2. \end{aligned}$$

Example 361. Evaluate $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$.

Solution If we complete the square in the denominator of this expression we find

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx.$$

Now, use the substitution $u = x+1$, $du = dx$. Then, by Table 6.7 with $\Box = u$,

$$\begin{aligned}\int \frac{1}{(x+1)^2+1}\,dx &= \int \frac{1}{u^2+1}\,du,\\ &= \text{Arctan}\,u + C,\\ &= \text{Arctan}\,(x+1)+C.\end{aligned}$$

Example 362. Evaluate $\displaystyle\int_0^1 \frac{1}{4x^2-4x+2}\,dx$.

Solution Completing the square in the denominator gives us

$$\int_0^1 \frac{1}{4x^2-4x+2}\,dx = \int_0^1 \frac{1}{4(x-1/2)^2+1}\,dx.$$

Next, use the substitution $u = 2x-1$, $du = 2dx$. Furthermore, when $x=0$, $u=-1$ while when $x=1$, $u=1$. Finally, $dx = du/2$, and by Table 6.7 with $\Box = u$,

$$\begin{aligned}\int_0^1 \frac{1}{(2x-1)^2+1}\,dx &= \frac{1}{2}\int_{-1}^1 \frac{1}{u^2+1}\,du,\\ &= \frac{1}{2}\text{Arctan}\,u\Big|_{-1}^{1},\\ &= \frac{1}{2}\left(\text{Arctan}\,1-\text{Arctan}\,(-1)\right),\\ &= \frac{1}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right),\\ &= \frac{\pi}{4}.\end{aligned}$$

Example 363. Evaluate $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$.

Solution If we complete the square inside the square root we find that $2x-x^2 = -(x^2-2x) = -((x-1)^2-1) = 1-(x-1)^2$. So,

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx.$$

This time we use the substitution $u = x-1$, $du = dx$. Then, by the first of Table 6.7 with $\Box = u$,

$$\begin{aligned}\int \frac{1}{\sqrt{1-(x-1)^2}}\,dx &= \int \frac{1}{\sqrt{1-u^2}}\,du,\\ &= \text{Arcsin}\,u + C,\\ &= \text{Arcsin}\,(x-1)+C.\end{aligned}$$

Example 364. Evaluate $\displaystyle\int \frac{1}{x^2-x+1}\,dx$.

Solution Let's use the result from Example 360, (4). When we complete the square in the denominator of this expression we find

$$\int \frac{1}{x^2-x+1}\,dx = \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx.$$

We let $u = x - 1/2$, $du = dx$. Then,

$$\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \int \frac{1}{u^2+\frac{3}{4}}\,du,$$

and we think about Table 6.7, that is, we think about the part of this Table which deals with the Arctan function. But instead of $3/4$ we need a 1. So, we factor out that number and see that,

$$\begin{aligned} \int \frac{1}{u^2+\frac{3}{4}}\,du &= \int \frac{1}{\frac{3}{4}\left(\frac{4u^2}{3}+1\right)}\,du, \\ &= \frac{4}{3}\int \frac{1}{\frac{4u^2}{3}+1}\,du, \\ &= \frac{4}{3}\int \frac{1}{\left(\frac{2u}{\sqrt{3}}\right)^2+1}\,du. \end{aligned}$$

OK, now we use another subtitution! Why? Because although we have the 1 in the right place we still can't integrate it directly using Table 6.7. So we set

$$v = \frac{2u}{\sqrt{3}},\ dv = \frac{2du}{\sqrt{3}} \ \Rightarrow du = \frac{\sqrt{3}dv}{2}.$$

Then

$$\begin{aligned} \frac{4}{3}\int \frac{1}{\left(\frac{2u}{\sqrt{3}}\right)^2+1}\,du &= \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{1}{v^2+1}\,dv, \\ &= \frac{2}{\sqrt{3}}\text{Arctan}\, v + C, \\ &= \frac{2}{\sqrt{3}}\text{Arctan}\left(\frac{2u}{\sqrt{3}}\right) + C, \\ &= \frac{2}{\sqrt{3}}\text{Arctan}\left(\frac{2(x-\frac{1}{2})}{\sqrt{3}}\right) + C, \\ &= \frac{2}{\sqrt{3}}\text{Arctan}\left(\frac{2x-1}{\sqrt{3}}\right) + C. \end{aligned}$$

Example 365. Evaluate $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx$, provided $x > 2$.

Solution Once again we complete the square inside the square root and we find that $x^2 - 2x = (x-1)^2 - 1$. So,

$$\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-1}}\,dx,$$

The substitution $u = x - 1$, $du = dx$, works once again. Then, by the Arcsecant formula in Table 6.7 with $\square = u$, we find

$$\begin{aligned} \int \frac{1}{(x-1)\sqrt{(x-1)^2-1}}\,dx &= \int \frac{1}{u\sqrt{u^2-1}}\,dx \\ &= \text{Arcsec}\, u + C, \quad (\text{since } u > 0 \Longrightarrow |u| = u), \\ &= \text{Arcsec}\,(x-1) + C. \end{aligned}$$

NOTES:

Exercise Set 37.

Evaluate the following integrals by "completing the square", (if need be).

1. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx.$
2. $\displaystyle\int \frac{2}{x^2-2x+2}\,dx.$
3. $\displaystyle\int \frac{1}{x^2-2x+5}\,dx.$
4. $\displaystyle\int \frac{1}{x^2-4x+3}\,dx.$
5. $\displaystyle\int \frac{4}{4x^2+4x+5}\,dx.$
6. $\displaystyle\int \frac{1}{4x-x^2-3}\,dx.$
7. $\displaystyle\int \frac{1}{\sqrt{4x-x^2}}\,dx.$

   • Complete the square. Factor out 4 from the square root, and then set $u = \dfrac{x-2}{2}$.
8. $\displaystyle\int_{-1}^0 \frac{1}{4x^2+4x+2}\,dx.$
9. $\displaystyle\int \frac{1}{\sqrt{2x-x^2+1}}\,dx.$
10. $\displaystyle\int \frac{1}{x^2+x+1}\,dx.$
11. $\displaystyle\int \frac{1}{x^2+x-1}\,dx.$
12. $\displaystyle\int \frac{1}{(2x+1)\sqrt{4x^2+4x}}\,dx$, for $x > -\frac{1}{2}$.

Suggested Homework Set 29. *Do problems 2, 4, 7, 9, 11*

7.6.2 Trigonometric Substitutions

A **trigonometric substitution** is particularly useful when the integrand has a particular form, namely, if it is, or it can be turned into, the sum or a difference of two squares, one of which is a constant. For example, we will see how to evaluate integrals of the form

$$\int \sqrt{x^2-4}\,dx, \quad \int_0^3 \sqrt{9+x^2}\,dx, \quad \int \frac{1}{\sqrt{3x^2-2x+1}}\,dx,$$

where the last one requires the additional use of the method of *completing the square.* In general, we'll be dealing with the integration of functions containing terms like

$$\sqrt{x^2-a^2}, \quad \sqrt{x^2+a^2}, \ or, \quad \sqrt{a^2-x^2}.$$

In this section many of the techniques that you've learned and used in the preceding sections come together in the evaluation of integrals involving square roots of quadratic functions. For instance, the function $3x^2-2x+1$ can be written as a sum or a difference of two squares once we use the method of *completing the square*, (see section 7.6.1).

Example 366. Write the function underneath the square root sign, $\sqrt{3x^2-2x+1}$, as a sum or a difference of *two squares.*

Solution To see this we note that, according to Table 7.10 in Section 7.6.1,

$$3x^2-2x+1 = 3\left[\left(x-\frac{1}{3}\right)^2+\frac{2}{9}\right],$$

so that

$$\begin{aligned}\sqrt{3x^2-2x+1} &= \sqrt{3}\sqrt{\left[\left(x-\frac{1}{3}\right)^2+\frac{2}{9}\right]},\\ &= \sqrt{3}\sqrt{u^2+a^2},\end{aligned}$$

if we use the substitutions

$$u = x-\frac{1}{3} \ \ \textit{and} \ \ a=\frac{\sqrt{2}}{3}.$$

The factor, $\sqrt{3}$, is not a problem as it can be factored out of the integral.

Example 367. Write the expression $\sqrt{x^2-2x-2}$ as the square root of a sum or a difference of *two squares.*

Solution We use Table 7.10 once again. Now,

$$\begin{aligned}\sqrt{x^2-2x-3} &= \sqrt{(x-1)^2-4},\\ &= \sqrt{u^2-a^2},\end{aligned}$$

provided we choose $u=x-1,\ a=\sqrt{4}=2$.

Example 368. Write the expression $\sqrt{2x-x^2}$ as the square root of a sum or a difference of *two squares.*

Solution We use Table 7.10 once again. Now,

$$\begin{aligned}\sqrt{2x-x^2} &= \sqrt{-(x^2-2x)},\\ &= \sqrt{1-(x^2-2x+1)},\\ &= \sqrt{1-(x-1)^2},\\ &= \sqrt{a^2-u^2},\end{aligned}$$

provided we choose $u=x-1,\ a=1$.

In the same way we can see that the method of *completing the square* can be used generally to write the quadratic ax^2+bx+c with $a\neq 0$, as a sum or a difference of two squares. Once we have written the general quadratic in this special form, we can use the substitutions in Table 7.11 in order to evaluate integrals involving these sums or differences of squares. Let's not forget that

Let $a \neq 0$ be a constant. If the integrand contains a term of the form:

- $\boldsymbol{\sqrt{a^2 - u^2}}$**,** substitute

$$u = a\sin\theta, \quad du = a\cos\theta\ d\theta, \quad \sqrt{a^2 - u^2} = a\cos\theta,$$

if $-\pi/2 < \theta < \pi/2$.

- $\boldsymbol{\sqrt{a^2 + u^2}}$**,** set

$$u = a\tan\theta, \quad du = a\sec^2\theta\ d\theta, \quad \sqrt{a^2 + u^2} = a\sec\theta,$$

if $-\pi/2 < \theta < \pi/2$.

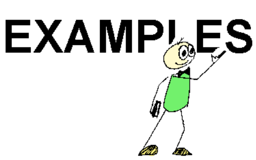
- $\boldsymbol{\sqrt{u^2 - a^2}}$**,** set

$$u = a\sec\theta, \quad du = a\sec\theta\tan\theta\ d\theta, \quad \sqrt{u^2 - a^2} = a\tan\theta,$$

if $0 < \theta < \pi/2$.

Table 7.11: Trigonometric Substitutions

$$ax^2 + bx + c = a\underbrace{\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)\right]}_{\textbf{“completing the square”}}.$$



Example 369. Evaluate $\displaystyle\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1 - 4x^2}}$.

Solution Note that we can evaluate this integral using Table 6.7 with $\square = 2x$ and obtain the value $\frac{\text{Arcsin}\,2x}{2}$. But let's see how a trigonometric substitution works. The integrand has a term of the form $\sqrt{a^2 - u^2}$ where $a = 1, u = 2x$. So, in accordance with Table 7.11, we set

$$2x = \sin\theta, \ 2dx = \cos\theta\ d\theta, \ \text{so that } \sqrt{1 - 4x^2} = \cos\theta.$$

It follows that

$$\theta = \text{Arcsin}\,2x, \ \text{and } dx = \frac{\cos\theta\ d\theta}{2}.$$

When $x = 0$, $\theta = \text{Arcsin}\,(2 \cdot 0) = \text{Arcsin}\,0 = 0$, while when $x = \frac{1}{4}$, $\theta = \text{Arcsin}\,\left(2 \cdot \frac{1}{4}\right) = \text{Arcsin}\,\frac{1}{2} = \frac{\pi}{6}$. The integral now becomes

$$\begin{aligned} \int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1 - 4x^2}} &= \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{\cos\theta}{\cos\theta} d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{6}} d\theta \\ &= \frac{\pi}{12}. \end{aligned}$$

Useful Integrals

A simple application of the two identities in (7.57) shows that

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C, \tag{7.69}$$

and

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C. \tag{7.70}$$

Table 7.12: Integrating the Square of the Cosine or Sine Function

Example 370. Evaluate $\displaystyle\int \frac{1}{x\sqrt{x^2-16}}\, dx$.

Solution This integrand has a term of the form $\sqrt{u^2-a^2}$ with $u = x, \ a = 4$. So, we let $x = 4\sec\theta$, which gives $\sqrt{x^2-16} = 4\tan\theta$, and $dx = 4\sec\theta \ \tan\theta \ d\theta$. So,

$$\begin{aligned} I &= \int \frac{4 \ \sec\theta \ \tan\theta \ d\theta}{4\sec\theta \cdot 4\tan\theta} \\ &= \frac{1}{4}\theta + C \\ &= \frac{1}{4}\text{Arcsec}\left(\frac{x}{4}\right) + C \end{aligned}$$

Example 371. Evaluate $\displaystyle\int \sqrt{2x-x^2}\, dx$.

Solution In this example, it isn't clear that this integrand is a square root of a *difference or sum of two squares*. But the method of *completing the square* can be used to show this (see Section 7.6.1). Remember that every quadratic function can be written as a sum or a difference of squares. We have already discussed this function in Example 368, above. We found by completing the square that

$$2x - x^2 = 1 - (x-1)^2,$$

and so,

$$\sqrt{2x-x^2} = \sqrt{1-(x-1)^2}.$$

Now $\sqrt{1-(x-1)^2}$ is of the form $\sqrt{a^2-u^2}$ provided we choose $a = 1, \ u = x-1$. This is our cue for the substitution! In accordance with Table 7.11 we use the substitution $x - 1 = \sin\theta$ so that $dx = \cos\theta \ d\theta$ and $\theta = \text{Arcsin}\,(x-1)$. In this case,

$$\sqrt{1-(x-1)^2} = \cos\theta,$$

and

$$\begin{aligned}
\int \sqrt{2x-x^2}\ dx &= \int \sqrt{1-(x-1)^2}\ dx \\
&= \int (\cos\theta)\ \cos\theta\ d\theta \\
&= \int \cos^2\theta\ d\theta \quad \text{(and by (7.69) in Table 7.12),} \\
&= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C, \\
&= \frac{\theta}{2} + \frac{(2\sin\theta\ \cos\theta)}{4} + C, \quad \text{(see above for } \cos\theta\text{),} \\
&= \frac{\theta}{2} + \frac{(x-1)\ \sqrt{1-(x-1)^2}}{2} + C, \\
&= \frac{1}{2}\text{Arcsin}\,(x-1) + \frac{(x-1)\ \sqrt{1-(x-1)^2}}{2} + C.
\end{aligned}$$

Example 372. Evaluate $I = \displaystyle\int \frac{1}{\sqrt{4x^2+4x+17}}\ dx.$

Solution. By completing the square we see that we can write

$$4x^2+4x+17 = (2x+1)^2 + 4^2 = u^2 + a^2,$$

where $u = 2x+1$ and $a = 4$. Using Table 7.11 we substitute

$$\begin{aligned}
2x+1 &= 4\tan\theta \\
\sqrt{(2x+1)^2+4^2} &= 4\sec\theta \\
dx &= 2\sec^2\theta\ d\theta.
\end{aligned}$$

Now the integral is simplified enough so that

$$\begin{aligned}
\int \frac{1}{\sqrt{4x^2+4x+17}}\ dx &= \int \frac{2\sec^2\theta\ d\theta}{4\sec\theta} = \frac{1}{2}\int \sec\theta\ d\theta \\
&= \frac{1}{2}\ln|\sec\theta+\tan\theta| + C \\
&= \frac{1}{2}\ln\left|\frac{\sqrt{(2x+1)^2+4^2}+2x+1}{4}\right| + C \\
&= \frac{1}{2}\ln\left|\sqrt{4x^2+4x+17}+2x+1\right| + C,
\end{aligned}$$

where the factor of $1/4$ disappears after noting that

$$\frac{1}{2}\ln\left|\frac{\square}{4}\right| = \frac{1}{2}\left(\ln|\square| - \ln 4\right).$$

The constant $-\frac{\ln 4}{2}$ is then absorbed into the (generic) constant of integration, C, where we use the *same symbol* C to denote this constant (don't worry, it's just a convention; we always denote a generic constant by C). So, for example, the constants $C-1, C+\ln 2, C-5^2+3.2\pi, \ldots$ are all denoted by the same symbol, C.

Example 373. Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\ dx.$

Solution This integrand has a term of the form $\sqrt{a^2-u^2}$ where $a=3$, $u=x$. So, an application of Table 7.11 shows that if we use the substitution $x = 3\sin\theta$, then $dx = 3\cos\theta\ d\theta$ and $3\cos\theta = \sqrt{\sqrt{9-x^2}}$, or,

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

So,

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \int \frac{(3\sin\theta)^2\ 3\cos\theta\ d\theta}{3\cos\theta}\\
&= 9\int \sin^2\theta\ d\theta\\
&= \frac{9}{2}\theta - \frac{9\sin 2\theta}{4} + C \quad \text{(by 7.70)},\\
&= \frac{9}{2}\theta - \frac{9\sin\theta\ \cos\theta}{2} + C\\
&= \frac{9}{2}\text{Arcsin}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\left(\frac{x}{3}\right)\cdot\frac{\sqrt{9-x^2}}{3} + C\\
&= \frac{9}{2}\text{Arcsin}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C,
\end{aligned}$$

where we have used the identity $\sin 2\theta = 2\sin\theta\ \cos\theta$ (just as before in Example 371).

Example 374. Evaluate $\int \sqrt{x^2-4}\ dx$.

Solution This integrand has the form $\sqrt{u^2-a^2}$ where $a=2, u=x$. In accordance with Table 7.11, we set

$$x = 2\sec\theta,\ \ dx = 2\sec\theta\tan\theta\ d\theta,\ \text{ so that }\ \sqrt{x^2-4} = 2\tan\theta.$$

It follows that $\theta = \text{Arcsec}\left(\frac{x}{2}\right)$, and

$$\begin{aligned}
\int \sqrt{x^2-4}\ dx &= \int (2\tan\theta)(2\sec\theta\tan\theta)\ d\theta\\
&= 4\int \sec\theta\ (\tan\theta)^2\ d\theta\\
&= 2\tan\theta\ \sec\theta - 2\ln|\sec\theta + \tan\theta| + C,
\end{aligned}$$

by Example 352. Back-substitution tells us that

$$\tan\theta = \frac{\sqrt{x^2-4}}{2},$$

and

$$\sec\theta = \frac{x}{2}.$$

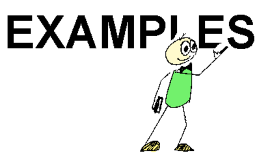
Combining these two relations into the last equality gives

$$\begin{aligned}
\int \sqrt{x^2-4}\ dx &= \frac{x}{2}\sqrt{x^2-4} - 2\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + C,\\
&= \frac{x}{2}\sqrt{x^2-4} - 2\ln\left|x+\sqrt{x^2-4}\right| + 2\ln 2 + C,\\
&= \frac{x}{2}\sqrt{x^2-4} - 2\ln\left|x+\sqrt{x^2-4}\right| + C,
\end{aligned}$$

where the last C is a generic constant denoted by the same symbol.

NOTES:

Example 375. Evaluate $I = \int \frac{dy}{\sqrt{25+9y^2}}$.



Solution The integrand has a term of the form $\sqrt{a^2+u^2}$ where $a = 5$ and $u = 3y$. This is our cue for the substitution. So, using Table 7.11, we set

$$3y = 5\tan\theta, \quad 3dy = 5\sec^2\theta\ d\theta, \quad \sqrt{a^2+u^2} = 5\sec\theta.$$

Then

$$\theta = \text{Arctan}\left(\frac{3y}{5}\right), \quad dy = \frac{5\sec^2\theta\ d\theta}{3},$$

and

$$\begin{aligned}
\int \frac{dy}{\sqrt{25+9y^2}} &= \frac{1}{3}\int \frac{5\sec^2\theta\ d\theta}{5\sec\theta} \\
&= \frac{1}{3}\int \sec\theta\ d\theta \\
&= \frac{1}{3}\ln|\sec\theta+\tan\theta| + C, \text{ (by Example 350)}, \\
&= \frac{1}{3}\ln\left|\frac{\sqrt{25+9y^2}}{5} + \frac{3y}{5}\right| + C,
\end{aligned}$$

and we can simplify this slightly by absorbing the constant $\frac{-\ln 5}{3}$ into the generic constant, C. This gives

$$\int \frac{dy}{\sqrt{25+9y^2}} = \frac{1}{3}\ln\left|\sqrt{25+9y^2}+3y\right| + C.$$

Example 376. Evaluate $\int \sqrt{3x^2-2x+1}\ dx$.

Solution We look up this function in Example 366, above. We found by completing the square that

$$\begin{aligned}
\sqrt{3x^2-2x+1} &= \sqrt{3}\sqrt{\left(x-\frac{1}{3}\right)^2+\frac{2}{9}}, &(7.71)\\
&= \sqrt{3}\sqrt{u^2+a^2}, &(7.72)
\end{aligned}$$

if we choose

$$u = x - \frac{1}{3} \quad \text{and} \quad a = \frac{\sqrt{2}}{3}.$$

So, we have to evaluate an integral of the form

$$\int \sqrt{3x^2-2x+1}\ dx = \sqrt{3}\int \sqrt{u^2+a^2}\ du.$$

Referring to Table 7.11, we let $u = a\tan\theta$, that is,

$$x - \frac{1}{3} = u = \frac{\sqrt{2}}{3}\tan\theta, \ dx = du = \frac{\sqrt{2}}{3}\sec^2\theta\ d\theta.$$

But our choice of substitution always gives $\sqrt{a^2+u^2} = a\sec\theta$. So, in actuality,

$$\begin{aligned}
\sec\theta &= \frac{\sqrt{a^2+u^2}}{a} \\
&\quad \text{from equation (7.72)}, \\
&= \sqrt{\frac{3}{2}}\sqrt{3x^2-2x+1}
\end{aligned}$$

So,

$$
\begin{aligned}
\int \sqrt{3x^2-2x+1}\; dx &= \int \left(\sqrt{\frac{2}{3}}\sec\theta\right)\left(\frac{\sqrt{2}}{3}\sec^2\theta\right)\, d\theta \\
&= \frac{2\sqrt{3}}{9}\int \sec^3\theta\; d\theta \\
&= \frac{2\sqrt{3}}{9}\cdot\frac{1}{2}\left(\tan\theta\sec\theta+\ln|\sec\theta+\tan\theta|\right)+C \\
&\quad \text{(by Example 351)}, \\
&= \frac{\sqrt{3}}{9}\left(\tan\theta\sec\theta+\ln|\sec\theta+\tan\theta|\right)+C,
\end{aligned}
$$

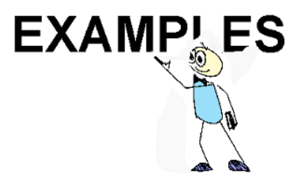
where, according to our definition of θ,

$$\tan\theta = \frac{3\sqrt{2}}{2}\left(x-\frac{1}{3}\right),$$

and

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{9}{2}\left(x-\frac{1}{3}\right)^2}.$$

You are invited to write down the complete formulation of the final answer!



SNAPSHOTS

Example 377. Evaluate $\displaystyle\int \frac{\sqrt{9x^2-1}}{x}\; dx$.

Solution This integrand contains a difference of two squares of the form $\sqrt{u^2-a^2}$ where $u = 3x$ and $a = 1$. So, we set $3x = \sec\theta$. Then $\sqrt{9x^2-1} = \tan\theta$, and $dx = \frac{1}{3}\sec\theta\ \tan\theta\ d\theta$. Hence

$$
\begin{aligned}
\int \frac{\sqrt{9x^2-1}}{x}\; dx &= \int \frac{\tan\theta\left(\frac{1}{3}\sec\theta\ \tan\theta\right)\ d\theta}{\frac{1}{3}\sec\theta} \\
&= \int \tan^2\theta\; d\theta \\
&= \int (\sec^2\theta - 1)\; d\theta \\
&= \tan(\theta) - \theta + C \\
&= \sqrt{9x^2-1} - \text{Arccos}\left(\frac{1}{3x}\right) + C.
\end{aligned}
$$

Example 378. Evaluate $\displaystyle\int_0^5 \sqrt{25-x^2}\; dx$.

Solution Let $x = 5\sin\theta$, $\Rightarrow$ $\sqrt{25-x^2} = 5\cos\theta$, $dx = 5\cos\theta\ d\theta$. When $x = 0$,

$\theta = 0$, while when $x = 5$, $\theta = \frac{\pi}{2}$. So,

$$\begin{aligned}
\int_0^5 \sqrt{25 - x^2}\, dx &= \int_0^{\frac{\pi}{2}} 5\cos\theta \cdot 5\cos\theta \; d\theta \\
&= 25\int_0^{\frac{\pi}{2}} \cos^2\theta \; d\theta \quad \text{(and by 7.69)}, \\
&= 25\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}} \\
&= 25(\frac{\pi}{4} + \frac{1}{4}\sin\pi - 0 - 0) \\
&= \frac{25\pi}{4}.
\end{aligned}$$

Example 379. Evaluate $\displaystyle\int \frac{4x^2\; dx}{(1-x^2)^{3/2}}$.

Solution Let $x = \sin\theta$, $dx = \cos\theta\; d\theta$. Then,

$$\cos\theta = \sqrt{1-x^2}, \quad \tan\theta = \frac{x}{\sqrt{1-x^2}},$$

and

$$\begin{aligned}
\int \frac{4x^2\; dx}{(1-x^2)^{3/2}} &= \int \frac{4\cdot\sin^2\theta\cdot\cos\theta\; d\theta}{\cos^3\theta} \\
&= 4\int \tan^2\theta\; d\theta \\
&= 4\int (\sec^2\theta - 1)\; d\theta \\
&= 4\int \sec^2\theta\; d\theta - 4\theta \\
&= 4\tan\theta - 4\theta + C \\
&= \frac{4x}{\sqrt{1-x^2}} - 4\mathrm{Arcsin}\, x + C
\end{aligned}$$

Exercise Set 38.

Evaluate the following integrals using any method.

1. $\displaystyle\int \sqrt{4-x^2}\; dx.$
2. $\displaystyle\int \sqrt{x^2+9}\; dx.$
3. $\displaystyle\int \sqrt{x^2-1}\; dx.$
4. $\displaystyle\int \sqrt{4x-x^2}\; dx.$
5. $\displaystyle\int \frac{dx}{(4-x^2)^{3/2}}.$
6. $\displaystyle\int \frac{x^2\; dx}{(9-x^2)^{3/2}}.$
7. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}.$

8. $\displaystyle\int \sqrt{4x^2 - 4x + 2}\ dx.$

9. $\displaystyle\int \frac{dx}{(9 + x^2)^2}.$

10. $\displaystyle\int \frac{\sqrt{4 - x^2}}{x}\ dx.$

11. $\displaystyle\int \frac{dx}{(x^2 + 25)^{3/2}}.$

12. $\displaystyle\int \frac{\sqrt{4 - x^2}}{x^2}\ dx.$

13. $\displaystyle\int \frac{dx}{x^4\sqrt{a^2 - x^2}}$, where $a \neq 0$ is a constant.

14. $\displaystyle\int \frac{dx}{x^4\sqrt{x^2 - a^2}}$, where $a \neq 0$ is a constant.

15. $\displaystyle\int \frac{\sqrt{x^2 + 2x - 3}}{x + 1}\ dx.$

16. $\displaystyle\int \frac{2x + 1\ dx}{\sqrt{x^2 + 2x + 5}}.$

Suggested Homework Set 30. *Do problems 2, 4, 5, 7, 8, 12*

NOTES:

7.7 Improper Integrals

The Big Picture

In some cases a definite integral may have one or both of its limits infinite. This occurs, for example, when we don't know how big a time or a frequency must be but we know it's BIG so we replace it by ∞ in the hope that we'll get a *sufficiently good* estimate for the problem at hand. For example, in a field of physics called *quantum mechanics* there is a law called the *Wien Distribution Law* which expresses the energy density as a function of frequency. It looks like this ...

$$Energy = \frac{8\pi h}{c^3} \cdot \int_0^{\infty} \nu^3 \ e^{-\frac{h\nu}{kT}} \ d\nu.$$

Here ν represents the frequency, T the temperature, and all other quantities are physical constants. We know that there is no particle having *infinite* frequency but this Law necessitates that we integrate over *all possible frequencies*, which must be a huge number, but finite nevertheless. We don't know how big it is, so we replace the upper limit by ∞. What else can we do? The answers we get in using this Law are quite accurate indeed. This integral is an example of an **improper integral**.

Such integrals can be grouped into two basic classes:

1. Those with one or both limits being infinite, or
2. Those with both limits being finite but the integrand being infinite somewhere inside the interval (or at the endpoints) of integration.

For example, the integrals

$$\int_0^{\infty} \frac{dx}{x^2+1}, \quad \int_{-\infty}^{\infty} e^{-x^2} \ dx,$$

are each of the first class, while the integrals

$$\int_0^{2} \frac{dx}{x^2-1}, \quad \int_0^{1} \frac{dx}{x}, \quad \int_{-1}^{1} \frac{dx}{x+1},$$

are each of the second class (why?).

Review

This subsection deals with the calculation of certain definite integrals and so you should review all the methods of integration in this Chapter.

The natural definition of these symbols involves interpreting the definite integral as a limit of a definite integral with suitable finite limits. In the evaluation of the resulting limit use may be made of L'Hospital's Rule in conjunction with the various techniques presented in this Chapter.

NOTES:

Integrals of the First Class

Let's assume that f is defined and integrable on every finite interval of the real line and let a be *any* fixed number in the domain of definition of f. We define an **improper integral with infinite limit(s)** as follows:

$$\int_a^\infty f(x)\,dx = \lim_{T\to\infty}\int_a^T f(x)\,dx, \qquad \int_{-\infty}^a f(x)\,dx = \lim_{T\to-\infty}\int_T^a f(x)\,dx,$$

whenever either one of these limits exists and is finite, in which case we say that the **improper integral converges**. In the event that the limit does not exist at all, we say the **improper integral diverges**. If the limit exists but its value is $\pm\infty$, we say that the **improper integral converges to $\pm\infty$**, respectively. A similar definition applies when both limits are infinite, *e.g.,* let a be any real number in the domain of f. Then

Some authors of Calculus texts prefer to include the case of an improper integral **converging to infinity** as one that is **divergent**. Our definitions are in the traditions of Mathematical Analysis.

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

provided both integrals exist and are finite. In this case, we say that the *improper integral converges*. If either integral does not exist (or is infinite), then the *improper integral diverges* (by definition).

Example 380. Determine the values of p for which the improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges or diverges.

Solution This is an improper integral since the upper limit is infinite. There are two cases: $p = 1$ which gives a natural logarithm, and $p \neq 1$ which gives a power function. Now, by definition, if $p \neq 1$,

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{T\to\infty}\int_1^T \frac{1}{x^p}\,dx \\
&= \lim_{T\to\infty}\left(\frac{x^{-p+1}}{1-p}\right)\Bigg|_1^T \\
&= \lim_{T\to\infty}\left(\frac{T^{-p+1}}{1-p} - \frac{1}{1-p}\right).
\end{aligned}$$

Now, if $p > 1$, then $p - 1 > 0$ and so,

$$\lim_{T\to\infty}\left(\frac{T^{-p+1}}{1-p} - \frac{1}{1-p}\right) = \lim_{T\to\infty}\left(\frac{1}{T^{p-1}(1-p)} - \frac{1}{1-p}\right) = \frac{1}{p-1}.$$

On the other hand, if $p < 1$, then $1 - p > 0$ and so,

$$\lim_{T\to\infty}\left(\frac{T^{1-p}}{1-p} - \frac{1}{1-p}\right) = \infty - \frac{1}{1-p} = \infty.$$

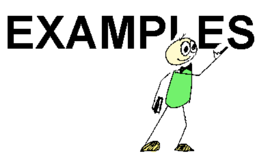
Finally, if $p = 1$, then

$$
\begin{aligned}
\int_1^\infty \frac{1}{x^p}\, dx &= \lim_{T\to\infty} \int_1^T \frac{1}{x}\, dx \\
&= \lim_{T\to\infty} (\ln|x|)\Big|_1^T \\
&= \lim_{T\to\infty} (\ln T - 0) \\
&= \infty.
\end{aligned}
$$

and so the improper integral converges to ∞. Summarizing these results we find

$$
\int_1^\infty \frac{1}{x^p}\, dx = \begin{cases} \infty, & \text{if} \quad p \le 1, \\ \frac{1}{p-1} & \text{if} \quad p > 1. \end{cases}
$$

Example 381. Evaluate $\displaystyle\int_{-\infty}^0 \sin x\ dx$.



Solution We know that

$$
\begin{aligned}
\int_{-\infty}^0 \sin x\ dx &= \lim_{T\to-\infty} \int_T^0 \sin x\ dx \\
&= \lim_{T\to-\infty} (-\cos T + 1) \\
&\quad \text{does not exist at all}
\end{aligned}
$$

due to the periodic oscillating nature of the cosine function on the real line. It follows that this improper integral is divergent.

Example 382. Evaluate the improper integral $\displaystyle\int_0^\infty x\ e^{-x}\ dx$.

Solution This is an improper integral since one of the limits of integration is infinite. So, by definition,

$$
\int_0^\infty x\ e^{-x}\ dx = \lim_{T\to\infty} \int_0^T x\ e^{-x}\ dx,
$$

and this requires Integration by Parts. We set $u = x,\ dv = e^{-x}$, $du = dx,\ v = -e^{-x}$ after which

$$
\begin{aligned}
\lim_{T\to\infty} \int_0^T x\ e^{-x}\ dx &= \left(-x\ e^{-x} - e^{-x}\right)\Big|_0^T \\
&= \lim_{T\to\infty} \Big(-\ (T+1)\ e^{-T}\ \ + 1\Big) \\
&= \lim_{T\to\infty} \left(-\frac{T+1}{e^T} + 1\right) \\
&= 1,
\end{aligned}
$$

by L'Hospital's Rule (Don't worry about the "T" here. Think of it as an "x" when using the Rule). If an antiderivative cannot be found using any method, one resorts to Numerical Integration (Section **??**).

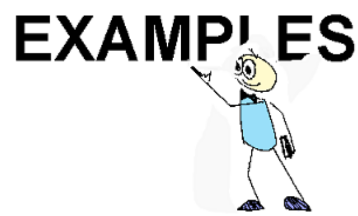
Example 383. Evaluate the improper integral

$$\int_0^\infty x^4\, e^{-x}\, dx \quad = \quad \lim_{T\to\infty} \int_0^T x^4\, e^{-x}\, dx.$$

Solution See Example 304. The Table Method gives

x^4	$+$	e^{-x}
$4x^3$	$-$	$-e^{-x}$
$12x^2$	$+$	e^{-x}
$24x$	$-$	$-e^{-x}$
24	$+$	e^{-x}
0		$-e^{-x}$

and so the most general antiderivative is given by



$$\begin{aligned} I &= -x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x} + C, \\ &= -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + C. \end{aligned}$$

Now, we evaluate the definite integral

$$\begin{aligned} \int_0^T x^4e^{-x}\, dx &= \left\{-e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24)\right\}\Big|_0^T, \\ &= \left\{-e^{-T}(T^4 + 4T^3 + 12T^2 + 24T + 24)\right\} - (-24), \\ &= 24 - \frac{T^4 + 4T^3 + 12T^2 + 24T + 24}{e^T}. \end{aligned}$$

Now, we simply take the limit as $T \to \infty$ and use L'Hospital's Rule 5 times on the expression on the right and we'll see that

$$\begin{aligned} \int_0^\infty x^4e^{-x}\, dx &= 24 - \lim_{T\to\infty} \frac{T^4 + 4T^3 + 12T^2 + 24T + 24}{e^T}, \\ &= 24 - 0, \\ &= 24. \end{aligned}$$

Example 384. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{2x^2}{(1+x^2)^2}\, dx$.

Solution Let's find an antiderivative, first. We set up the following table,

x	$+$	$\dfrac{2x}{(1+x^2)^2}$
1	$-$	$-\dfrac{1}{1+x^2}$
0		$-\operatorname{Arctan} x$

since the second entry on the right is obtained from the first via the simple substitution $u = 1 + x^2$, $du = 2x\ dx$. The third such entry is a basic Arctangent integral. It follows that,

$$\begin{aligned}
\int_0^T \frac{2x^2}{(1+x^2)^2}\ dx &= \left\{-\frac{x}{1+x^2} + \operatorname{Arctan} x\right\}\bigg|_0^T, \\
&= \left\{-\frac{T}{1+T^2} + \operatorname{Arctan} T\right\},
\end{aligned}$$

and so, for example (we can use ANY number as the lower limit of integration),

$$\begin{aligned}
\int_0^\infty \frac{2x^2}{(1+x^2)^2}\ dx &= \lim_{T\to\infty} \int_0^T \frac{2x^2}{(1+x^2)^2}\ dx, \\
&= \lim_{T\to\infty} \left\{-\frac{T}{1+T^2} + \operatorname{Arctan} T\right\}, \\
&= 0 + \frac{\pi}{2}, \\
&= \frac{\pi}{2},
\end{aligned}$$

where we used L'Hospital's Rule (or a more elementary method) in the first limit and a basic property of the Arctangent function (see Chapter 3). A similar argument shows that

$$\begin{aligned}
\int_{-\infty}^0 \frac{2x^2}{(1+x^2)^2}\ dx &= \lim_{T\to-\infty} \int_T^0 \frac{2x^2}{(1+x^2)^2}\ dx, \\
&= \lim_{T\to-\infty} \left\{-\int_0^T \frac{2x^2}{(1+x^2)^2}\ dx\right\}, \\
&= \lim_{T\to-\infty} \left\{+\frac{T}{1+T^2} - \operatorname{Arctan} T\right\}, \\
&= 0 + \frac{\pi}{2}, \\
&= \frac{\pi}{2},
\end{aligned}$$

and we conclude that

$$\begin{aligned}
\int_{-\infty}^\infty \frac{2x^2}{(1+x^2)^2}\ dx &= \frac{\pi}{2} + \frac{\pi}{2}, \\
&= \pi.
\end{aligned}$$

Example 385.

Find the area enclosed by the infinite funnel given by $y^2 = e^{-2x}$, where $x \geq 0$, (see Figure 142).

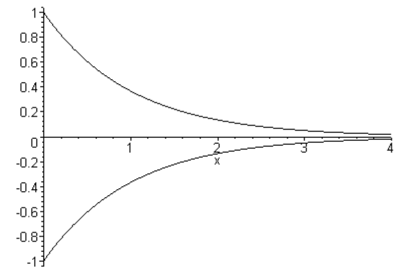


Figure 142.

Solution We use symmetry and realize that the area enclosed by this region is twice the area enclosed by either part lying above or below the $x-$axis. Note that the graph is given by

$$y = \begin{cases} e^{-x}, & \text{if } \ y > 0, \\ -e^{-x}, & \text{if } \ y < 0. \end{cases}$$

Let's say that we use that part of the graph lying above the $x-$axis (that is, $y > 0$) in our area calculation. Since the integrand is positive, the total area is given by a definite integral, namely,

$$\text{Area} = 2 \int_0^\infty e^{-x}\ dx.$$

This integral is straightforward since

$$\int_0^\infty e^{-x}\ dx = \lim_{T\to\infty} \int_0^T e^{-x}\ dx = 1.$$

Hence,

$$\text{Area} = 2 \int_0^\infty e^{-x}\ dx = 2,$$

(see the margin for a numerical treatment).

The moral here is that an *infinite region may enclose a finite area!*

Numerical estimation of the integral in Example 385 as a function of the upper limit, T.

T	Area $\approx 2 \int_0^T e^{-x}\ dx$
5	1.986524106
10	1.999909200
100	1.999999999
$1,000$	1.999999999
$10,000$	1.999999999
10^5	1.999999999
$\cdots$	$\cdots$

Example 386. Evaluate the improper integral that defines Wien's Distribution Law

$$Energy = \frac{8\pi h}{c^3} \cdot \int_0^\infty \nu^3\ e^{-\frac{h\nu}{kT}}\ d\nu.$$

explicitly.

Solution The first thing to do is to reduce this integral to something more recognizable. This is most readily done by applying a *substitution*, in this case,

$$x = \frac{h\nu}{kT}, \quad dx = \frac{hd\nu}{kT}, \Longrightarrow \ d\nu = \frac{kT\ dx}{h}.$$

When $\nu = 0,\ x = 0$ and when $\nu = \infty,\ x = \infty$. Thus,

$$\begin{aligned} \int_0^\infty \nu^3\ e^{-\frac{h\nu}{kT}}\ d\nu &= \frac{k^3T^3}{h^3} \int_0^\infty x^3\ e^{-x}\ \frac{kT}{h}\ dx \\ &= \frac{k^4T^4}{h^4} \int_0^\infty x^3\ e^{-x}\ dx \\ &= \frac{k^4T^4}{h^4} \lim_{T\to\infty} \int_0^T x^3\ e^{-x}\ dx. \end{aligned}$$

But the Table Method applied to the integral on the right gives us

x^3	$+$	e^{-x}
$3x^2$	$-$	$-e^{-x}$
$6x$	$+$	e^{-x}
6	$-$	$-e^{-x}$
0		e^{-x}

So,

$$\int_0^T x^3\ e^{-x}\ dx = -\left(\frac{x^3+3x^2+6x+6}{e^x}\right)\Big|_0^T = 6 - \left(\frac{T^3+3T^2+6T+6}{e^T}\right).$$

A few applications of L'Hospital's Rule gives us

$$\lim_{T\to\infty} \int_0^T x^3\ e^{-x}\ dx = 6,$$

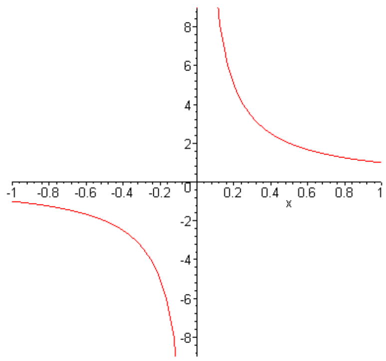
and so

$$\begin{aligned}\int_0^\infty \nu^3\ e^{-\frac{h\nu}{kT}}\ d\nu &= \frac{k^4T^4}{h^4} \lim_{T\to\infty}\int_0^T x^3\ e^{-x}\ dx \\ &= \frac{6k^4T^4}{h^4}.\end{aligned}$$

Collecting terms we find an expression for the Energy in terms of the temperature, T:

$$\begin{aligned}\text{Energy} &= \frac{8\pi h}{c^3}\cdot\int_0^\infty \nu^3\ e^{-\frac{h\nu}{kT}}\ d\nu \\ &= \frac{8\pi h}{c^3}\ \frac{6k^4T^4}{h^4} \\ &= \frac{48\pi k^4T^4}{h^3c^3}.\end{aligned}$$

NOTE: In this example we made use of the Susbtitution Rule, Integration by Parts and L'Hospital's Rule!



The graph of $y = \frac{1}{x}$ over the interval $[-1, 1]$. Note that the area under the curve to the left of 0 equals the negative of the area to the right of 0 but each area is infinite.

Figure 143.

Integrals of the Second Class

Now we look at the case where the limits of integration are finite but the integrand has an *infinite* discontinuity inside the range of integration. In this case, we need to modify the definition of such an improper integral. The reason we call it *improper* can be gathered from the following examples.

A **common error** in Calculus is the flawed procedure of **integrating across a discontinuity**. This is bad news! You can't normally get away with this! For example,

$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} \frac{1}{x}\ dx \neq 0,$$

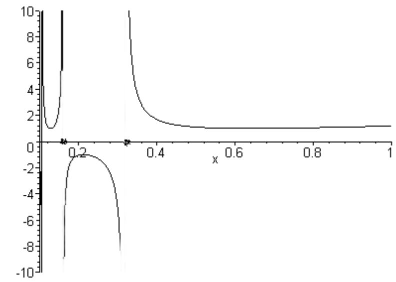
yet one would expect this to be equal to zero (by integrating without much thought using the natural logarithm and evaluating it between the limits, -1 and 1). You see, a quick look at the graph, Figure 143, shows that the areas under the graph

don't cancel because they are each infinite, and we can't subtract infinities unless we do this in a limiting way. So the answer isn't zero (necessarily).

So, what's going on? The point is that the interval of integration contains an **infinite discontinuity** of f, in this case, at $x = 0$, since (see Figure 143),

$$\lim_{x \to 0} f(x) = +\infty.$$

If one *forgets* this, then one can get into trouble as we have seen. What do we do? We simply redefine the notion of an improper integral for functions of this class.



$y = \csc(\frac{1}{x})$ for $0.1 \le x \le 1$.

Figure 144.

Improper Integrals of functions with an Infinite Discontinuity

Let f be continuous over $[a, b)$ and assume that f has an **infinite discontinuity at $x = b$**. We define the improper integral of f over $[a, b)$ by the symbol

$$\int_a^b f(x)\, dx = \lim_{T \to b^-} \int_a^T f(x)\, dx.$$

This is a *one-sided limit*, actually a limit from the left, at $x = b$. In fact, **all the improper integrals of the second class will be defined in terms of one-sided limits!**

Let f be continuous over $(a, b]$ and assume that f has an **infinite discontinuity at $x = a$**. Then we define the improper integral of f over $(a, b]$ by the symbol

$$\int_a^b f(x)\, dx = \lim_{T \to a^+} \int_T^b f(x)\, dx,$$

and this is now a limit from the right at $x = a$.

NOTE: In the event that f has *many infinite discontinuities* inside the interval (see Figure 144) of integration $[a, b]$, then we just break up this interval into those pieces in which f is continuous and apply the definitions above over each piece separately. In the case of Figure 144, the function defined by $y = \csc(\frac{1}{x})$ has *infinitely many infinite discontinuities* in $(0, 1]$! We just showed a few of them here, namely those at $x = \frac{1}{\pi} \approx 0.3$ and at $x = \frac{1}{2\pi} \approx 0.17$

For this function,

$$\int_{0.1}^{1} \csc\left(\frac{1}{x}\right)\, dx = \int_{0.1}^{\frac{1}{2\pi}} \csc\left(\frac{1}{x}\right)\, dx + \int_{\frac{1}{2\pi}}^{\frac{1}{\pi}} \csc\left(\frac{1}{x}\right)\, dx + \int_{\frac{1}{\pi}}^{1} \csc\left(\frac{1}{x}\right)\, dx.$$

The integral in the middle, on the right of the last display has two discontinuities, one at each end-point, $x = \frac{1}{2\pi}$ and $x = \frac{1}{\pi}$ (Why?). In this case we define the improper integral naturally as follows:

Let f be continuous over an interval (a, b) and assume that f has an **infinite discontinuity at both $x = a$ and $x = b$**. Let c be any point inside (a, b). Then we define the improper integral of f over (a, b) by the symbol

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx,$$

where the first integral on the right is defined by a limit from the right, while the second integral is defined by a limit from the left in accordance with our definitions, above.

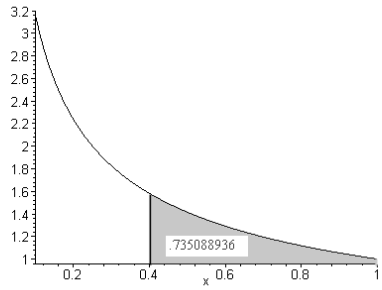
Example 387. Evaluate the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

Solution Note that is an improper integral since the integrand has an infinite discontinuity at $x = 0$. Using our definitions we see that we can give meaning to the symbol

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

only if we interpret it as a limit, namely,

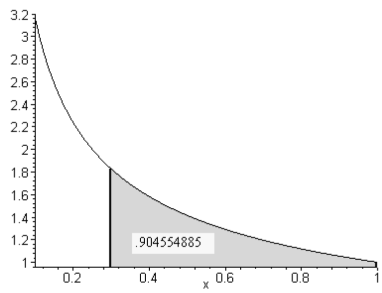
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{T\to 0^+} \int_T^1 \frac{1}{\sqrt{x}}\,dx.$$



$T = 0.4$

Now,

$$\begin{aligned}\int_T^1 \frac{1}{\sqrt{x}}\,dx &= \int_T^1 x^{-1/2}\,dx \\ &= 2\left(x^{1/2}\right)\Big|_T^1 \\ &= 2(1-\sqrt{T}).\end{aligned}$$



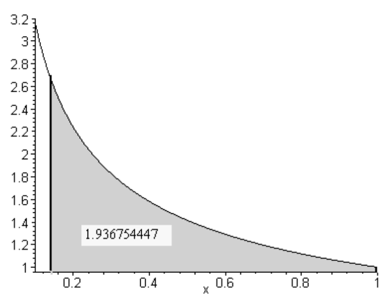
$T = 0.3$

It follows that

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{T\to 0^+} 2(1-\sqrt{T}) = 2.$$

Remark In the margin, we can see the effect of passing to the limit $T \to 0^+$ graphically. The actual values of the corresponding integrals obtained can be interpreted geometrically as the shaded areas seen there. So, for instance,

T	Area $= \int_T^1 \frac{1}{\sqrt{x}}\,dx$
0.4	.735088936
0.3	.904554885
0.001	1.936754447
0.0001	1.980000000
0.000001	1.998000000
...	...
0	2.0000000000



$T = 0.001$

For this example, our answer actually corresponds to a *limiting area*!

Exercise Convince yourself that if $p > 1$, then

$$\int_0^1 \frac{1}{\sqrt[p]{x}}\,dx = \frac{p}{p-1}.$$

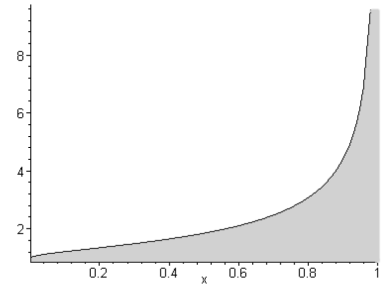
Example 388. Evaluate the integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$.

Solution This is an improper integral since the integrand has an infinite discontinuity at $x = 1$. So, we can give meaning to the symbol

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$$

only if we interpret it as a limit once again, namely,

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{T\to 1^-} \int_0^T \frac{1}{\sqrt{1-x^2}}\, dx.$$



The graph of $y = \frac{1}{\sqrt{1-x^2}}$.

Figure 145.

But we recall from our chapter on Integration that

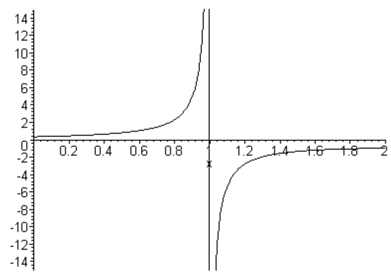
$$\begin{aligned}\int_0^T \frac{1}{\sqrt{1-x^2}}\, dx &= \operatorname{Arcsin} x \Big|_0^T \\ &= \operatorname{Arcsin} T - \operatorname{Arcsin} 0 \\ &= \operatorname{Arcsin} T,\end{aligned}$$

since $\operatorname{Arcsin} 0 = 0$. Since the Arcsin function is continuous at $x = 1$ we conclude that

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{T\to 1^-} \operatorname{Arcsin} T = \operatorname{Arcsin} 1 = \frac{\pi}{2}.$$

Thus, the (actually infinite) shaded region in Figure 145 has area equal to $\frac{\pi}{2}$.

Example 389. Evaluate the integral $\int_0^2 \frac{1}{x^2-4x+3}\, dx$.



The graph of $y = \frac{1}{x^2-4x+3}$ on $[0, 2]$.

Figure 146.

Solution At first sight this doesn't look improper. But since we are thinking about using the method of Partial Fractions (what else?), we should factor the denominator and then inspect it for any zeros inside the interval $[0, 2]$. Observe that

$$\int_0^2 \frac{1}{x^2-4x+3}\, dx = \int_0^2 \frac{1}{(x-3)(x-1)}\, dx,$$

and so the point $x = 1$ is an infinite discontinuity of our integrand that is *inside* the interval (not at the endpoints), see Figure 146. According to our definition we can write

$$\begin{aligned}\int_0^2 \frac{1}{x^2-4x+3}\, dx &= \int_0^1 \frac{1}{x^2-4x+3}\, dx + \int_1^2 \frac{1}{x^2-4x+3}\, dx \\ &= \lim_{T\to 1^-} \int_0^T \frac{1}{(x-3)(x-1)}\, dx + \\ &\quad \lim_{T\to 1^+} \int_T^2 \frac{1}{(x-3)(x-1)}\, dx \\ &= \lim_{T\to 1^-} \left(\frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1|\right) \Big|_0^T + \\ &\quad + \lim_{T\to 1^+} \left(\frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1|\right) \Big|_T^2 \\ &= \lim_{T\to 1^-} \left(\frac{1}{2}\ln\left|\frac{T-3}{T-1}\right| - \frac{\ln 3}{2}\right) + \\ &\quad \lim_{T\to 1^+} \left(-\frac{1}{2}\ln\left|\frac{T-3}{T-1}\right|\right) \\ &= \infty - \infty,\end{aligned}$$

so the improper integral diverges (by definition) since each one of these limits actually exists but is infinite (recall that $\ln|1/0| = \infty$).

Example 390. Evaluate the integral $\int_0^1 x \ \ln x \ dx$.

Solution The integrand is undefined at the left end-point $x = 0$, since we have an indeterminate form of type $0 \cdot (-\infty)$ here. There are no other discontinuities here since $\ln 1 = 0$ and this is fine. So, it is best to treat this integral as an improper integral and take a limiting approach. We integrate by parts: That is, we set

$$u = \ln x, \ du = \frac{1}{x} \ dx, \qquad dv = x \ dx, \ v = \frac{x^2}{2}.$$

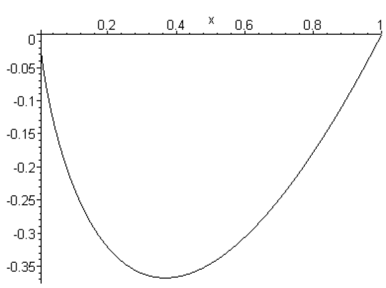
Then,

$$\begin{aligned}
\int_0^1 x \ \ln x \ dx &= \lim_{T \to 0^+} \int_T^1 x \ \ln x \ dx \\
&= \lim_{T \to 0^+} \left(\frac{x^2 \ln x}{2} \Big|_T^1 - \int_T^1 \frac{x}{2} \ dx \right) \\
&= \lim_{T \to 0^+} \left(-\frac{T^2 \ln T}{2} - \frac{1 - T^2}{4} \right) \\
&= -\frac{1}{2} \cdot \left(\lim_{T \to 0^+} T^2 \ln T \right) - \frac{1}{4} \\
& \quad \text{provided the first limit exists} \ldots \\
&= -\frac{1}{2} \cdot \left(\lim_{T \to 0^+} \frac{\ln T}{T^{-2}} \right) - \frac{1}{4} \\
& \quad \text{provided this limit exists} \ldots \\
&= -\frac{1}{2} \cdot \left(\lim_{T \to 0^+} \frac{1/T}{-2T^{-3}} \right) - \frac{1}{4} \\
& \quad \text{where we used L'Hospital's Rule,} \\
&= -\frac{1}{2} \cdot \left(\lim_{T \to 0^+} \frac{T^2}{(-2)} \right) - \frac{1}{4} \\
&= -\frac{1}{4}.
\end{aligned}$$

NOTE: This is an interesting integrand because

$$\lim_{x \to 0^+} x \ \ln x = 0,$$

as can be gathered from either L'Hospital's Rule (similar to the above calculation) or its graph (see Figure 147). We would have expected the integrand to be infinite but this is not always the case. Indeterminate forms can arise as well at the end-points and the integral should still be treated as improper.



The graph of $y = x \ln x$ on $(0, 1]$.

Figure 147.

The moral is:

If an integral looks improper, treat it like an improper integral

because, even if it isn't, you'll get the *right* answer anyhow, and then you won't have to worry! Here's such an example.

Example 391. Evaluate $\int_{-1}^1 \frac{3x^2 + 2}{\sqrt[3]{x^2}} \ dx$.

Solution The integrand is infinite when $x = 0$ (which is between -1 and 1). So, the integral is improper. Its evaluation is straightforward, however. Let's find an antiderivative, first.

$$\begin{aligned}
\int \frac{3x^2+2}{\sqrt[3]{x^2}}\,dx &= \int \left(3x^2x^{-2/3} + 2x^{-2/3}\right)\,dx \\
&= \int \left(3x^{4/3} + 2x^{-2/3}\right)\,dx \\
&= \frac{9}{7}x^{7/3} + 6x^{1/3}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{-1}^{1} \frac{3x^2+2}{\sqrt[3]{x^2}}\,dx &= \int_{-1}^{0} \frac{3x^2+2}{\sqrt[3]{x^2}}\,dx + \int_{0}^{1} \frac{3x^2+2}{\sqrt[3]{x^2}}\,dx \\
&\quad \text{by definition of the improper integral} \ldots \\
&= \lim_{T\to 0^-} \int_{-1}^{T} \frac{3x^2+2}{\sqrt[3]{x^2}}\,dx + \lim_{T\to 0^+} \int_{T}^{1} \frac{3x^2+2}{\sqrt[3]{x^2}}\,dx \\
&\quad \text{once again, by definition} \\
&= \lim_{T\to 0^-} \left(\frac{9}{7}x^{7/3} + 6x^{1/3}\right)\Big|_{-1}^{T} + \lim_{T\to 0^+} \left(\frac{9}{7}x^{7/3} + 6x^{1/3}\right)\Big|_{T}^{1} \\
&= \lim_{T\to 0^-} \left(\frac{9}{7}T^{7/3} + 6T^{1/3} - \left((-1)\frac{9}{7} + (-1)6\right)\right) + \\
&\quad + \lim_{T\to 0^+} \left(\left(\frac{9}{7} + 6\right) - \left(\frac{9}{7}T^{7/3} + 6T^{1/3}\right)\right) \\
&= \left(\frac{9}{7} + 6\right) + \left(\frac{9}{7} + 6\right) \\
&\quad \text{by continuity of our antiderivative at } T = 0 \\
&= \frac{102}{7} \approx 14.57,
\end{aligned}$$

NOTE: In this example, the infinite discontinuity at $x = 0$ is "virtual" insofar as the improper integral is concerned, in the sense that we could evaluate the integral directly without resorting to the improper integral definitions and still obtain the right answer! The point is, YOU JUST DON'T KNOW when you're starting out. So, you see, you have nothing to lose (except a few more minutes of your time) in treating such an integral (that is, one that *looks* improper) *as* an improper integral.

NOTES:

Example 392. Evaluate the improper integral $\int_0^{\infty} x \cos x \, dx$.

Solution This is improper because of the infinite upper limit of integration. Furthermore, Integration by Parts (using the Table Method) gives us

$$\begin{aligned}
\int_0^{\infty} x \cos x \, dx &= \lim_{T\to\infty} \int_0^T x \cos x \, dx \\
&= \lim_{T\to\infty} (x \sin x + \cos x)\Big|_0^T \\
&= \lim_{T\to\infty} ((T \sin T + \cos T) - (0+1)) \\
&= \lim_{T\to\infty} (T \sin T + \cos T - 1)
\end{aligned}$$

and this limit does not exist!

The function "$T \sin T + \cos T - 1$" has no limit at ∞ as you can gather from its erratic behavior as T increases

T	$\int_0^T x \cos x \, dx$
10	-7.279282638
100	-50.77424524
1000	826.4419196
$10,000$	-3058.096044
10^5	3572.880436
10^6	-349993.5654
...	...

It folows that the improper integral diverges (see the margin).

Exercise Set 39.

Determine which of the following integrals is improper and give reasons: Do not evaluate the integrals

1. $\int_{-1}^{2} \frac{1}{2x} \, dx$
2. $\int_{-1}^{1} \frac{1}{1+x^2} \, dx$
3. $\int_{0}^{1} \frac{1}{x^p} \, dx$, for $p > 1$
4. $\int_{-1}^{\infty} \frac{1}{(1+x)^p} \, dx$, for $p > 1$
5. $\int_{-1}^{1} \frac{1}{(1+x)^p} \, dx$, for $p > 1$
6. $\int_{-1}^{1} e^{-x^2} \, dx$
7. $\int_{-\pi}^{\pi} \csc x \, dx$
8. $\int_{-\infty}^{\infty} x^2 e^{-2x} \, dx$
9. $\int_{0}^{1} 2x^3 \ln 2x \, dx$
10. $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$

Evaluate the following improper integrals using any method

11. $\int_{1}^{\infty} \frac{1}{x^{1.5}} \, dx$
12. $\int_{2}^{\infty} \frac{1}{\sqrt{x}} \, dx$

13. $\displaystyle\int_0^2 \frac{1}{2x}\ dx$

14. $\displaystyle\int_0^\infty x^2 e^{-x}\ dx$

15. $\displaystyle\int_{-\infty}^\infty \frac{2x}{(1+x^2)^2}\ dx$

16. $\displaystyle\int_{-1}^0 \frac{x}{\sqrt{1-x^2}}\ dx$
• Use a substitution, first.

17. $\displaystyle\int_0^2 \frac{1}{x^2-1}\ dx$

18. $\displaystyle\int_1^2 \frac{1}{1-x^2}\ dx$

19. $\displaystyle\int_0^\infty e^{-x}\ \sin x\ dx$
• See Section 7.3.4.

20. $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

21. $\displaystyle\int_{-1}^1 \frac{x+1}{\sqrt[5]{x^3}}\ dx$

22. $\displaystyle\int_{-1}^1 \frac{x-1}{\sqrt[3]{x^5}}\ dx$

23. $\displaystyle\int_{-\infty}^\infty e^{-|x|}\ dx$
• Consider the cases $x < 0$ and $x > 0$ separately when removing the absolute value; that is, rewrite this integral as a sum of two improper integrals (without absolute values) each having one finite limit of integration (say, $c = 0$).

24. For what values of p does the improper integral
$$\int_2^\infty \frac{dx}{x(\ln x)^p}$$
converge to a finite number?

25. Can you find a value of p such that
$$\int_0^\infty x^p\ dx$$
converges to a finite number?

26. Let f be a continuous function on $(0, \infty)$. Define the improper integral
$$v(\lambda) = \sqrt{\frac{2}{\pi}}\ \int_0^\infty f(t)\ \cos(\lambda t)\ dt$$
This is called the **Fourier Cosine Transform** of f and is of fundamental importance in the study of electromagnetic waves, wavelets, and medical imaging techniques. Evaluate
$$v(\lambda) = \sqrt{\frac{2}{\pi}}\ \int_0^\infty e^{-2t}\ \cos(\lambda t)\ dt.$$

27. **Hard** Let f be a continuous function defined on the interval $[a, \infty)$ and assume that
$$\lim_{x\to\infty} f(x) = 2.$$
Can the improper integral $\displaystyle\int_0^\infty f(x)\ dx$ converge to a *finite* number? Give reasons.

Hint Proceed as follows:

- Show that the hypothesis on the limit forces the existence of a number X such that $f(x) > 1$ for each $x > X$.
- Let T be any number with $T > X$. Show that $\displaystyle\int_X^T f(x)\ dx > T - X$.
- Now let $T \to \infty$ in the last integral and state your conclusion.

28. **Hard** Let f, g each be continuous functions over the interval $[a, \infty)$ where a is some fixed number. Suppose that $0 \le f(x) \le g(x)$ for each x, where $a \le x < \infty$, and that the improper integral
$$\int_a^\infty g(x)\ dx$$
converges to a finite number.
Prove that
$$\int_a^\infty f(x)\ dx$$
also converges to a finite number.
Hints This is called a **Comparison Theorem for improper integrals** as it allows you to test for the convergence of an improper integral by comparing to another one whose convergence you already know! To prove this proceed as follows:
    - Since $f(x) \ge 0$ conclude that the integral $\mathcal{F}(x) = \int_a^x f(t)\ dt$ is an increasing function for $x \ge a$ (Use the Fundamental Theorem of Calculus).
    - Use the fact that an increasing function must have a limit to deduce that the improper integral $\displaystyle\int_a^\infty f(t)\ dt$ actually exists
    - By comparing the definite integrals of f and g, conclude that the *limit* of the improper integral of f must be finite.

29. Use the Comparison Theorem above with $f(x) = e^{-x^2}$ and $g(x) = e^{-x}$ to show that the improper integral
$$\int_0^\infty e^{-x^2}\ dx$$
converges to a finite limit.
• Show that $x^2 \ge x$ if $x \ge 1$ and conclude that $f(x) \le g(x)$ for $x \ge 1$. Now show that the improper integral of g converges and $\int_1^\infty e^{-x}\ dx = 1$. Use the Comparison Theorem to arrive at your conclusion.

30. **Long but not hard** Use Simpson's Rule with $n = 22$ over the interval $[-5, 5]$ to estimate the value of the Gaussian-type integral
$$L = \int_{-\infty}^\infty e^{-x^2}\ dx.$$
Now multiply your numerical result by itself (the result is an estimate of L^2). Do you recognize this number? Now *guess* the value of L.

Suggested Homework Set 31. *Problems 1, 2, 4, 14, 15, 19, 21, 22*

7.8 Chapter Exercises

The exercises in this section are chosen randomly among all the techniques you have seen in this Chapter. Note that levels of difficulty vary from exercise to exercise.

Prove the following trigonometric identities using the basic identities in Section 7.1.

1. $\cos^2 x - \sin^2 x = \cos 2x$
2. $\cos^4 x - \sin^4 x = \cos 2x$
3. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$
4. $\sqrt{1+\cos x} = \sqrt{2} \cdot \cos\left(\frac{x}{2}\right)$, if $-\pi \leq x \leq \pi$.
5. $\sqrt{1-\cos x} = \sqrt{2} \cdot \sin\left(\frac{x}{2}\right)$, if $0 \leq x \leq 2\pi$
6. $\sqrt{1+\cos 5x} = \sqrt{2} \cdot \cos\left(\frac{5x}{2}\right)$, if $-\pi \leq 5x \leq \pi$.

Use numerical integration to evaluate the following integrals.

7. $\int_0^2 (2x-1)\, dx$, using the Trapezoidal Rule with $n = 6$. Compare your answer with the exact answer obtained by direct integration.
8. $\int_0^4 (3x^2 - 2x + 6)\, dx$, using Simpson's Rule with $n = 6$. Compare your answer with the exact answer obtained by direct integration.
9. $\int_{-\pi}^{\pi} (\cos^2 x + \sin^2 x)\, dx$, using the Trapezoidal Rule with $n = 6$. Compare your answer with the exact answer obtained by direct integration.
10. $\int_{-\pi}^{\pi} (\cos^2 x - \sin^2 x)\, dx$, using Simpson's Rule with $n = 6$. Compare your answer with the exact answer obtained by direct integration.
11. $\int_0^1 e^{-x^2}\, dx$, using Simpson's Rule with $n = 6$
12. $\int_{-1}^2 \frac{1}{1+x^6}\, dx$, using Simpson's Rule with $n = 4$
13. $\int_{-2}^2 \frac{x^2}{1+x^4}\, dx$, using the Trapezoidal Rule with $n = 6$. Compare your answer with the exact answer obtained by direct integration.
14. $\int_1^2 (\ln x)^3\, dx$, using Simpson's Rule with $n = 6$.

Evaluate the following integrals using any method

NOTES:

15. $\int \sqrt{3x+2}\ dx$

16. $\int \frac{1}{x^2+4x+4}\ dx$

17. $\int \frac{dx}{(2x-3)^2}$

18. $\int \frac{dx}{\sqrt{a+bx}}$

19. $\int (\sqrt{a}-\sqrt{x})^2\ dx$

20. $\int \frac{x\ dx}{\sqrt{a^2-x^2}}$

21. $\int x^2\sqrt{x^3+1}\ dx$

22. $\int \frac{(x+1)}{\sqrt[3]{x^2+2x+2}}\ dx$

23. $\int (x^4+4x^2+1)^2(x^3+2x)\ dx$

24. $\int x^{-\frac{1}{3}}\sqrt{x^{\frac{2}{3}}-1}\ dx$

25. $\int \frac{2x\ dx}{(3x^2-2)^2}$

26. $\int \frac{dx}{4x+3}$

27. $\int \frac{x\ dx}{2x^2-1}$

28. $\int \frac{x^2\ dx}{1+x^3}$

29. $\int \frac{(2x+3)\ dx}{x^2+3x+2}$

30. $\int \sin(2x+4)\ dx$

31. $\int 2\cos(4x+1)\ dx$

32. $\int \sqrt{1-\cos 2x}\ dx$

33. $\int \sin\frac{3x-2}{5}\ dx$

34. $\int x\cos ax^2\ dx$

35. $\int x\sin(x^2+1)\ dx$

36. $\int \sec^2\frac{\theta}{2}\ d\theta$

37. $\int \frac{d\theta}{\cos^2 3\theta}$

38. $\int \frac{d\theta}{\sin^2 2\theta}$

39. $\int x\csc^2(x^2)\ dx$

40. $\int \tan\frac{3x+4}{5}\ dx$

41. $\int \frac{dx}{\tan 2x}$

42. $\int \sqrt{1+\cos 5x}\ dx$

43. $\int \csc(x+\frac{\pi}{2})\ \cot(x+\frac{\pi}{2})\ dx$

44. $\int \cos 3x\ \cos 4x\ dx$

45. $\int \sec 5\theta\tan 5\theta\ d\theta$

46. $\int \frac{\cos x}{\sin^2 x}\ dx$

47. $\int x^2\cos(x^3+1)\ dx$

48. $\int \sec\theta(\sec\theta+\tan\theta)\ d\theta$

49. $\int (\csc\theta-\cot\theta)\csc\theta\ d\theta$

50. $\int \cos^{-4}x\sin(2x)\ dx$

51. $\int \frac{\tan^2\sqrt{x}}{\sqrt{x}}\ dx$

52. $\int \frac{1+\sin 2x}{\cos^2 2x}\ dx$

53. $\int \frac{dx}{\cos 3x}$

54. $\int \frac{dx}{\sin(3x+2)}$

55. $\int \frac{1+\sin x}{\cos x}\ dx$

56. $\int (1+\sec\theta)^2\ d\theta$

57. $\int \frac{\csc^2 x\ dx}{1+2\cot x}$

58. $\int e^x\ \sec e^x\ dx$

59. $\int \frac{dx}{x\ln x}$

60. $\int \frac{dt}{\sqrt{2-t^2}}$

61. $\int \frac{dx}{\sqrt{3-4x^2}}$

62. $\int \frac{(2x+3)\ dx}{\sqrt{4-x^2}}$

63. $\int \frac{dx}{x^2+5}$

64. $\int \frac{dx}{4x^2+3}$

65. $\int \frac{dx}{x\sqrt{x^2-4}},\quad x>0.$

66. $\int \frac{dx}{x\sqrt{4x^2-9}},\quad x>0$

67. $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

68. $\displaystyle\int \frac{dx}{\sqrt{4x^2+3}}$

69. $\displaystyle\int \frac{dx}{\sqrt{x^2-16}}$

70. $\displaystyle\int \frac{e^x}{1+e^{2x}}\ dx$

71. $\displaystyle\int \frac{1}{x\sqrt{4x^2-1}}\ dx$

72. $\displaystyle\int \frac{dx}{\sqrt{4x^2-9}}$

73. $\displaystyle\int e^{-3x}\ dx$

74. $\displaystyle\int \frac{dx}{e^{2x}}$

75. $\displaystyle\int (e^x-e^{-x})^2\ dx$

76. $\displaystyle\int xe^{-x^2}\ dx$

77. $\displaystyle\int \frac{\sin\theta\ d\theta}{\sqrt{1-\cos\theta}}$

78. $\displaystyle\int \frac{\cos\theta\ d\theta}{\sqrt{2-\sin^2\theta}}$

79. $\displaystyle\int \frac{e^{2x}\ dx}{1+e^{2x}}$

80. $\displaystyle\int \frac{e^x\ dx}{1+e^{2x}}$

81. $\displaystyle\int \frac{\cos\theta\ d\theta}{2+\sin^2\theta}$

82. $\displaystyle\int \sin^3 x\cos x\ dx$

83. $\displaystyle\int \cos^4 5x\sin 5x\ dx$

84. $\displaystyle\int (\cos\theta+\sin\theta)^2\ d\theta$

85. $\displaystyle\int \sin^3 x\ dx$

86. $\displaystyle\int \cos^3 2x\ dx$

87. $\displaystyle\int \sin^3 x\cos^2 x\ dx$

88. $\displaystyle\int \cos^5 x\ dx$

89. $\displaystyle\int \sin^3 4\theta\cos^3 4\theta\ d\theta$

90. $\displaystyle\int \frac{\cos^2 x\ dx}{\sin x}$

91. $\displaystyle\int \frac{\cos^3 x\ dx}{\sin x}$

92. $\displaystyle\int \tan^2 x\sec^2 x\ dx$

93. $\displaystyle\int \sec^2 x\tan^3 x\ dx$

94. $\displaystyle\int \frac{\sin x\ dx}{\cos^3 x}$

95. $\displaystyle\int \frac{\sin^2 x\ dx}{\cos^4 x}$

96. $\displaystyle\int \sec^4 x\ dx$

97. $\displaystyle\int \tan^2 x\ dx$

98. $\displaystyle\int (1+\cot\theta)^2\ d\theta$

99. $\displaystyle\int \sec^4 x\tan^3 x\ dx$

100. $\displaystyle\int \csc^6 x\ dx$

101. $\displaystyle\int \tan^3 x\ dx$

102. $\displaystyle\int \frac{\cos^2 t dt}{\sin^6 t}$

103. $\displaystyle\int \tan\theta\csc\theta\ d\theta$

104. $\displaystyle\int \cos^2 4x\ dx$

105. $\displaystyle\int (1+\cos\theta)^2\ d\theta$

106. $\displaystyle\int (1-\sin x)^3\ dx$

107. $\displaystyle\int \sin^4 x\ dx$

108. $\displaystyle\int \sin^2 2x\cos^2 2x\ dx$

109. $\displaystyle\int \sin^4\theta\cos^2\theta\ d\theta$

110. $\displaystyle\int \cos^6 x\ dx$

111. $\displaystyle\int \cos x\sin 2x\ dx$

112. $\displaystyle\int \sin x\cos 3x\ dx$

113. $\displaystyle\int \sin 2x\sin 3x\ dx$

114. $\displaystyle\int \cos 2x\cos 4x\ dx$

115. $\displaystyle\int \sin^2 2x\cos 3x\ dx$

116. $\displaystyle\int \sec x\csc x\ dx$, **hard** 

117. $\displaystyle\int \frac{dx}{1-\cos x}$, **hard** 

118. $\displaystyle\int \frac{dx}{\sqrt{2+2x-x^2}}$

119. $\displaystyle\int \frac{dx}{\sqrt{1+4x-4x^2}}$

120. $\displaystyle\int \frac{dx}{\sqrt{2+6x-3x^2}}$, **hard.**

121. $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+13}}$

122. $\displaystyle\int \frac{dx}{2x^2-4x+6}$

123. $\displaystyle\int \frac{dx}{(1-x)\sqrt{x^2-2x-3}}$, for $x>1$. **hard.**

124. $\displaystyle\int \frac{(2x+3)\ dx}{x^2+2x-3}$

125. $\displaystyle\int \frac{(x+1)\ dx}{x^2+2x-3}$

126. $\displaystyle\int \frac{(x-1)\ dx}{4x^2-4x+2}$

127. $\displaystyle\int \frac{x\ dx}{\sqrt{x^2-2x+2}}$

128. $\displaystyle\int \frac{(4x+1)\ dx}{\sqrt{1+4x-4x^2}}$

129. $\displaystyle\int \frac{(3x-2)\ dx}{\sqrt{x^2+2x+3}}$

130. $\displaystyle\int \frac{e^x\ dx}{e^{2x}+2e^x+3}$

131. $\displaystyle\int \frac{x^2\ dx}{x^2+x-6}$

132. $\displaystyle\int \frac{(x+2)\ dx}{x^2+x}$

133. $\displaystyle\int \frac{(x^3+x^2)\ dx}{x^2-3x+2}$

134. $\displaystyle\int \frac{dx}{x^3-x}$

135. $\displaystyle\int \frac{(x-3)\ dx}{x^3+3x^2+2x}$

136. $\displaystyle\int \frac{(x^3+1)\ dx}{x^3-x^2}$

137. $\displaystyle\int \frac{x\ dx}{(x+1)^2}$

138. $\displaystyle\int \frac{(x+2)\ dx}{x^2-4x+4}$

139. $\displaystyle\int \frac{(3x+2)\ dx}{x^3-2x^2+x}$

140. $\displaystyle\int \frac{8\ dx}{x^4-2x^3}$

141. $\displaystyle\int \frac{dx}{(x^2-1)^2}$

142. $\displaystyle\int \frac{(1-x^3)\ dx}{x(x^2+1)}$

143. $\displaystyle\int \frac{(x-1)\ dx}{(x+1)(x^2+1)}$

144. $\displaystyle\int \frac{4x\ dx}{x^4-1}$

145. $\displaystyle\int \frac{3(x+1)\ dx}{x^3-1}$

146. $\displaystyle\int \frac{(x^4+x)\ dx}{x^4-4}$

147. $\displaystyle\int \frac{x^2\ dx}{(x^2+1)(x^2+2)}$

148. $\displaystyle\int \frac{3\ dx}{x^4+5x^2+4}$

149. $\displaystyle\int \frac{(x-1)\ dx}{(x^2+1)(x^2-2x+3)}$

150. $\displaystyle\int \frac{x^3\ dx}{(x^2+4)^2}$

151. $\displaystyle\int \frac{(x^4+1)\ dx}{x(x^2+1)^2}$

152. $\displaystyle\int \frac{(x^2+1)\ dx}{(x^2-2x+3)^2}$

153. $\displaystyle\int \frac{x\ dx}{\sqrt{x+1}}$

154. $\displaystyle\int x\sqrt{x-a}\ dx$

155. $\displaystyle\int \frac{\sqrt{x+2}}{x+3}\ dx$

156. $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$

157. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}}$

158. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}}$

159. $\displaystyle\int x^3\sqrt{x^2+a^2}\ dx$

160. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+a^2}}$

161. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}}$

162. $\displaystyle\int \frac{x^2\ dx}{\sqrt{x^2+a^2}}$

163. $\displaystyle\int \frac{x^2\ dx}{(x^2+a^2)^2}$

164. $\displaystyle\int x\cos x\ dx$

165. $\displaystyle\int x\sin x\ dx$

166. $\displaystyle\int x\sec^2 x\ dx$

167. $\displaystyle\int x\sec x\tan x\ dx$

168. $\displaystyle\int x^2e^x\ dx$

169. $\displaystyle\int x^4\ln x\ dx$

170. $\int x^3 e^{x^2}\ dx$

171. $\int \sin^{-1} x\ dx$

172. $\int \tan^{-1} x\ dx$

173. $\int (x-1)^2 \sin x\ dx$

174. $\int \sqrt{x^2 - a^2}\ dx$

175. $\int \sqrt{x^2 + a^2}\ dx$

176. $\int \frac{x^2\ dx}{\sqrt{x^2 - a^2}}$

177. $\int e^{2x} \sin 3x\ dx$

178. $\int e^{-x} \cos x\ dx$

179. $\int \sin 3x \cos 2x\ dx$

180. $\int_0^{\frac{\pi}{8}} \cos^3(2x)\sin(2x)\ dx$

181. $\int_1^4 \frac{2^{\sqrt{x}}}{2\sqrt{x}}\ dx$

182. $\int_0^{\infty} x^3 e^{-2x}\ dx$

183. $\int_{-\infty}^{+\infty} e^{-|x|}\ dx$

184. $\int_0^{\infty} \frac{4x}{1+x^4}\ dx$

185. $\int_{-1}^{1} x^2 \cos(n\pi x)\ dx$, where $n \geq 1$, is an integer.

186. $\frac{1}{2}\int_{-2}^{2} x^2 \sin\left(\frac{n\pi x}{2}\right)\ dx$, where $n \geq 1$, is an integer.

187. $\frac{1}{L}\int_{-L}^{L} (1-x)\sin\left(\frac{n\pi x}{L}\right)\ dx$, where $n \geq 1$, is an integer and $L \neq 0$.

188. $\int_0^2 (x^3+1)\cos\left(\frac{n\pi x}{2}\right)\ dx$, where $n \geq 1$, is an integer.

189. $\int_{-1}^{1} (2x+1)\cos(n\pi x)\ dx$, where $n \geq 1$, is an integer.

190. $\frac{1}{L}\int_{-L}^{L} \sin x\ \cos\left(\frac{n\pi x}{L}\right)\ dx$, where $n \geq 1$, is an integer and $L \neq 0$.

191. A manufacturing company forecasts that the yearly demand x for its product over the next 15 years can be modelled by

$$x = 500(20 + te^{-0.1t}), \quad 0 \leq t \leq 15$$

where x is the number of units produced per year and t is the time in years (see Section **??**). What is the total demand over the next 10 years?

192. Suppose the time t, in hours, for a bacterial culture to grow to y grams is modelled by the **logistic growth model**

$$t = 25\int \frac{1}{y(10-y)}\ dy.$$

If one gram of bacterial culture is present at time $t = 0$,

(a) Solve for t

(b) Find the time it takes for the culture to grow to 4 grams

(c) Show that solving for y in terms of t gives

$$y = \frac{10}{1 + 9e^{-0.4t}}$$

(d) Find the weight of the culture after 10 hours.

NOTES:

Evaluate the following integrals using any method.

193. $\displaystyle\int \frac{\cos t}{3+\sin t}\,dt$

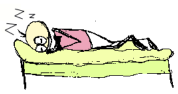
194. $\displaystyle\int \frac{\sqrt{x}}{1+2\sqrt{x}}\,dx$

195. $\displaystyle\int \frac{t^{2/3}}{1+t}\,dt$ 

196. $\displaystyle\int_0^{\pi/2} \frac{\sin 2t}{2+\cos t}\,dt$

197. $\displaystyle\int \frac{1}{2+3\sqrt{x}}\,dx$

198. $\displaystyle\int \frac{\sin x}{\tan x+\cos x}\,dx$



199. $\displaystyle\int \frac{2\sec t}{3\tan t+\cot t}\,dt$

200. $\displaystyle\int \frac{2-\sqrt{x}}{2+\sqrt{x}}\,dx$

201. $\displaystyle\int \sqrt{\frac{1+u}{1-u}}\,du$

202. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

203. $\displaystyle\int \frac{1}{2x(1-\sqrt[4]{x})}\,dx$

204. $\displaystyle\int \frac{1}{x^2(1+\sqrt[3]{x})}\,dx$ 

205. $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

206. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}\,dx$

207. $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$

208. $\displaystyle\int \frac{1}{\sqrt[3]{x}-\sqrt[4]{x}}\,dx$

209. $\displaystyle\int_0^1 \frac{1}{1+\sqrt{x}}\,dx$

210. $\displaystyle\int \sqrt{x}\sin\sqrt{x}\,dx$

211. $\displaystyle\int_0^1 \frac{1}{1+\sqrt[5]{x}}\,dx$

212. $\displaystyle\int_0^{\pi/2} \frac{1}{\cos x+2\sin x}\,dx$ 

213. $\displaystyle\int_0^{\infty} \frac{1}{x^2+\sqrt{x}}\,dx$ 

214. $\displaystyle\int_0^{\pi/4} \frac{1}{\cos 2x+\sin 2x}\,dx$ 

215. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}\,dx$ 

216. $\displaystyle\int_0^{\pi/4} \frac{1}{1+\sin^2 x}\, dx$

217. $\displaystyle\int_0^{4\pi} \frac{1}{3+\cos x}\, dx$

218. $\displaystyle\int_{-\pi}^{\pi} \frac{1}{2-\sin x}\, dx$

219. $\displaystyle\int_{2\pi}^{5\pi} \frac{\cos^2 x}{3+4\sin^2 x}\, dx$

220. The Legendre Polynomials can be defined by *Rodriguez's formula*,

$$P_n(x) = \frac{1}{n!2^n} D^n((x^2-1)^n)$$

where D^n is the n-th derivative of the function $(x^2-1)^n$. If g is any n-times differentiable function, show that

$$\int_{-1}^{1} g(x)P_n(x)\, dx = \frac{(-1)^n}{n!2^n}\int_{-1}^{1} g^{(n)}(x)(x^2-1)^n\, dx.$$

7.9 Using Computer Algebra Systems

Use your favorite Computer Algebra System (CAS), like Maple, MatLab, etc., or even a graphing calculator to answer the following questions:

1. Evaluate $\displaystyle\int_{-2}^{-1} \frac{dx}{x}$ exactly.

2. Show that the value of $\displaystyle\int \sin mt \sin nt\, dt$ where m, n are integers and $m \neq n$ does not depend on the choice of m, n. Do you get the same value if m, n are NOT integers? Explain.

3. Show that the value of $\displaystyle\int \cos mt \cos nt\, dt$ where m, n are integers does not depend on the choice of m, n. Do you get the same value if m, n are NOT integers? Explain.

4. Evaluate $\displaystyle\int_1^3 \sin x^2\, dx$ using Simpson's Rule with $n = 50$. How close is your answer to the real answer?

5. Compare the values of the integral $\displaystyle\int_0^1 \sqrt{x}\, dx$ with its approximations obatined by using the Trapezoidal Rule with $n = 25$ and Simpson's Rule with $n = 30$.

6. Estimate the value of
$$\int_0^{2\pi} \frac{dt}{2+\cos t}$$
using Simpson's Rule with $n = 20$. Is this an improper integral? Explain.

7. Find the area under the curve $y = x|\sin x|$ between $x = 0$ and $x = 4\pi$.

8. Estimate the improper integral $\int_0^\infty 2e^{-x^2}\,dx$ using Simpson's Rule with $n = 40$ over the interval $[0, 10]$. Does the number you obtain remind you of a specific relation involving π?
9. Show that $\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}$. Explain, without actually calculating the value, why this integral must be a negative number.
10. The Gamma function, $\Gamma(x)$, is defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

for $x > 0$. Show that if $n \geq 1$ is an integer, then $\Gamma(n+1) = n!$ where $n!$ represents the product of the first n numbers, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$.
11. Guess the value of the following limit per referring to the preceding exercise:

$$\lim_{x\to\infty} \frac{e^x\,\Gamma(x)}{x^{x-\frac{1}{2}}\sqrt{2\pi}}.$$

This is called *Stirling's Formula*.

NOTES:

Chapter 8

Applications of the Integral

8.1 Motivation

In this chapter we describe a few of the main applications of the definite integral. You should note that these applications comprise only a minuscule fraction of the totality of applications of this concept. Many Calculus books would have to be written in order to enumerate other such applications to the wealth of human knowledge including the social sciences, the physical sciences, the arts, engineering, architecture, etc.

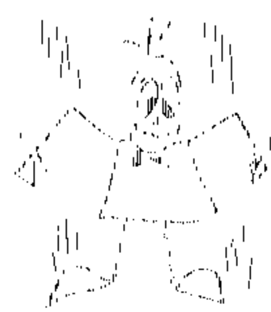
Review

You should be familiar with each one of the methods of integration desribed in Chapter 7. A thorough knowledge of those principal methods such as the Substitution Rule and Integration by Parts will help you work out many of the problems in this chapter.

Net change in the position and distance travelled by a moving body: The case of rectilinear motion.

Recall that the words "rectilinear motion" mean "motion along a line". If a particle moves with velocity, $v(t)$, in a specified direction along a line, then by physics,

$$v(t) = \frac{d}{dt}s(t)$$



where $s(t)$ is its displacement or distance travelled in time t, from some given point of reference. The *net change in position* as the particle moves from A to B from time $t = a$ to time $t = b$ is given by

$$\text{Net change in position} = \int_a^b v(t)\ dt.$$

Furthermore, the *total distance* that it travels along the line is given by

$$\text{Total distance travelled} = \int_a^b |v(t)|\ dt,$$

where, by definition of the absolute value,

$$|v(t)| = \begin{cases} v(t), & if\ v(t) \geq 0, \\ -v(t), & if\ v(t) < 0. \end{cases}$$

Example 393. A particle starting at the origin of some given coordinate system moves to the right in a straight line with a velocity $v(t) = 4\cos 2t$, for $0 \le t \le \pi$.

a) Sketch the graph of $v(t)$.

b) Sketch the graph of $s(t)$, its distance at time t.

c) Find the total distance travelled.

d) Find the net change in position.

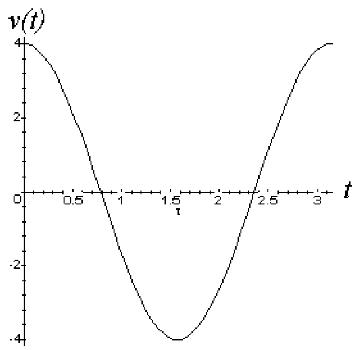


Figure 148.

Solution a) Recall that the velocity is the derivative of the distance, $s(t)$, at time t. So, the velocity is given by $s'(t) = v(t)$. The graph of this motion is given by Figure 148, but this sketch doesn't show what the particle is doing along the line, right? Still, the graph does give us some interesting information. For example, when the velocity is positive (or when the curve is above the $x-$axis), the particle moves to the right. On the other hand, when the velocity is negative (or when the curve is below the $x-$axis), the particle moves to the left. When the velocity is zero, the particle stops and may or may not 'reverse' its motion.

b) So, we need to re-interpret the information about the velocity when dealing with the line. The point is that the particle's position is really a plot of the distance, $s(t)$, rather than the velocity, right? But, from the Fundamental Theorem of Calculus, we also know that

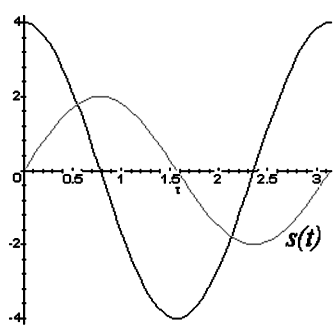


Figure 149.

$$\begin{aligned} s(t) &= s(0) + \int_0^t s'(x)\ dx, \\ &= s(0) + \int_0^t 4\cos 2x\ dx, \\ &= 2\sin 2x|_0^t, \\ &= 2\sin 2t, \end{aligned}$$

where $0 \le t \le \pi$. So, the position of this particle along the line segment is given by the graph of $s(t)$ in Figure 149. Note that when $s(t) < 0$ the particle has 'reversed' its motion (*i.e.,* the velocity has changed its sign).

c) Now, the total distance traveled is given by

$$\int_0^\pi |4\cos 2t|\ dt,$$

where

$$|4\cos 2t| = \begin{cases} 4\cos 2t, & if\ 0 \le t \le \frac{\pi}{4}\ and\ \frac{3\pi}{4} \le t \le \pi \\ -4\cos 2t, & if\ \frac{\pi}{4} \le t \le \frac{3\pi}{4} \end{cases}$$

You also noticed that the *total distance travelled* is actually equal to the *area* under the curve $y = |4\cos 2t|$ between the lines $t = 0$ and $t = \pi$, because it is given by the definite integral of a positive continuous function. Referring to Figure 148 we see that

$$\begin{aligned}
\int_0^\pi |4\cos 2t|\ dt &= \left(\int_0^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \int_{\frac{3\pi}{4}}^{\pi}\right)|4\cos 2t|\ dt,\\
&= \int_0^{\frac{\pi}{4}} 4\cos 2t\ dt - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 4\cos 2t\ dt + \int_{\frac{3\pi}{4}}^{\pi} 4\cos 2t\ dt,\\
&= \frac{4}{2}\sin 2t\Big|_0^{\frac{\pi}{4}} - \frac{4}{2}\sin 2t\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \frac{4}{2}\sin 2t\Big|_{\frac{3\pi}{4}}^{\pi},\\
&= \frac{1}{2}[(4\sin\frac{\pi}{2} - 0) - (4\sin\frac{3\pi}{2} - 4\sin\frac{\pi}{2}) + (4\sin 2\pi - 4\sin\frac{3\pi}{2})]\\
&= \frac{1}{2}[4 - 0 - (-4) + 4 + 0 + 4] = 8\ units
\end{aligned}$$

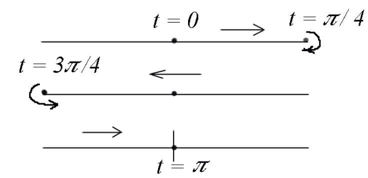


Figure 150.

d) Finally, the net change in position is simply given by $\int_a^b v(t)\ dt$, or

$$\begin{aligned}
\text{Net change in position} &= \int_0^\pi 4\cos 2t\ dt,\\
&= 4\sin 2t|_{t=0}^{t=\pi} = 4\sin 2\pi - 4\sin 0\\
&= 0 - 0 = 0
\end{aligned}$$

This means that the particle is at the same place (at time $t = \pi$) that it was when it started (at $t = 0$).

Let's recap. Imagine the following argument in your mind but keep your eyes on Figure 149. As t goes from $t = 0$ to $t = \pi/4$, the particle's distance increases and, at the same time, its speed decreases (see Figure 149). When $t = \pi/4$ its speed is zero and its distance is at a maximum (equal to 2 units). As t passes $t = \pi/4$ we see that the speed is negative and the distance is decreasing (so the particle has reversed its motion and is returning to the origin). Next, when $t = \pi/2$ we see that the particle is at the origin (since $s(\pi/2) = 0$), and its speed is -2 units/s. At this point in time the speed picks up again (but it is still negative) and so the particle keeps going left (of the origin) until its speed is zero again (which occurs when $t = 3\pi/4$). Now, at this time, the speed picks up again (it becomes positive), and so the particle reverses its motion once more but it proceeds to the right (towards the origin again) until it stops there when $t = \pi$. This argument is depicted in Figure 150.

NOTES:

8.2 Finding the Area Between Two Curves

The Big Picture

The purpose of this section is to develop machinery which we'll need to correctly formulate the **solution of an area problem using definite integrals**. In other words, we will describe a method for finding the area of an arbitrary closed region in two-dimensional space (or, if you like, the $xy-$plane). For example, what is the area of a given swimming pool? What if you're a contractor and you need to have an estimate on the amount of asphalt that a given stretch of road surface will require? You'll need to know its area first, right? How big should a solar panel be in order to take in a certain amount of solar energy? Many more questions like these can be formulated all over the place and each one requires the knowledge of a certain **area**.

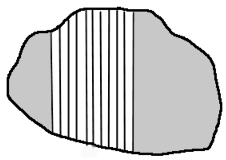


Figure 151.

Later on, we'll adapt the method in this section to problems in three-dimensional space and use the ideas here to formulate the solution of the problem of finding the volume of a specified three-dimensional region. So, the material in this section is really important for later use!

But first, we have to understand how to estimate the area of a given region bounded by two or more curves using vertical or horizontal "slices". Then we need to use this information to set up one or more definite integrals including the limits of integration. Thirdly, we use the methods of the preceding Chapter to evaluate the required integral. Of course, we can always use a formula and we'll write one down in case you prefer this particular method in order to solve the problem.

Review

Look over the material dealing with **Curve sketching**, **Inverse functions** and **Absolute values**. In particular, you should know *how to find the form of an inverse function* (when you know there is one) and you should remember the procedure for "removing an absolute value". Finally, a review of **Newton's method for finding the roots of equations** may come in handy in more general situations. This stuff is crucial if you want to know how to solve these area problems! Finally, don't forget to review the techniques of integration (by parts, substitution, etc.) covered earlier .

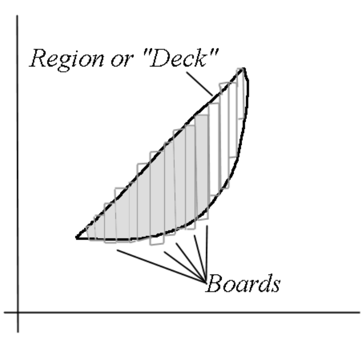


Figure 152.

Finding an area

You know that decks (the home and garden variety-type) come in all shapes and sizes. What's common to all of them, however, is that they're mostly made up of long cedar slices of different lengths all joined together in a parallel fashion and then fastened down (with nails, glue, etc. see a top-view in Figure 151). So, what's the area of this surface? You get this by finding the area of each cedar slice and adding up all the areas, right? Then *voilà*, you're done! Now, we can apply this principle to a general region in the plane. Given a closed region (a deck) call it $\mathcal{R}$, we can describe it by cutting it up into thin slices of line (you can think of cedar boards) in such a way that the totality of all such lines makes up the region $\mathcal{R}$ (and so its area can be found by adding up the areas of all the boards, see Figure 152). In this section we'll always assume that f, g are two given functions defined and continuous on their common

domain of definition, usually denoted by an interval $[a, b]$.

In the following paragraphs, it is helpful to think of a "slice" as a wooden board the totality of which make up some irregularly shaped deck. The first thing we'll do is learn how to **set up the form of the area of a typical slice** of a specified region in the plane. This is really easy if the slice is just a cedar board, right? It's the same idea in this more general setting. The things to remember are that **you need to find the coordinates of the endpoints (or extremities) of a typical slice**, that **areas are always positive numbers**, and that **the area of a rectangle is the product of its height and its width**. The symbol for the width of a "board" will be denoted by dx (or dy), in order to tie this concept of area to the definite integral (see Chapter 6).

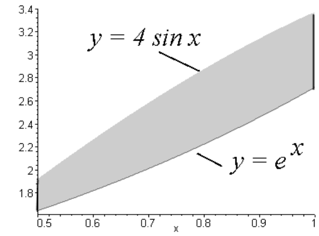


Figure 153.

Example 394. The region $\mathcal{R}$ is bounded by the curves $y = e^x$, $y = 4\sin x$, $x = 0.5$ and $x = 1$. Find the area of a typical vertical slice and the area of a typical horizontal slice through $\mathcal{R}$.

Solution A careful sketch (see Figure 153) will convince you that the curve $y = e^x$ lies *below* $y = 4\sin x$ in this range. Don't worry, this isn't obvious and you really have to draw a careful sketch of the graph to see this. The extremities of a typical vertical slice are then given by (x, e^x) and $(x, 4\sin x)$ while its width will be denoted by the more descriptive symbol, dx. So, its area is given by (see Figure 154),

$$\begin{aligned}\text{Vertical slice area} &= (\text{height}) \cdot (\text{width}) \\ &= (\text{difference in the y-coordinates}) \cdot (\text{width}) \\ &= (4\sin x - e^x)\ dx\end{aligned}$$

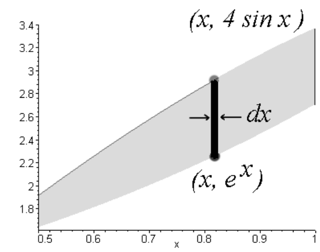


Figure 154.

When describing the *area of a horizontal slice*, the idea is to use the inverse function representation of each one of the curves, defined by some given functions, making up the outline of the region. Now, recall that to get the form of the inverse functions we simply **solve for the $x-$ variable in terms of the $y-$ variable**. The coordinates of the extremities of the slices must then be **expressed as functions of y**, (no " $x's$" allowed at all, okay?).

Now, the extremities of a typical horizontal slice (see Figure 155), are given by

$$(\text{Arcsin}\,(y/4), y) \quad \textit{and} \quad (\ln y, y),$$

provided that $y \leq e$. Why? Because if we solve for x in the expression $y = 4\sin x$ we get $x = \text{Arcsin}\,(y/4)$. Similarly, $y = e^x$ means that $x = \ln y$ so this explains the other point, $(\ln y, y)$. Combining this with the fact that the *height* of such a slice is denoted by the more descriptive symbol, dy, we get

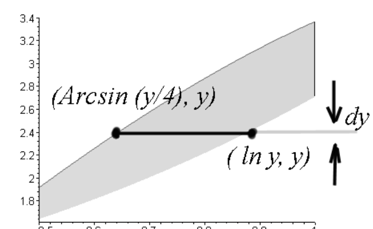


Figure 155.

$$\begin{aligned}\text{Horizontal slice area} &= (\text{width}) \cdot (\text{height}) \\ &= (\text{difference in } x\text{−coordinates}) \cdot (\text{height}) \\ &= \left(\ln y - \text{Arcsin}\left(\frac{y}{4}\right)\right)\ dy, \quad \text{if } \ y \leq e \\ &= \left(1 - \text{Arcsin}\left(\frac{y}{4}\right)\right)\ dy, \quad \text{if } \ 1 < y \leq 4\sin 1.\end{aligned}$$

Example 395. A region in the xy-plane has a vertical slice with coordinates (x, x^2e^x) and (x, e^{2x}) with $0 \leq x \leq 1$. What is its area?

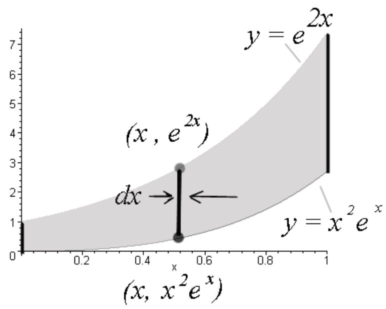


Figure 156.

Solution First, we note that if $0 \leq x \leq 1$, then $x^2\ e^x < e^{2x}$ (so that we know which one of the two points is at the top!). You get this inequality by comparing the graph of each function on the interval $0 \leq x \leq 1$, (see Figure 189). By definition, the area of a slice is given by

$$\begin{aligned} \text{Vertical slice area} &= (\text{height}) \cdot (\text{width}) \\ &= (\text{difference in the y-coordinates}) \cdot (\text{width}) \\ &= (e^{2x} - x^2e^x)\ dx \end{aligned}$$

NOTE: This is one problem you wouldn't want to convert to horizontal slices because the inverse function of the function with values $x^2\ e^x$ is very difficult to write down.

Example 396. A region in the xy-plane has a horizontal slice whose extremities have coordinates $(-\sqrt{1-y^2}, y)$ and $(\sqrt{1-y^2}, y)$, where $0 \leq y \leq 1$. Find the area of a typical slice and determine the shape of the region.

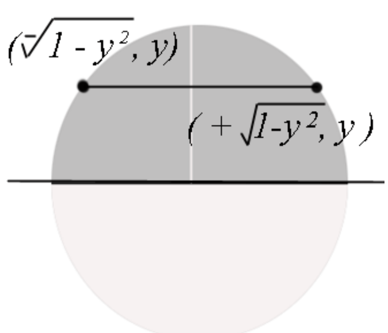


Figure 157.

Solution By definition, the area of a typical horizontal slice is given by

$$\begin{aligned} \text{Horizontal slice area} &= (\text{width}) \cdot (\text{height}) \\ &= (\text{difference in the } x\text{−coordinates})\ \ (\text{height}) \\ &= \left(\sqrt{1-y^2} - (-\sqrt{1-y^2})\right)\ dy \\ &= \left(2 \cdot \sqrt{1-y^2}\right)\ dy \end{aligned}$$

To **find the shape of the region**, just note that the $x-$coordinate of every horizontal slice is of the form $x = \sqrt{1-y^2}$. Solving for y^2 and rearranging gives the equivalent description, $x^2 + y^2 = 1$. Since $0 \leq y \leq 1$, it follows that the region is the upper half-circle (semi-circle) of radius equal to 1, (see Figure 157).

In some cases, it's not so easy to see what a typical slice looks like, because there may be more than one of them. Indeed, **there may be two, three, or more of such typical slices**. So, how do you know how many there are? Well, a *rule of thumb* is given in Table 8.1.

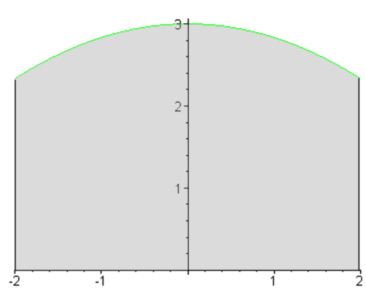


Figure 158.

Example 397. Find the area of a typical slice(s) for the (closed) region $\mathcal{R}$ bounded by the curves $y = 0$, $x = -2$, $x = 2$ and the curve $y = 3 - x^2/6$.

Solution The graph of this region is shown in Figure 158 and resembles the cross-section of an aircraft hangar. Now refer to Table 8.1. The first step is to find the points of intersection of the curves so that we can "see" the closed region and use this information to find the limits of integration. The required points of intersection are given by setting $x = \pm 2$ into the expression for $y = 3 - x^2/6$. This gives the values $y = 3 - (\pm 2)^2/6 = 7/3$. So, the "roof" of the hangar starts at the point $(-2, 7/3)$ and ends at $(2, 7/3)$.

Now let's take a horizontal slice starting along the line $y = 0$ (the bottom-most portion of $\mathcal{R}$). As we slide this slice "up" along the region $\mathcal{R}$ (see Figure 159), we see that its extremities are points of the form $(-2, y), (2, y)$ at least until we reach the curve where $y = 3 - x^2/6$. We saw above that this happens when $y = 7/3$. Now, as the

How many "typical" slices are there?

1. **Draw the region** carefully, call it $\mathcal{R}$, and **find all the points of intersection** of the curves making it up.
2. **Choose a vertical (resp. horizontal) slice** close to the left-most (resp. bottom-most) extremity, say $x = a$ (resp. $y = c$), of $\mathcal{R}$
3. **Find the coordinates** of its extremities
4. In your mind's eye (imagine this) ... Slide this slice from left to right (resp. bottom to top) through $\mathcal{R}$ and see if the coordinates of the extremities of the slice change form as you proceed through the whole region. Record the first such change, let's say that it happens when $x = x_0$ (resp. $y = y_0$)
5. Each time the coordinates of the typical slice change form as you proceed from left to right (resp. bottom to top) you repeat this rule with Item 2, above, and
6. Continue this procedure until you've reached the right-most (resp. top-most) extremity of $\mathcal{R}$.

Table 8.1: Finding the Number of Typical Slices

slice slides up through this part of the region (past $y = 7/3$), its extremities change and now have coordinates $(-\sqrt{18-6y}, y)$ and $(\sqrt{18-6y}, y)$. Why? (Just solve the equation $y = 3 - x^2/6$ for x). Eventually, we'll reach the top-most part of the region with such slices and they'll all end when $y = 3$ (the highest peak).

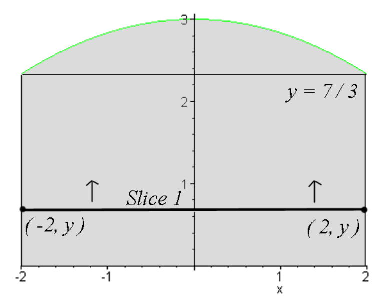


Figure 159.

Let's recap. In this case, if we use horizontal slices we get two "typical slices": Those that have endpoints of the form $(-2, y), (2, y)$ (name them Slice 1) and those that have endpoints of the form $(-\sqrt{18-6y}, y)$ and $(\sqrt{18-6y}, y)$, (call them Slice 2). See Figures 159 and 160 in the margin.

So, the area of a typical horizontal slice, Slice 1, can be found as follows:

$$\begin{aligned}\text{Area of Slice 1} &= (\text{width}) \cdot (\text{height}) \\ &= (\text{difference in the } x\text{−coordinates}) \cdot (\text{height}) \\ &= ((2) - (-2))\ dy \\ &= 4\ dy,\end{aligned}$$

and this formula for a typical Slice 1 is valid whenever $0 \le y \le 7/3$.

On the other hand, the area of typical horizontal slice, Slice 2, is

$$\begin{aligned}\text{Area of Slice 2} &= (\text{width}) \cdot (\text{height}) \\ &= (\text{difference in the } x\text{−coordinates}) \cdot (\text{height}) \\ &= 2\ \sqrt{18-6y}\ dy\end{aligned}$$

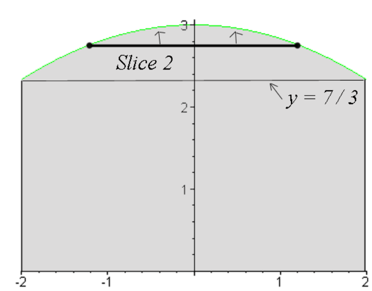


Figure 160.

and this formula for a typical Slice 2 is valid whenever $7/3 \le y \le 3$.

NOTE: We'll see below that these typical intervals $0 \le y \le 7/3$ and $7/3 \le y \le 3$ are related to the "limits of the definite integrals" which give the area of the region.

Exercise Set 40.

Find the areas of the following slices of various regions in the plane. It helps to sketch them first.

1. A vertical slice of the closed region bounded by the curves $y = x^2 - 1$ and $y = 0$, between $x = 0$ and $x = 1$.
2. A horizontal slice of the closed region bounded by the curves $y = x^2 - 1$ and $y = 0$, between $x = 0$ and $x = 1$.
3. A vertical slice of the closed region bounded by the curves $y = x^2 + 5x + 6$, $y = e^{2x}$, $x = 0$, and $x = 1$.
   • Find out which one is bigger, first!
4. **Hard**. A horizontal slice of the closed region bounded by the curves $y = x^2 + 5x + 6$, $y = e^{2x}$, and $x = 0$.
   • Be careful here, there are really *two* sets of such slices. Identify each one separately. Furthermore, you'll need to use **Newton's Method** to estimate the points of intersection of the two curves!

5. Refer to the preceding exercise: Now find the area of a typical vertical slice of the region bounded on the top by the curve $y = x^2 + 5x + 6$, on the right by $y = e^{2x}$, below by $y = 5$, and to the left by $x = 0$.
   • Once again, be careful here as there are really *two* sets of such slices. Identify each one separately.

Suggested Homework Set 32. *Work out problems 1, 3, 5*

1. Use your knowledge of curve sketching here
2. Vertical or horizontal slices, but which one? See the previous section.
3. Remember that areas are positive numbers, and the area of a rectangle is (base)(height). This part should look like $(f(x) - g(x))\,dx$ or $(F(y) - G(y))\,dy$.
4. This gives you the "limits of integration"
5. This area may be a sum of different definite integrals depending on the number of typical slices within the region.
6. Use the methods of integration of the previous sections here

Figure 161.

NOTES:

Now that we know how to set up the form of the area of a typical slice we can derive the form of the general area integral as in Table 8.3.

This is basically the way it's done! Let's look at some specific examples which combine all the steps needed in the setup and evaluation of an area integral. In these examples, steps refer to the outline in Table 8.2 at the beginning of this section.

Finding the area of a region $\mathcal{R}$ (see Figure 161)

1. **Sketch the region** $\mathcal{R}$ whose area you want to find.

2. **Divide** $\mathcal{R}$ **into "typical slices"** (like the cedar boards...)
   • Find the coordinates of each extremity of such a typical slice.

3. Find the **area of one of these typical slices** (think of each one as a very thin rectangle);
   • Its area is equal to the difference between the $y-$ or $x-$ coordinates of the extremities you found above multiplied by the width of the slice (either dx or dy);

4. If the width is dx, find the the **left-most point** $(x = a)$ and the **right-most point** $(x = b)$ of that part of $\mathcal{R}$ corresponding to the chosen slice;
   • Otherwise, the height is dy and you find the the **bottom-most point** $(y = c)$ and the **top-most point** $(y = d)$ of that part of $\mathcal{R}$ corresponding to the chosen slice.

5. **Set up the definite integral** for the area by adding up all the areas of each typical slice making up $\mathcal{R}$ and

6. **Evaluate the integral.**

Table 8.2: Finding the Area of a Region $\mathcal{R}$

Anatomy of an area integral

The area integral for a region with only one typical slice will look like either

$$\int_{\underbrace{\boxed{\text{a}}}_{\text{left-most slice}}}^{\overbrace{\boxed{\text{b}}}^{right-most\ slice}} \overbrace{\boxed{(\text{ height of a typical slice})\ dx}}^{typical\ slice\ area}$$

or

$$\int_{\underbrace{\boxed{\text{c}}}_{\text{bottom-most slice}}}^{\overbrace{\boxed{\text{d}}}^{top-most\ slice}} \overbrace{\boxed{(\text{ width of a typical slice})\ dy}}^{typical\ slice\ area}$$

Table 8.3: Anatomy of a Definite Integral for the Area Between Two Curves

Example 398. Find the area of the region $\mathcal{R}$ enclosed by the curves $y = x^2$, $y = 4 - 3x^2$ and the vertical lines $x = -1$ and $x = +1$.

Solution Refer to Table 8.2 for the overall philosophy ...

1. First, we sketch the two parabolic curves (see Figure 162) to obtain a region which looks like an inverted shield.

2. Next, we'll divide our $\mathcal{R}$ into thin *vertical* slices, say. To do this we slice $\mathcal{R}$ with a line segment which begins on the "lower curve" (namely $y = x^2$) and ends on the "upper curve", (namely, $y = 4 - 3x^2$). Now, the coordinates of each extremity of this slice are given by (x, x^2), the lower point, and $(x, 4 - 3x^2)$, the upper point.

3. Now, the area of this slice is given by multiplying its width by its height, right? So, (see Figure 163),

$$\begin{aligned}
\text{Typical slice area} &= (\text{height}) \cdot (\text{width}) \\
&= (\text{difference in the y-coordinates}) \cdot (\text{width}) \\
&= ((4 - 3x^2) - (x^2))\ dx \\
&= (4 - 4x^2)\ dx
\end{aligned}$$

Remember that you get the points of intersection by equating the x or y coordinates and then solving the resulting equation. In this case you get $y = 4 - 3x^2 = x^2 = y$ which forces $4 = 4x^2$, or $x^2 = 1$, from which $x = \pm 1$. You get the y-coordinate of these points by setting $y = x^2$ (or $y = 4 - 3x^2$) with $x = \pm 1$.

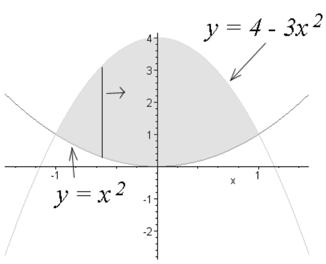


Figure 162.

4. Next, all these slices start at the left-most $x = a = -1$, right? And they all end at the right-most $x = b = +1$. Note that these two curves intersect at the two points $(-1, 1)$ and $(1, 1)$. Normally, you have to find these points!

5. Since $a = -1$ and $b = 1$, we can set up the integral for the area as

$$\begin{aligned}
\text{Sum of the areas of all slices} &= \int_{-1}^{1} (4 - 4x^2)\ dx \\
&= 4 \cdot \int_{-1}^{1} (1 - x^2)\ dx \\
&= \frac{16}{3}.
\end{aligned}$$

Example 399. Find the area enclosed by the triangle bounded by the lines $y = x$, $y = 4 - x$, and the $x-$axis (or the line $y = 0$, which is the same thing) .

Solution The region is represented as Figure 164 in the margin. Without actually sketching the graph, you could tell that it would be a triangle because the region is bounded by three straight lines. True! Of course, you're probably thinking " Why can't I just find the area of the triangle using the usual formula?". Well, you're right. Let's find its area using geometry: This gives us its area as $(1/2) \cdot (base) \cdot (height) = (1/2) \cdot (4) \cdot (2) = 4$. The area should be equal to 4, but this is only a check, alright? Because we want to know how to find the areas of general regions and not just triangles!

The next step is to find the points of intersection of all these lines, because these points are usually important in our finding the "limits of the definite integral(s)". As before, we equate the $y-$coordinates and find $4 - x = x$ from which we obtain $x = 2$ and then $y = 4 - 2 = 2$, as well. So the peak of the triangle has coordinates $(2, 2)$. The line $y = 4 - x$ intersects the line $y = 0$ when $4 - x = 0$ or $x = 4$, which now forces $y = 0$. Combining these results we see that the required region is a triangle with vertices at $(0, 0), (2, 2)$ *and* $(4, 0)$.

Next, we need to draw a "typical slice" in our triangle, right? Let's try a vertical slice again, just for practice (we'll look at horizontal slices later). All our vertical slices start on the "lower curve", namely, $y = 0$ and end on the "upper curve" given by ... ? Wait, but there are "2" such "upper curves", that is, $y = x$ and $y = 4 - x$. So, we'll have to divide our typical slice into two classes: 1) Those that end on the curve $y = x$ and 2) Those that end on the curve $y = 4 - x$ (see Figure 165). So, it looks like this problem can be broken down into a problem with "two" typical slices.

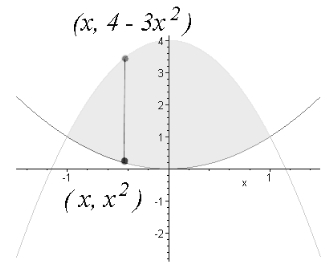


Figure 163.

Now we need to find the extremities of each of these two sets of slices. Look at the slice, call it "A", which goes from $(x, 0)$ to (x, x) (since $y = x$ on that curve). Its width is dx while its height is equal to the difference between the $y-$coordinates of its extremities, namely,

$$\textit{slice height} = (x) - (0) = x.$$

Its area is then given by

$$\begin{aligned}\text{The area of typical slice, A} &= (\text{height}) \cdot (\text{width}) \\ &= (x) \cdot (dx) \\ &= x\ dx\end{aligned}$$

Okay, now all such "A-slices" start at the point $(0, 0)$, or the line $x = 0$, and end along the line $x = 2$. This means that $a = 0, b = 2$ for these slices. The area of the triangle, Δ_1, with vertices at $(0, 0), (2, 2), (2, 0)$ is now given by

$$\boxed{\text{Area of } \Delta_1 = \int_0^2 x\ dx.}$$

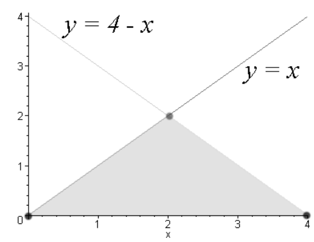


Figure 164.

Now look at the slice, call it "B", which goes from $(x, 0)$ to $(x, 4-x)$ (since, $y = 4-x$ on the other side). Its width is still dx but its height is given by

$$\textit{slice height} = (4 - x) - (0) = 4 - x.$$

The area of a "B-slice" is now given by

$$\begin{aligned}\text{The area of typical slice, B} &= (\text{height}) \cdot (\text{width}) \\ &= (4 - x) \cdot (dx) \\ &= (4 - x)\ dx\end{aligned}$$

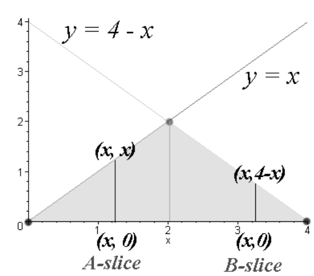


Figure 165.

In this case, all such "B-slices" start along the line $x = 2$, and end along the line $x = 4$, right? This means that $a = 2, b = 4$ for these slices. The area of the triangle, Δ_2, with vertices at $(2, 2), (2, 0), (4, 0)$ is then given by

$$\boxed{\text{Area of } \Delta_2 = \int_2^4 (4 - x)\ dx.}$$

The total area is now equal to the sum of the areas of the triangles, Δ_1, Δ_2. So, the

$$\begin{aligned}
\text{Area of the region} &= \text{Area of } \Delta_1 + \text{Area of } \Delta_2 \\
&= \int_0^2 x\ dx + \int_2^4 (4-x)\ dx = \left.\frac{x^2}{2}\right|_0^2 + \left.\left(4x - \frac{x^2}{2}\right)\right|_2^4 \\
&= \left(\frac{(2^2)}{2} - 0\right) + \left((4\cdot 4) - \frac{(4)^2}{2}\right) - \left((4\cdot 2) - \frac{(2^2)}{2}\right) \\
&= 2 + 8 - 6 = 4
\end{aligned}$$

which agrees with our earlier geometrical result!

So, why all the trouble? Because most regions are not triangles, and this example describes the idea behind finding the area of more general regions.

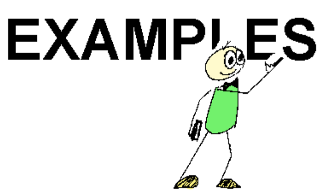
Okay, now what about horizontal slices ?

Well, this is where your knowledge of inverse functions will come in handy. The "rule of thumb" is this ...

We want the area of a region $\mathcal{R}$. There are **always two ways of getting this area**.

If $\mathcal{R}$ is "easily described" using **functions of x only** then **use vertical slices**.

On the other hand, if $\mathcal{R}$ is more "easily described" using **functions of y only** then **use horizontal slices**.



NOTE: By "easily described" we mean that the functions that are obtained are comparatively "easier to integrate" than their counterpart. For example, if the area of a typical vertical slice for a region is given by the expression $(4-4x^2)\ dx$ while the corresponding area of its counterpart horizontal slice is given by $(2\cdot\sqrt{4-y}/\sqrt{3})\ dy$ then use the vertical slice to solve the area problem, because that expression is *relatively easier* to integrate.

Example 400. Solve the area problem of Example 399 using "horizontal slices".

Solution The idea here is to use the inverse function representation of each one of the functions making up the outline of the triangle. That is, we need to find the inverse function of each of the functions $y = x$ and $y = 4 - x$. Now, to get these functions we simply **solve for the $x-$variable in terms of the $y-$variable**. The coordinates of the extremities of the slices must then be **expressed as functions of** y (no "$x's$" allowed at all!).

Okay, in our case, $y = x$ means that $x = y$ and $y = 4 - x$ implies that $x = 4 - y$. Have a look at Figure 166. You see that the extremities of the horizontal slice are (y, y), on the left, with coordinates as functions of y, and $(4 - y, y)$ on the right. All we did to get these functions of y was to "leave the $y's$ alone" and whenever we see an x we solve for it in terms of y, as we did above.

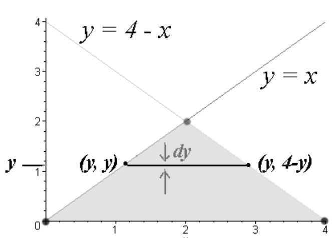


Figure 166.

Now, something really neat happens here! The horizontal slice of Figure 166 is really typical of any such horizontal slice drawn through the triangular region. This means that we only need one integral to describe the area, instead of two, as in Example 399.

Proceeding as before, we can calculate the area of this typical horizontal slice. In

fact,

$$\begin{aligned}
\text{The area of a horizontal slice} &= (\text{height}) \cdot (\text{width}) \\
&= (\text{height}) \cdot (\text{difference in the } x\text{−coordinates}) \\
&= (\text{height}) \cdot ((\text{right } x\text{−coord.}) - (\text{left } x\text{−coord.})) \\
&= (dy) \cdot ((4-y) - (y)) \\
&= (4-2y)\ dy.
\end{aligned}$$

Now, all such slices start at the bottom-most point where $y = 0$ and end at the top-most point where $y = 2$. This means that $a = 0, b = 2$. Thus, the area of our region is given by

$$\begin{aligned}
\text{Area} &= \int_0^2 (4-2y)\ dy \\
&= \left(4y - \frac{2y^2}{2}\right)\Bigg|_0^2 \\
&= (8-4) - (0-0) \\
&= 4.
\end{aligned}$$

which shows you that either method gives the same answer!

SNAPSHOTS

Example 401. Find an expression for the area of the region bounded by the curves $y = x^2$ and $y = x^2 + 5$ and between the lines $x = -2$ and $x = 1$.

Solution

- Use vertical slices (as y is given nicely as a function of x).
- Area of a typical vertical slice $= ((x^2+5) - (x^2))\ dx = 5\ dx$.
- Vertical slices start at $x = -2$ and end at $x = 1$.
- The integral for the area is given by

$$\begin{aligned}
\text{Area} &= \int_{-2}^{1} (\text{Area of a typical vertical slice}) \\
&= \int_{-2}^{1} 5\ dx.
\end{aligned}$$

Example 402. Find an expression for the area of the region bounded by the curves $y = 2x - x^2$ and $y = x^3 - x^2 - 6x$ and between the lines $x = -1$ and $x = 0$.

Solution

- Use vertical slices.
- Sketch the graphs carefully, (see Figure 167).
- Area of a typical vertical slice $= ((x^3 - x^2 - 6x) - (2x - x^2))\ dx = (x^3 - 8x)\ dx$.
- Vertical slices start at $x = -1$ and end at $x = 0$.
- The integral for the area is given by

$$\begin{aligned}\text{Area} &= \int_{-1}^{0} (\text{Area of a typical vertical slice}) \\ &= \int_{-1}^{0} (x^3 - 8x)\ dx.\end{aligned}$$

The next example is based on the preceding one...

Example 403. Find an expression for the area of the region bounded by the curves $y = 2x - x^2$ and $y = x^3 - x^2 - 6x$ and between the lines $x = 0$ and $x = 3$, (see Figure 168).

Solution

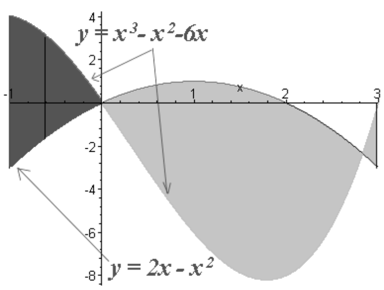


Figure 167.

- Use vertical slices.
- Sketch the graphs carefully on the interval $[0, 3]$.
- Points of intersection in $[0, 3]$: The two curves intersect when $x = \sqrt{8} < 3$.
- Two typical slices here! Slice 1: Those to the left of $x = \sqrt{8}$ and ... Slice 2: Those to the right of $x = \sqrt{8}$.
- Area of a typical vertical Slice 1 $= ((2x - x^2) - (x^3 - x^2 - 6x))\ dx = (8x - x^3)\ dx$.
- Vertical Slice 1's start at $x = 0$ and end at $x = \sqrt{8}$.
- Area of a typical vertical Slice 2 $= ((x^3 - x^2 - 6x) - (2x - x^2))\ dx = (x^3 - 8x)\ dx$.
- Vertical Slice 2's start at $x = \sqrt{8}$ and end at $x = 3$.
- The integral for the area is given by a sum of two integrals ...

$$\begin{aligned}\text{Area} &= \int_{0}^{\sqrt{8}} (\text{Area of a typical vertical Slice 1}) + \\ &\quad \int_{\sqrt{8}}^{3} (\text{Area of a typical vertical Slice 2}) \\ &= \int_{0}^{\sqrt{8}} (8x - x^3)\ dx + \int_{\sqrt{8}}^{3} (x^3 - 8x)\ dx\end{aligned}$$

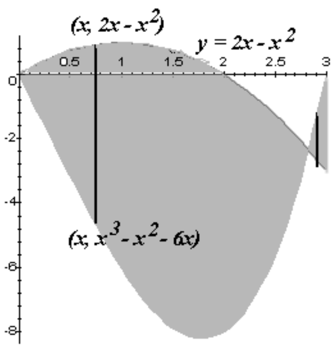


Figure 168.

Example 404. Find an expression for the area of the region bounded by the curves $y^2 = 3 - x$ and $y = x - 1$. Evaluate the integral.

Solution

- Use horizontal slices: It is easier to solve for x in terms of y.
- Sketch the region (see Figure 169).
- Points of intersection: $(2, 1)$ and $(-1, -2)$.
- Area of a typical horizontal slice: $((3 - y^2) - (y + 1))\ dy = (2 - y^2 - y)\ dy$.
- Horizontal slices start at $y = -2$ and end at $y = 1$.
- The integral for the area is given by

$$\begin{aligned}\text{Area} &= \int_{-2}^{1} (\text{Area of a typical horizontal slice}) \\ &= \int_{-2}^{1} (2 - y^2 - y)\ dy = \left(2y - \frac{y^3}{3} - \frac{y^2}{2}\right)\Bigg|_{-2}^{1} \\ &= \frac{9}{2}.\end{aligned}$$

Example 405. Let $\mathcal{R}$ be the shaded region in Figure 170. It is bounded by the two curves whose equations are $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$ where $a < b$. What is the area of this region $\mathcal{R}$?

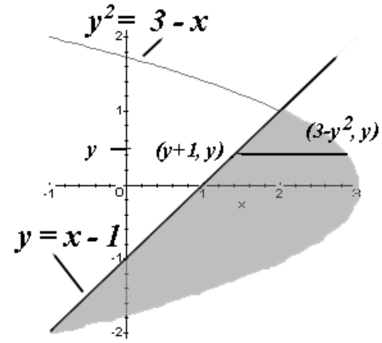


Figure 169.

Solution Well, the quick formula we spoke of is this:

The area of $\mathcal{R}$ is given by the definite integral of the absolute value of the difference of the two functions in question ...

$$\text{The area of } \mathcal{R} = \int_a^b |f(x) - g(x)|\,dx \tag{8.1}$$

You know, formulae are just that,− formulae, and you have to "know when and how to use them". So, the moral is, "If you don't want to use slices, you'll have to remember how to remove absolute values "!

If you think about it, this formula is intuitively true, right? Take a vertical slice, find its typical area (which is always a positive number) and add them all up by using the integral.

Now, there is a corresponding formula for "horizontal slices". It's not obvious though and it does need some careful assumptions due to the nature of inverse functions. You can probably believe it because you can use "horizontal slices" instead of vertical ones, etc.

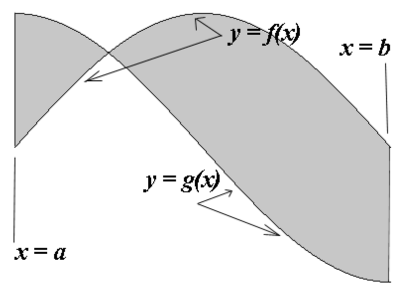


Figure 170.

Example 406. Let f, g be continuous functions defined on a common interval $[a, b]$ with the property that $f(a) = g(a)$ and $f(b) = g(b)$, (see Figure 171). If f and g are one-to-one functions on $[a, b]$ then the area of the closed region $\mathcal{R}$ bounded by these two curves is given by

$$\text{The area of } \mathcal{R} = \int_c^d |F(y) - G(y)|\,dy \tag{8.2}$$

where F, G are the **inverse functions** of f, g respectively and $[c, d]$ is the common range of f and g, that is, $c = f(a) = g(a)$, and $d = f(b) = g(b)$.

Example 407. Use the formula in Example 405 above to find the area of the region bounded by the curves $y = x$ and $y = x^3$ between the lines $x = 0$ and $x = 1$.

Solution By the result above, we know that

$$\begin{aligned}
\text{The area of } \mathcal{R} &= \int_0^1 |x - x^3|\,dx \\
&= \int_0^1 (x - x^3)\,dx, \text{ (because } x \geq x^3 \text{ on this interval)} \\
&= \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_0^1 \\
&= (\frac{1}{2} - \frac{1}{4}) - (0 - 0) \\
&= \frac{1}{4}.
\end{aligned}$$

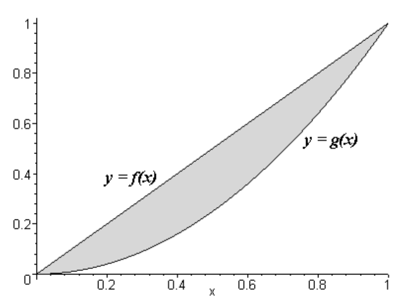


Figure 171.

Even though we didn't have to draw the graphs here, we still have to know "which one is bigger" so that we can remove the absolute value sign in the integrand. A

slight variant of the example above is given next. This is an example where we need to think about the "removal of the absolute value" process ...

Example 408. Use the formula in Example 405 above to find the area of the region bounded by the curves $y = x$ and $y = x^3$ between the lines $x = -1$ and $x = 1$.

Solution In this example, we know that (by definition of the absolute value),

$$|x - x^3| = \begin{cases} x - x^3, & \text{if} \quad x \geq x^3, \\ x^3 - x, & \text{if} \quad x \leq x^3, \end{cases}$$

or, since $x \geq x^3$ when $0 \leq x \leq 1$, and $x \leq x^3$ when $-1 \leq x \leq 0$, we get that

$$|x - x^3| = \begin{cases} x - x^3, & \text{if} \quad 0 \leq x \leq 1, \\ x^3 - x, & \text{if} \quad -1 \leq x \leq 0, \end{cases}$$

Now, we break up the domain of integration in order to reflect these different "parts". We find

$$\begin{aligned} \text{The area of } \mathcal{R} &= \int_{-1}^{1} |x - x^3| \, dx \\ &= \int_{-1}^{0} (x^3 - x) \, dx + \int_{0}^{1} (x - x^3) \, dx \\ &= \left(\frac{x^4}{4} - \frac{x^2}{2} \right) \Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_{0}^{1} \\ &= \frac{1}{4} + \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

Sometimes studying the regions for "symmetry" properties (see Chapter 5) can be very useful. In this case, the two regions (the one to the left and the one to the right of the $y-$axis) really have the same area so, finding the area of one and doubling it, gives the answer we want.

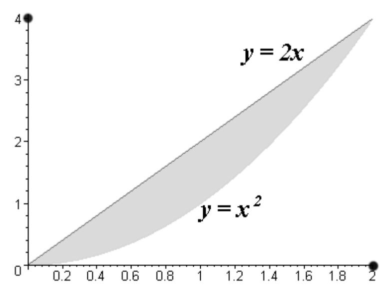


Figure 172.

Example 409. Use the formula in Example 406 above to find the area of the region $\mathcal{R}$ bounded by the curves $y = x^2$ and $y = 2x$.

Solution The points of intersection of these graphs occur when $x^2 = 2x$, that is, when $x = 0$ or $x = 2$, see Figure 172. This gives the two points $(0, 0)$ and $(2, 2)$. Note that each of the functions f, g, where $f(x) = 2x$ and $g(x) = x^2$, is one-to-one on the interval $0 \leq x \leq 2$. Furthermore, $f(0) = g(0) = 0$ and $f(2) = g(2) = 4$. Since each one of these is continuous, it follows that the assumptions in Example 406 are all satisfied and so the area of the shaded region $\mathcal{R}$ between these two curves is given by

$$\text{The area of } \mathcal{R} = \int_{0}^{4} |F(y) - G(y)| \, dy$$

where F, G are the **inverse functions** of f, g respectively and $[0, 4]$ is the common range of f and g. In this case, the *inverse functions* are given, (see Chapter 3), by the functions $F(y) = y/2$ and $G(y) = \sqrt{y}$. From the theory of Inverse Functions we

Let $\mathcal{R}$ be the region bounded by the vertical lines $x = a$, $x = b$ and between the two curves defined by the functions $y = f(x)$ and $y = g(x)$ as in Figure 170. Then the area of this region is given by

$$\text{The area of } \mathcal{R} = \int_a^b |f(x) - g(x)|\ dx. \tag{8.3}$$

Table 8.4: The Area of a Region Between Two Curves

know that the common domain of these inverses is the common range of the original functions, namely, the interval $[0, 4]$. So, according to formula in Example 406, the area of the shaded region in Figure 172 is given by

$$\begin{aligned}
\int_0^4 |F(y) - G(y)|\ dy &= \int_0^4 \left|\frac{y}{2} - \sqrt{y}\right|\ dy \\
&= \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)\ dy, \quad \text{(because } \sqrt{y} \geq \frac{y}{2} \text{ here)}, \\
&= \frac{4}{3}.
\end{aligned}$$

If we use *vertical slices* we would find that the same area is given by,

$$\int_0^2 \left(2x - x^2\right)\ dx = \frac{4}{3},$$

which agrees with the one we just found (as it should!).

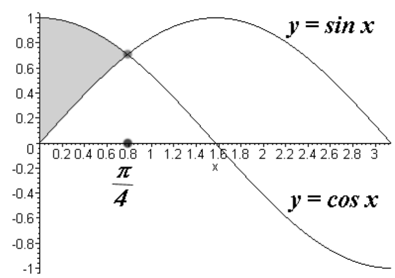


Figure 173.

Example 410. Find the form of the integral for the area of the "curvilinear triangle" in the first quadrant bounded by the $y-$axis and the curves $y = \sin x$, $y = \cos x$, see Figure 173.

Solution The points of intersection of these two curves are given by setting $\cos x = \sin x$ and also finding where $\cos x$, $\sin x$ intersect the $y-$axis. So, from Trigonometry, we know that $x = \frac{\pi}{4}$ is the *first* point of intersection of these two curves in the first quadrant. This, in turn, gives $y = \frac{\sqrt{2}}{2}$. Finally, $x = 0$ gives $y = 1$ and $y = 0$ respectively for $y = \cos x$ and $y = \sin x$. Using Table 8.4 we see that the area of this region is given by

$$\begin{aligned}
\int_a^b |f(x) - g(x)|\ dx &= \int_0^{\frac{\pi}{4}} |\sin x - \cos x|\ dx, \\
&= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\ dx, \\
&= \sqrt{2} - 1.
\end{aligned}$$

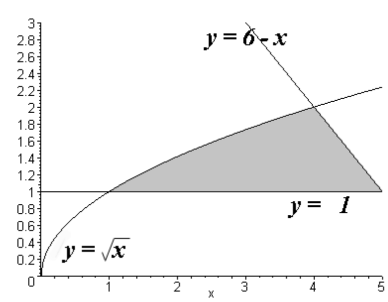


Figure 174.

Example 411. Find the form of the integral for the area of the region bounded on the right by $y = 6 - x$, on the left by $y = \sqrt{x}$ and below by $y = 1$.

Solution The first thing to do is to sketch the region, see Figure 174, because it involves more than two curves. The points of intersection of all these curves are given by $\sqrt{x} = 1 \Rightarrow x = 1$, and $6 - x = 1 \Rightarrow x = 5$, and $6 - x = \sqrt{x} \Rightarrow (\sqrt{x} + 3)(\sqrt{x} - 2) = 0$, $\Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$. The points are therefore $(1, 1), (5, 1)$, and $(4, 2)$.

This region may be split up into two smaller regions, one to the left of the line $x = 4$ and the other one to the right of $x = 4$. We may then use Table 8.4 or, what comes to the same thing, *vertical slices*, to find

$$\begin{aligned} \text{Area} &= \int_1^4 (\sqrt{x} - 1)\ dx + \int_4^5 (5 - x)\ dx, \\ &= \frac{13}{6}, \end{aligned}$$

or, if we use *horizontal slices* and take advantage of the shape of the region, we get the simpler formula for the same area, namely,

$$\begin{aligned} \text{Area} &= \int_1^2 \left(6 - y - y^2\right)\ dy, \\ &= \frac{13}{6}. \end{aligned}$$

Exercise Set 41.

Find an expression for the area of the following closed regions in the plane. It helps to sketch them first; also, you can refer to the first set of exercises in this section for additional help. Evaluate the integrals.

1. The region bounded by the curves $y = x^2 - 1$ and $y = 0$.
2. The region bounded by the curves $y = x^2 - 1$ and $y = 3$.
3. The region bounded by the curves $y = x^2 + 5x + 6$, $y = e^{2x}$, $x = 0$, and $x = 1$.
   • This is a long problem: See which function is bigger, then use Newton's method to find the points of intersection of these two functions on the required interval.
4. The region bounded by the curves $y = x^2 + 5x + 6$, $y = e^{2x}$, and $x = 0$.
   • See the preceding Exercise Set.
5. The region bounded by the curve $x = ye^y$ and the lines $y = 0$ and $y = 1$.
   • Hint: Sketch the curve $y = xe^x$ on the usual $xy-$axes first, then reflect this curve about the line $y = x$.
6. The region bounded by the curve $y = x^2 \sin x$ between $x = 0$ and $x = \pi$.
7. The region bounded by the curve $y = \cos^2 x \cdot \sin x$ between $x = 0$ and $x = \pi$.
   • Watch out here!
8. The region bounded by the curve $y = \sin 3x \cdot \cos 5x$ between the lines $x = \frac{\pi}{10}$ and $x = \frac{3\pi}{10}$.
   • This region is in the lower half-plane, but your area will still be positive!
9. The region bounded by the curves given by $x + y^2 = 2$ and $x + y = 0$.
   • The easy way is to solve for x in terms of y, sketch the graphs, and use "horizontal slices".
10. The region bounded by the curves $y = 2$, $y = -2$, $y = x + 5$ and $x = y^2$.
    • Use horizontal slices.
11. Find an expression for the area of the region enclosed by $y = \sin |x|$ and the $x-$axis for $-\pi \le x \le \pi$.
    • This graph is "V"-shaped; take advantage of symmetry.
12. Find an expression for the area of the region enclosed by the curves $y = \ \cos x$, and $y = \sin x$ for $\frac{\pi}{4} \le x \le \frac{9\pi}{4}$.

Suggested Homework Set 33. *Do problems 1, 2, 4, 6, 9*

NOTES:

8.3 The Volume of a Solid of Revolution

The Big Picture

The ideas in the preceding section can be modified slightly to solve the problem of finding the volume of a, so-called, **solid of revolution** obtained by rotating a region in the plane about an axis or even an arbitrary line. For example, if we rotate a region given by a circle of any radius lying in the plane about any line (not intersecting it), we'll get a doughnut-shaped region called a **torus**. This torus is an example of a solid of revolution because it is obtained from the original planar region by *revolving* it about an axis (any axis) by a full 2π radians. Another natural example is obtained by rotating a line segment about an axis parallel to it. This generates a cylinder as in Figure 175 whose axis of rotation is through its center. Another example is furnished by rotating the parabola whose equation is $y = x^2$ about the $y-$axis, a full $2\pi-$radians. This generates the solid in Figure 176, (think of it as full of water).

Figure 175.

Review

Review Chapter 7 on Techniques of Integration and the previous section on finding the area between two curves (the geometric way).

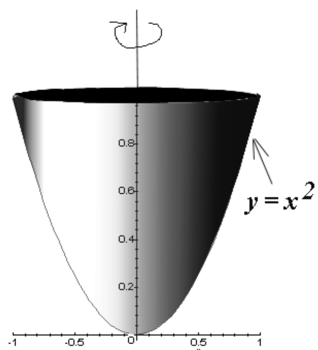


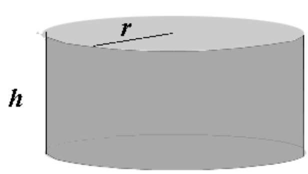
Figure 176.

The volume of the solid so obtained can be found by slicing up the solid into thin slices much like we did in the previous section, and then rotating the slices themselves about the same axis. The volume of each one of these *thin* slices can be approximated by the volume of a *thin cylinder* and the definite integral for the volume can then be found by *adding up* the volumes of each contribution. Okay, this sounds like a lot, but it's not as difficult as you think.

The principle for finding the volume of a solid of revolution is simple enough. **We can always reconstruct a solid from all its slices**, much like the collection of all the slices in a loaf of bread can be used to reconstruct the shape of the original loaf! The total volume of bread is approximately equal to the sum of the volumes of each one of the slices, right? Another analogy can be found in a roll of coins. The sum of the volumes of each coin gives the volume of the roll, and so on. The same idea is used here. The world around us is full of solids of revolution; *e.g.,* wheels, car tires, styrofoam cups, most drinking glasses, many lamp shades, eggs, any cylindrical object (pencil, high-lighter, pens, ...), tin cans, buckets, CD-s, records, etc. Of course, we could go on for quite a while and find many, many, more such objects.

Preliminaries

Let's recall the volume of simple cylindrical regions. The volume of a right circular cylinder is, the area of its base times its height, or,



$$\begin{aligned} \text{Volume of Cylinder} &= \pi\ (radius)^2\ (height), \\ &= \pi r^2 h, \end{aligned}$$

where r is its radius and h is its height, see Figure 177.

Figure 177.

OK, now starting with this simple fact we can derive the volume of a cylinder with a cylindrical hole, as in Figure 178, right? Since the hole is a cylinder in its own right, the resulting volume is given by subtracting the volume of "air" inside the hole from the volume of the cylinder. If we denote the **inner radius**, that is, the radius of the inside hole, by r_{in}, and we denote the radius of the **outer radius** by r_{out}, then the volume, V_{hole}, of the remaining solid which is a *shell*, is given by

$$\begin{aligned} V_{hole} &= \pi\, {r_{out}}^2\ (height) - \pi\, {r_{in}}^2\ (height), \\ &= \pi \left({r_{out}}^2 - {r_{in}}^2\right)\ h, \\ &= \pi(r_{out}+r_{in})(r_{out}-r_{in})\ h, \\ &= 2\pi\frac{(r_{out}+r_{in})}{2}(r_{out}-r_{in})\ h, \\ &= 2\pi(\text{average radius})\ (\text{width of wall})\ (\text{height}), \end{aligned}$$

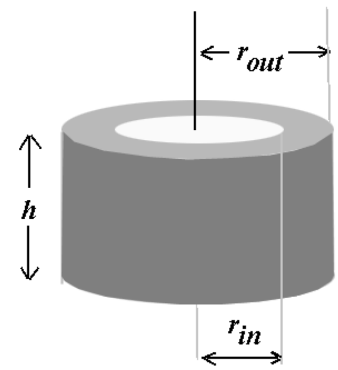


Figure 178.

and this is the key equation, that is,

$$\boxed{\begin{aligned} V_{hole} &= 2\pi(\text{average radius})\ (\text{width of wall})\ (\text{height}) \qquad (8.4) \\ &= \pi \left({r_{out}}^2 - {r_{in}}^2\right)\ h, \end{aligned}}$$

This formula (8.4) is applicable to cylinders of any orientation, size, or thinness or thickness. It is a basic generic formula which is valid for **any** cylinder with (or even without) a hole in it! See, for example, Figures 179, 180, 181 in the margins to see that this generic formula is verified regardless of the orientation or the size of the cylinder.

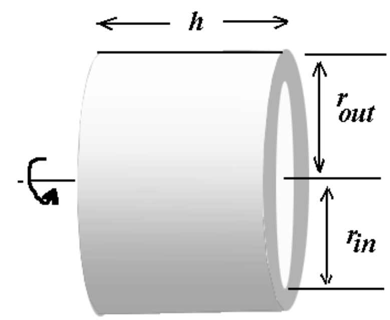


Figure 179.

Remarks The **width of the wall** of such a cylinder with a hole may be very thin and denoted by dx, (in this case, think of a tin can with a very thin metal wall), or it may be very thick so that the hole is nonexistent (in the case where $r_{in} = 0$).

When the hole is nonexistent the cylinder may look like a coin or a stack of coins. In this case, the outer radius,r_{out}, is just the radius of the cylinder (and $r_{in} = 0$). So its volume is given by the usual formula, $V = \pi r^2 h$. The point is, regardless of how thick or thin the wall is, we still use the same formula (8.4) to find the volume of the remaining cylinder (or shell). Next, the **Average Radius** is, by definition, the average of the two radii, the inner and the outer radii defined above. So, we can write

$$\text{average radius} = \frac{r_{out}+r_{in}}{2},$$

and

$$\text{wall width} = r_{out} - r_{in}.$$

The **height** is a given quantity and has nothing to do with the inner or outer radii (both of which relate together to form the "wall" mentioned above). Let's look at a few examples to see how this formula is used.

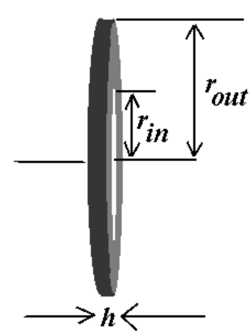


Figure 180.

Example 412. Find the volume of the cylindrical solid (of revolution) obtained by **rotating the vertical** line segment whose ends are at (x, x^2) and $(x, 1)$, and whose width is defined by the symbol dx, about the $y-axis$. Here we take it that $0 < x < 1$, is some given number.

Solution See Figure 182. The geometry of this situation tells us that

$$\begin{aligned} r_{in} &= x - dx, \\ r_{out} &= x, \\ \text{wall width} &= r_{out} - r_{in}, \\ &= dx, \\ \text{height} &= \text{Difference in the } y\text{−coordinates}, \\ &= 1 - x^2. \end{aligned}$$

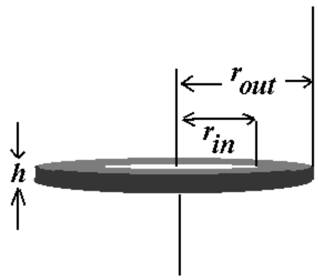


Figure 181.

The volume, let's call it dV, of this very thin shell (analogous to the amount of metal making up a tin can) is then equal to

$$\begin{aligned} dV &= 2\pi(\text{average radius}) \text{ (width of wall) (height)}, \\ &= 2\pi \frac{(x - dx) + x}{2} (dx)\ (1 - x^2), \\ &= 2\pi x(1 - x^2)dx - \boxed{\pi(1 - x^2)(dx)^2}. \end{aligned}$$

NOTE: Do you see the quadratic term in dx, on the right, the one in the box? This will be very important later, since its contribution is negligible and we'll be able to *forget about it*! We'll just keep the first-order term in dx and using this term we'll be able to write down the definite integral for a solid of revolution. It's really slick and it always works, (this idea of *forgetting* second order terms goes back to Newton and Leibniz).

Example 413. Find the volume of the cylindrical solid of revolution obtained by **rotating the horizontal line segment** whose ends are at $(0, y)$ and $(\sqrt{y}, y)$, and whose height is defined by the symbol dy, about the $y - axis$. Here we take it that $0 < y \leq 1$, is some given number.

Solution In this case, the geometry tells us that

$$\begin{aligned} r_{in} &= 0, \quad \text{(the left end-point is ON the } y\text{−axis)}, \\ r_{out} &= \sqrt{y}, \\ \text{wall width} &= r_{out} - r_{in}, \\ &= \sqrt{y}, \\ \text{height} &= dy, \end{aligned}$$

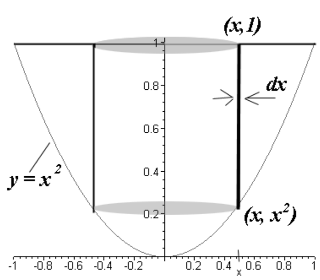


Figure 182.

So, the Volume, dV, of this very thin solid of revolution (which looks like a coin) is then equal to

$$\begin{aligned} dV &= 2\pi(\text{average radius}) \text{ (width of wall) (height)}, \\ &= 2\pi \frac{\sqrt{y} + 0}{2} (\sqrt{y})(dy), \\ &= \pi(\sqrt{y})^2 dy, \\ &= \pi y dy. \end{aligned}$$

See Figure 183 for a geometric interpretation of this question.

Example 414. Find the volume of the cylindrical solid of revolution obtained by **rotating the vertical line segment** whose ends are at $(x, 0)$ and $(x, 2x - x^2)$,

and whose width is defined by the symbol dx, about the y-axis. Here we take it that $0 < x \le 2$, is some given number.

Solution In this case, the geometry (see Figure 184) tells us that

$$\begin{aligned} r_{in} &= x - dx, \quad \text{(as } dx \text{ is the wall width)}, \\ r_{out} &= x, \\ \text{wall width} &= r_{out} - r_{in}, \\ &= dx, \\ \text{height} &= \text{Difference in the } y-\text{coordinates}, \\ &= 2x - x^2. \end{aligned}$$

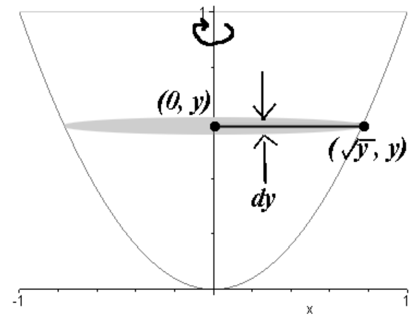


Figure 183. *Note that $r_{in} = 0$ as there is no central "hole" in this example. It follows that the wall width is the same as the radius of the thin coin-like solid generated. The height is small quantity which we denote by dy.*

So, the Volume, dV, of this very thin solid of revolution (which looks like an empty tin can) is then equal to

$$\begin{aligned} dV &= 2\pi(\text{average radius}) \text{ (width of wall) (height)}, \\ &= 2\pi\frac{2x - dx}{2}(dx)(2x - x^2), \\ &= \pi(2x - dx)(2x - x^2)\, dx, \\ &= 2\pi x(2x - x^2)dx - \pi(2x - x^2)(dx)^2. \end{aligned}$$

Example 415. Refer to Example 414 above. Find the volume of the cylindrical solid of revolution obtained by **rotating the vertical line segment** whose ends are at $(x, 0)$ and $(x, 2x - x^2)$, and whose width is defined by the symbol dx, about the x-axis.

Solution Watch out! This same vertical line segment in Figure 184 is now being rotated about the $x-$axis, OK? In this case, draw a picture of the slice rotating about the $x-$axis, and convince yourself that, this time,

$$\begin{aligned} r_{in} &= 0, \quad \text{(as there is NO hole now)}, \\ r_{out} &= \text{Difference in the } y-\text{coordinates of ends}, \\ &= (2x - x^2) - (0), \\ &= 2x - x^2, \\ \text{wall width} &= r_{out} - r_{in}, \\ &= 2x - x^2, \\ \text{height} &= dx. \end{aligned}$$

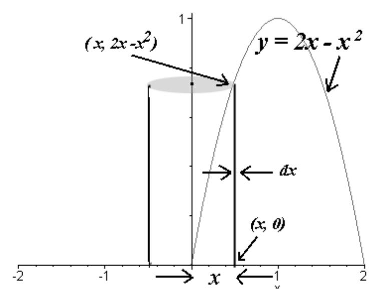


Figure 184.

It follows that the Volume, dV, of this very thin solid of revolution (which looks like a coin on its side) is equal to

$$\begin{aligned} dV &= 2\pi(\text{average radius}) \text{ (width of wall) (height)}, \\ &= 2\pi\frac{2x - x^2}{2}(2x - x^2)\, (dx), \\ &= \pi(2x - x^2)^2\, dx, \end{aligned}$$

and there is NO $(dx)^2-$term this time! This is OK, they don't *always* show up.

Example 416. Once again, refer to Example 414 above. Find the volume of the cylindrical solid of revolution obtained by **rotating the horizontal line segment**

whose ends are at $(1-\sqrt{1-y}, y)$ and $(1+\sqrt{1-y}, y)$, and whose height is defined by the symbol dy, about the x-axis.

Solution Watch out! Now we are rotating a horizontal line segment about the $x-$axis, OK? When we do this, **we have to remember to write all the coordinates of the ends in terms of the variable, y**. In this case, draw a picture of the slice rotating about the $x-$axis, and convince yourself that, this time, you get something that looks like an empty tin can on its side (like Figure 179), above, where

$$\begin{aligned} r_{in} &= y - dy, \quad \text{(a really big hole)}, \\ r_{out} &= \text{Distance from slice to } x-\text{axis}, \\ &= y, \\ \text{wall width} &= r_{out} - r_{in}, \\ &= dy, \\ \text{height} &= \text{Difference between the } x-\text{coordinates of ends}, \\ &= \left(1+\sqrt{1-y}\right) - \left(1-\sqrt{1-y})\right), \\ &= 2\sqrt{1-y}. \end{aligned}$$

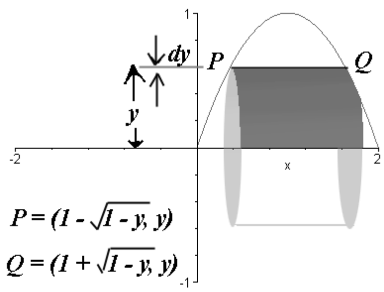


Figure 185.

So, the Volume, dV, of this very thin solid of revolution (which looks like a tin can on its side) is given by

$$\begin{aligned} dV &= 2\pi(\text{average radius})\ (\text{width of wall})\ (\text{height}), \\ &= 2\pi\frac{2y-dy}{2}(dy)(2\sqrt{1-y}), \\ &= \pi(2y-dy)(2\sqrt{1-y})\ dy, \\ &= 4\pi y\sqrt{1-y}\ dy - \boxed{2\pi\sqrt{1-y}\ (dy)^2} \end{aligned}$$

and now there is a $(dy)^2-$term! Don't worry, this is OK, we'll forget about it later! Compare your picture of this slice with Figure 185.

Example 417. One last time, refer to Example 414 above. Find the volume of the cylindrical solid of revolution obtained by **rotating the horizontal line segment** whose ends are at $(1-\sqrt{1-y},\ y)$ and $(1+\sqrt{1-y},\ y)$, and whose height is defined by the symbol dy, about the y-axis.

Solution OK, now we are rotating a horizontal line segment about the $\boldsymbol{y}-$**axis**! Just like before, **we have to remember to write all the coordinates of the ends in terms of the variable, y**. Once again, we draw a picture of the slice rotating about the $y-$axis, and you'll note that this time, you get something that looks like a very thin washer, one with a hole in the middle. Now, the calculation is the same but there won't be a $(dy)^2$-term here. We note that the center of the washer is roughly around the point $(0, y)$. So,

$$\begin{aligned} r_{in} &= 1-\sqrt{1-y}, \ \text{(distance from } (0,y) \text{ to } (1-\sqrt{1-y},\ y)), \\ r_{out} &= 1+\sqrt{1-y}, \ \text{(distance from } (0,y) \text{ to } \textit{outer rim}), \\ \text{wall width} &= r_{out} - r_{in}, \\ &= 2\sqrt{1-y}, \\ \text{height} &= dy, \end{aligned}$$

and the Volume, dV, of this very thin solid of revolution (which looks like a very

thin washer) is given by

$$\begin{aligned}
dV &= 2\pi(\text{average radius})\ (\text{width of wall})\ (\text{height}),\\
&= 2\pi\frac{\left(1-\sqrt{1-y}\right)+\left(1+\sqrt{1-y}\right)}{2}(2\sqrt{1-y})\ (dy),\\
&= 2\pi\left(\frac{1+1}{2}\right)2\sqrt{1-y}\ dy,\\
&= 4\pi\ \sqrt{1-y}\ dy,
\end{aligned}$$

and there is no $(dy)^2-$term! Compare your picture of this slice with Figure 186.

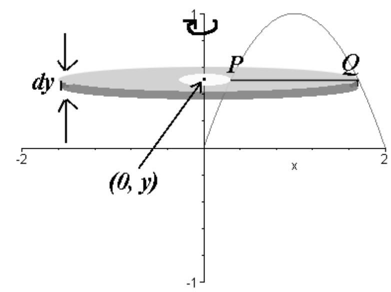


Figure 186.

Finding the Volume of a Solid of Revolution

Now that we know how to find the volume of a slice when it is rotated about either one of the principal axes (x or y), we can produce the volume of the whole solid of revolution using a definite integral. The neat thing about the method we're using is that **we DON'T have to draw the three-dimensional solid whose volume we want!** All we need is some closed planar region (called a projection or profile) as a starting point. Let's look at how this is done.

Example 418. Find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$, the $y-$axis and the line $y = 1$ about the $y-$axis.

Solution Refer to Example 412 and Figure 182. The region whose volume we wish to find looks roughly like the solid Figure 176, when it is rotated about the $y-$axis. The procedure for finding this volume is described in the adjoining Table 8.5.

1. **Sketch the region**. This was already done in Figure 182.
2. **Decide on a typical slice**. Let's try a vertical slice as in Example 412.
3. **Find the volume of the slice**. This was done in Example 412. Indeed, its volume is given by the expression

$$dV = 2\pi x(1-x^2)\ dx - \boxed{\pi(1-x^2)\ (dx)^2} \tag{8.5}$$

4. **Find the limits of integration**. Okay, all such slices begin at $x = 0$ and end at $x = 1$ (because we want to describe the whole region). So, the limits of integration are $x = 0$ and $x = 1$.
5. **Write down the definite integral for the volume**. Remember to **DROP** any $(dx)^2$ or $(dy)^2$ terms! In this case we refer to (8.5), drop the square term in the box, and write the definite integral for the volume as

$$Volume = \int_0^1 2\pi x(1-x^2)\ dx.$$

6. **Evaluate the definite integral**. This integral is straightforward as it is a polynomial. Its value is found as

$$\begin{aligned}
Volume &= \int_0^1 2\pi x(1-x^2)\ dx,\\
&= 2\pi\int_0^1 (x-x^3)\ dx,\\
&= \boldsymbol{\frac{\pi}{2}}.
\end{aligned}$$

Finding the Volume of a Solid of Revolution

(a) **Sketch the region** Use the methods of Chapter 5.

(b) **Decide on a typical slice** The rule of thumb here is just like the one for areas in the previous section: For rotation about either the $x-$ or $y-$axis,

   i. If it easier to describe the region with functions of x, use a vertical slice, see Examples 412, 414, 415, otherwise

   ii. Write all the expressions as functions of y (by finding the inverse functions) and use a horizontal slice. See Examples 413, 416, 417.

(c) **Find the volume of the slice, dV** Use Equation (8.4) and the Examples in this section

(d) **Find the limits of integration** These are obtained by finding the *extremities* of the region, see the Examples.

(e) **Write down the definite integral for the volume** DROP all terms containing the square of either dx or dy from the expression for dV in item 3c.

(f) **Evaluate the definite integral** Use the methods of Chapter 8.

Table 8.5: Setting up the Volume Integral for a Solid of Revolution

This value is the volume of the full (of water, oil ?) solid of revolution which looks like Figure 176.

Example 419. Find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$, the $y-$axis and the line $y = 1$ about the $y-$axis, using **horizontal slices!**.

Solution We just worked out this one in Example 418 using vertical slices. The nature of the problem makes it clear that it shouldn't matter whether we choose vertical or horizontal slices, right? Both should theoretically give the same answer! The region still looks like the solid Figure 176, when it is rotated about the $y-$axis. This time we use horizontal slices, as required. Fortunately, we found the volume of such a typical horizontal slice in Example 413, namely

$$dV = \pi y \, dy,$$

and, in order for all these slices to cover the whole region, we note that all such horizontal slices start at $y = 0$ and end at $y = 1$. It follows that the definite integral for the volume is given by

$$\begin{aligned} Volume &= \int_0^1 \pi y \, dy, \\ &= \pi \int_0^1 y \, dy, \\ &= \boldsymbol{\frac{\pi}{2}}, \end{aligned}$$

which, of course, *agrees* with the answer we found in the previous problem using

vertical slices!.

Example 420. Find the volume of the cylindrical solid of revolution obtained by rotating the region bounded by the curves $y = 2x - x^2$ and $y = 0$ about the $\boldsymbol{y-axis}$, using vertical slices.

Solution We prepared the solution of this problem in Example 414 by finding the volume of a typical vertical slice, dV, where

$$dV \quad = \quad 2\pi x(2x - x^2)\ dx - \pi(2x - x^2)(dx)^2,$$

We drop the $(dx)^2-$term as required and note that all such slices start at $x = 0$ and end at $x = 2$, (see Figure 184). So, the expression for the volume is given by

$$\begin{aligned} Volume &= \int_0^2 2\pi x(2x - x^2)\ dx, \\ &= 2\pi \int_0^2 (2x^2 - x^3)\ dx, \\ &= \boldsymbol{\frac{8\pi}{3}}. \end{aligned}$$

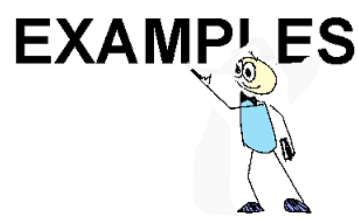
Example 421. Find the volume of the cylindrical solid of revolution obtained by rotating the region bounded by the curves $y = 2x - x^2$ and $y = 0$ about the $\boldsymbol{x-axis}$.

Solution Remember that we are rotating about the $x-$axis! We examined this problem in Example 415 by finding the volume of a typical vertical slice, dV, where

$$dV \quad = \quad \pi(2x - x^2)^2\ dx$$

Note that, in this case, all such slices start at $x = 0$ and end at $x = 2$, (see Figure 184). So, the expression for this volume, is given by

$$\begin{aligned} Volume &= \int_0^2 \pi(2x - x^2)^2\ dx, \\ &= \pi \int_0^2 (4x^2 - 4x^3 + x^4)\ dx, \\ &= \boldsymbol{\frac{16\pi}{15}}. \end{aligned}$$



This answer differs from that in Example 419 because now we are rotating about the $x-$axis and not the $y-$axis! The solids even "look different".

Example 422. Find the volume of the cylindrical solid of revolution obtained by rotating the region bounded by the curves $y = 2x - x^2$ and $y = 0$ about the $\boldsymbol{x-axis}$, using horizontal slices.

Solution We are still rotating about the $x-$axis but this time we are using *horizontal* slices! So, our answer must be exactly the same as the one we found in Example 8.3. We examined this problem in Example 416 by finding the volume of a typical horizontal slice, dV, where

$$dV \quad = \quad 4\pi y\sqrt{1-y}\ dy - \boxed{2\pi\sqrt{1-y}\ (dy)^2}.$$

We drop the boxed term as usual, and note that, in this case, all such horizontal slices start at $y = 0$ and end at $y = 1$, (see Figure 185). So, the expression for this

volume, is given by

$$\begin{aligned}
Volume &= \int_0^1 4\pi y\sqrt{1-y}\ dy, \\
&= 4\pi \int_0^1 y\sqrt{1-y}\ dy, \\
&= -4\pi \int_1^0 (1-u)\sqrt{u}\ du, \text{ (use the substitution } 1-y=u, \text{ etc.)}, \\
&= 4\pi \int_0^1 \left(u^{1/2}-u^{3/2}\right)\ du, \text{(and use the Power Rule for Integrals)} \\
&= 4\pi \left(\frac{2}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right)\Big|_0^1, \\
&= 8\pi\left(\frac{1}{3}-\frac{1}{5}\right), \\
&= \mathbf{\frac{16\pi}{15}}.
\end{aligned}$$

Example 423. Determine the volume of the solid of revolution obtained by rotating the region bounded by the line $y=0$ and the curve $y=2x-x^2$ about the $y-$axis.

Solution Let's use a horizontal slice: see Figure 186 for such a typical slice. Its volume is given in Example 417 as

$$dV = 4\pi\ \sqrt{1-y}\ dy,$$

and the limits of integration are given by noting that all such slices start at $y=0$ and end at $y=1$. It follows that the volume of the resulting solid of revolution is given by

$$\begin{aligned}
Volume &= 4\pi \int_0^1 \sqrt{1-y}\ dy, \\
&= -4\pi \int_1^0 \sqrt{u}\ du, \text{ (use the substitution } 1-y=u, \text{ etc.)}, \\
&= 4\pi \int_0^1 u^{1/2}\ du, \text{(and use the Power Rule for Integrals)} \\
&= 4\pi \left(\frac{2}{3}u^{3/2}\right)\Big|_0^1, \\
&= 8\pi\left(\frac{2}{3}-0\right), \\
&= \mathbf{\frac{8\pi}{3}},
\end{aligned}$$

in accordance with the answer given in Example 420 where we used vertical slices in order to solve the problem.

SNAPSHOTS

Example 424. A region is bounded by the curves $y=\ln x$, the lines $x=1$, $x=2$ and the $x-$axis. Find an expression for the volume of the solid of revolution obtained by revolving this region about the $x-$axis. DO NOT EVALUATE the integral.

Solution A typical vertical slice has the end-points $(x, 0)$ and $(x, \ln x)$. When this slice is rotated about the $x-$axis its volume, dV, is given by

$$dV = \pi(\ln x)^2 \, dx.$$

So, the volume of the solid of revolution is given by adding up all the volumes of such slices (which begin at $x = 1$ and end at $x = 2$). This gives rise to the definite integral,

$$Volume = \pi \int_1^2 (\ln x)^2 \, dx,$$

and this must be integrated using Integration by Parts.

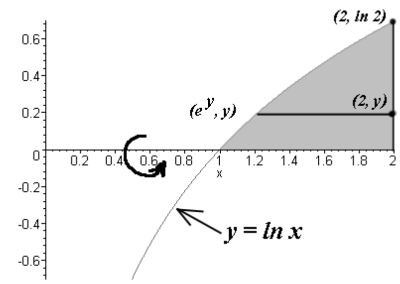


Figure 187.

What if we had used a horizontal slice? See Figure 187: In this case we write $x = e^y$ (using the inverse function of the natural logarithm), and so such a typical horizontal slice has end-points (e^y, y) and $(2, y)$. We also have $r_{in} = y - dy$, $r_{out} = y$, wall width $= dy$ and height $= 2 - e^y$, where $0 \leq y \leq \ln 2$, (since this is the interval which corresponds to the original interval $1 \leq x \leq 2$, when we set $y = \ln x$). So, the limits of integration are from $y = 0$ to $y = \ln 2$. Neglecting terms in $(dy)^2$ we find the following expression for the volume of the resulting solid:

$$Volume = 2\pi \int_0^{\ln 2} \left(2y - ye^y\right) \, dy.$$

In case you want to practice your integration techniques, the common answer to these two integrals is $\pi \left(2(\ln 2)^2 - 4\ln(2) + 2\right) \approx 0.5916$.

Example 425. Find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$ and $y = 2x$ about the $\boldsymbol{x}-$**axis**.

Solution Sketch the region and choose a typical vertical slice. Its endpoints have coordinates (x, x^2) and $(x, 2x)$. Note that $y = 2x$ lies above $y = x^2$. Their points of intersection are $x = 0$ and $x = 2$, and these give the limits of integration. In this case, $r_{in} = x^2$, $r_{out} = 2x$, wall width $= 2x - x^2$ and height $= dx$. So, the required volume is given by

$$\begin{aligned} Volume &= 2\pi \int_0^2 \left(4x^2 - x^4\right) \, dx, \\ &= \frac{128\pi}{15}. \end{aligned}$$

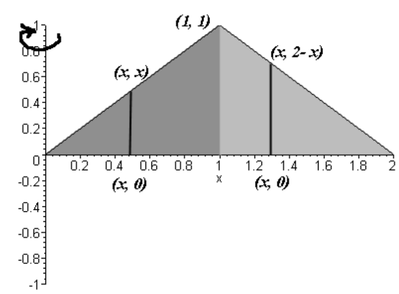


Figure 188.

Example 426. A region is bounded by the curves $y = x$, $y = 2 - x$, and the $x-$axis. Find the volume of the solid of revolution obtained by revolving this region about the $y-$axis.

Solution This region looks like an inverted "v" (it is also similar to the Greek upper case letter, Λ, called *Lambda*) and is, in fact, a triangle with vertices at $(0, 0)$, $(1, 1)$, and $(2, 0)$, (see Figure 188). If you decide to choose a vertical slice you'll need to set up two definite integrals as there are two typical vertical slices, one typical slice lies to the left of $x = 1$ and one lies to the right of $x = 1$. The slices' end-points are respectively, $(x, 0)$, (x, x) and $(x, 0)$, $(x, 2 - x)$. Notice that in both cases, we have $r_{in} = x - dx$, and $r_{out} = x$. The heights change, that's all. The volume of

the required solid is given by adding up the two integrals, (remember to drop the $(dx)^2-$terms),

$$\begin{aligned} Volume &= 2\pi \int_0^1 x^2\ dx + 2\pi \int_1^2 x(2-x)\ dx, \\ &= \frac{2\pi}{3} + \frac{4\pi}{3}, \\ &= 2\pi. \end{aligned}$$

The best way to solve this problem is by using a horizontal slice. This is because the geometry of the picture makes it clear that only ONE such slice is needed. Writing all required expressions as functions of y (by solving all $x's$ in terms of y), we find that a typical horizontal slice has end-points (y, y) and $(2-y, y)$. In this case, $r_{in} = y$, $r_{out} = 2-y$, $Wall\ width = 2-2y$, height $= dy$. So, the required volume is given by

$$\begin{aligned} Volume &= 2\pi \int_0^1 (2-2y)\ dy, \\ &= 2\pi, \end{aligned}$$

as we expect (since the answers must be the same).

$\boxed{\textbf{Example 427.}}$ Find an expression for the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = 4\sin x$, $y = e^x$, $x = 0.5$ and $x = 1$ about the $y-$axis. Evaluate your expression to three significant digits.

Solution Use a vertical slice, see Figure 154. In this case we see that the volume, dV of the small solid of revolution obtained when the slice is rotated about the $y-$axis is

$$dV = 2\pi\ x\ (4\sin x - e^x)\ dx - \pi(4\sin x - e^x)(dx)^2,$$

and so the volume of required the solid of revolution is given by

$$\begin{aligned} Volume &= \int_{0.5}^1 2\pi\ x\ (4\sin x - e^x)\ dx, \\ &= 8\pi \int_{0.5}^1 x\sin x\ dx - 2\pi \int_{0.5}^1 xe^x\ dx, \end{aligned}$$

and two successive "integrations by parts" give

$$\begin{aligned} Volume &= 8\pi \int_{0.5}^1 x\sin x\ dx - 2\pi \int_{0.5}^1 xe^x\ dx, \\ &= 8\pi(\sin x - x\cos x)\ |_{0.5}^1 - 2\pi(xe^x - e^x)\ |_{0.5}^1, \\ &= 2.0842\pi - 1.6487\pi, \\ &= 0.4355\pi = 1.3681. \end{aligned}$$

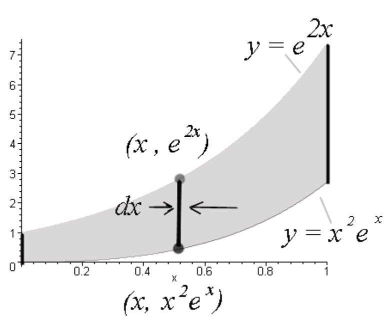


Figure 189.

$\boxed{\textbf{Example 428.}}$ Find an expression for the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2e^x$, $y = e^{2x}$, $x = 0$ and $x = 1$ about the $x-$axis. DO NOT EVALUATE THE INTEGRAL.

Solution The region is sketched in Figure 189 using a vertical slice. In this case,

$$\begin{aligned} r_{in} &= x^2e^x \\ r_{out} &= e^{2x} \\ \text{width of wall} &= e^{2x} - x^2e^x, \\ \text{height} &= dx. \end{aligned}$$

So, the volume of the solid of revolution is given by

$$Volume = \pi \int_0^1 \left(e^{4x} - x^4 e^{2x}\right)\, dx,$$

which, incidentally, can be integrated without much difficulty (the first integral is easy, the second integral is done using integration by parts). If you DO decide you want to integrate this expression your answer should be

$$Volume = \pi \left(\frac{1}{4}e^4 - \frac{1}{4}e^2 + \frac{1}{2}\right).$$

NOTES:

Exercise Set 42.

Find the form (do not evaluate) of the definite integral for the solid of revolution obtained by rotating the following regions about the specified axis.

1. The region bounded by the curve $y = x$, the $x-$axis, and the line $x = 1$. Rotate this about the $x-$axis.
2. The loop enclosed by the curves $y = x^2$ and $y = x$. Rotate this about the $y-$axis.
3. The region bounded by the curves $y = x$, $y = 2x$, and the line $x = 1$ on the right. Rotate this about the $x-$axis.
   • *Hint:* Use a vertical slice.
4. The triangular region bounded by the curve $y = 2x$, the $x-$axis and the lines $x = 0$ and $x = 2$. Rotate this about the $y-$axis.
5. The region bounded by the curves $y = x$, $y = 2x$, and the line $x = 1$ on the right. Rotate this about the $y-$axis.
   • Use both types of slices here: A vertical one first as it is easier to set up, and then use a horizontal slice (dividing the region into two pieces). Base yourself on Example 425.
6. Evaluate the definite integral for the solid of revolution obtained by rotating each one of the previous regions about the specified axis.

Suggested Homework Set 34. *Do problems 1, 3, 5*

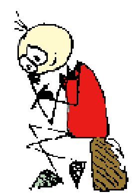
NOTES:

8.4 Measuring the length of a curve

The Big Picture

Another important application of the integral consists in finding the length of a curve in the plane (we call these *planar curves*). By the *plane* we usually mean the ordinary $xy-$plane (it is also called the *Cartesian plane*), but it *can* be any *other* two-dimensional plane as well, in that different *coordinate systems* may be used on that plane. There is then a slight change in the formula for the expression of its length (but don't worry about this now, we'll see this much later). Now, all we really need to know in order to handle the material in this section is a working knowledge of how to manipulate a *square root* function and some algebra (see Chapter 1). The reason we may want to calculate the length of a curve is because it may be representing the length of the path of an object (car, satellite, animal, etc) or it may be the length of a strand of DNA. We will motivate the main result in this section with a topic from astronomy.

Long, long ago in a land not far away, Sam Shmidlap discovered an asteroid one evening while observing the night sky. Since Sam had a lot of time on his hands he decided he was going to find out how far this asteroid had travelled over the next few days. Now, Sam thought the orbit of this asteroid was a parabola (it really isn't, but in reality it's pretty close to one). So, all he had to do was to find three reference points to determine this parabola completely, right? (see the Chapter on numerical integration and, in particular, Simpson's Rule, Section **??**). Okay, so he observed the sky, and using a standard Cartesian coordinate system (we'll leave out the details for now) he wrote down its position at three consecutive time intervals. Now that he knew the actual equation of the parabola he called on his friend, Sama, who is a Calculus wiz and asked her to find the length of this celestial parabolic arc that he had just found. She, of course, remembered all the results of this section and provided him with a speedy answer. He then bought her flowers. So the story goes ...



Sounds too theoretical? Let's use some numbers... We'll assume, for simplicity, that Sam's parabola is given by a quadratic equation of the form

$$f(t) = at^2 + bt + c, \quad 0 \le t \le 3,$$

where t is measured in days, say, and a, b, c are some numbers which arise from the observed position of the celestial object. In this case there are three consecutive days, and the interval from $t = 0$ to $t = 1$ represents the time interval corresponding to the first complete day, etc. Now comes Sama who looks at this parabola and writes down the equation of its *length* using the **arc-length formula** (which we'll see below), namely,

$$\text{The arc length of } f(t) \text{ between } t = a \text{ and } t = b \quad = \quad \int_a^b \sqrt{1 + (f'(t))^2}\ dt$$

where, in this case,

$$\begin{aligned} \int_0^3 \sqrt{1 + (f'(t))^2}\ dt &= \int_0^3 \sqrt{1 + (2at + b)^2}\ dt \\ &= \frac{1}{2a} \int_b^{6a+b} \sqrt{1 + u^2}\ du, \end{aligned}$$

where she used the substitution $u = 2at + b$ (note that a, b are just *numbers*) in the definite integral in order to simplify its form. Since Sama never worries, she then used the *trigonometric substitution* $u = \tan\theta$ to bring the last integral to the form

$$\frac{1}{2a} \int_b^{6a+b} \sqrt{1 + u^2}\ du \quad = \quad \frac{1}{2a} \int_c^d \sec^3\theta\ d\theta,$$

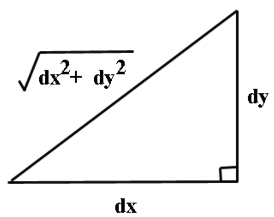


Figure 190.

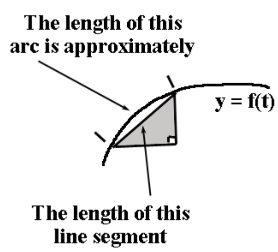


Figure 191.

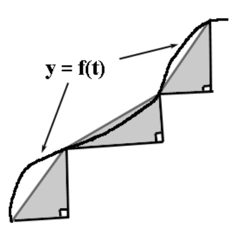


Figure 192.

where now $c = \text{Arctan}\, b$ and $d = \text{Arctan}\,(6a + b)$ (remember these *inverse functions* ?). This last integral in θ was already solved using the methods of Chapter 8, see Example 351.

This simple example shows that the length of the arc of a planar curve has one main interpretation; that is, it represents a physical *length*, such as a distance travelled, or the actual length of a road section, or path, or string, etc. Let's see how we can derive this *simple-to-write-down-but-not-so-easy-to-use* arc length formula. We won't do this rigourously right now, but it should be *believable*.

Review

You'll need to remember the **Theorem of Pythagoras** on the connection between the lengths of the three sides of a right-angled triangle. Next, you'll need to review your techniques of integration, especially **trigonometric substitutions**, (Section 7.6). If these fail to give you some answer then you can use **numerical integration**, which is always good for an approximation to the actual answer.

Let's motivate the derivation of the arc-length formula, the one Sama used in our introduction, above. To see this, all we need to use is the Theorem of Pythagoras applied to a right-angled triangle with sides dx, dy and $\sqrt{dx^2 + dy^2}$, see Figure 190.

We'll be applying this easy formula to the calculation of the sums of the lengths of a sequence of line segments that can be used to approximate the required length of an arc of a given *nice* curve (see Figure 191). The smaller the arc the better the approximation. Since every finite arc of a curve can be thought of as being composed of the union of a finite number of *smaller* arcs (just like a ruler is made up of a finite number of sections), its length is then the sum of the lengths of all these smaller arcs. So its length can be approximated as the sum of the lengths of the line segment approximations to each one of the smaller arcs, see Figure 192. The point is that we just apply the construction in Figure 191 over and over again on every smaller arc until we get something like Figure 192. In this business, remember that *a straight line is also a curve.*

So, a combination of the ideas in Figs. 190, 191 shows that the length of a really small arc in the graph of a differentiable curve is given by the expression

$$
\begin{aligned}
\text{Length of a tiny arc} &\approx \sqrt{dx^2 + dy^2}, \\
&\approx \sqrt{dx^2 \left(1 + \frac{dy^2}{dx^2}\right)}, \\
&\approx dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}, \\
&\approx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.
\end{aligned}
$$

Now, the amazing thing about Leibniz's notation for the *derivative* of a function is that it allows us to interpret the symbol $\frac{dy}{dx}$ appearing in the last equation as the derivative $y'(x)$ of the function. The point is that even though we started out with dx, dy as being independent numbers (actually *variables*), their quotient can be interpreted as the derivative of the function y.

Since every such arc is a sum of a large number of smaller or *tiny* arcs, the total length of the curve given by $y = f(t)$ between the points $t = a$ and $t = b$ can be thought of as being given by a *definite integral*. This means that the length of a plane curve given by $y = f(t)$ from $t = a$ to $t = b$, provided f is differentiable is denoted by the symbol, $L(f; a, b)$, where this symbol is defined in Equation (8.6) below:

$$L(f; a, b) \quad = \quad \int_a^b \sqrt{1 + (f'(t))^2}\, dt. \tag{8.6}$$

As usual, it doesn't matter whether we denote the free variables in the definite integral above by t or by x, or by any other symbol as this will always give the same value. For example, it is true that

$$L(f; a, b) \quad = \quad \int_a^b \sqrt{1 + (f'(x))^2}\, dx. \tag{8.7}$$

Okay, now so far we are convinced that if $y = f(x)$ is a differentiable curve defined on the interval $[a, b]$, then its length is given by (8.7). The actual proof of this result will appear on the web site.

But, you see, whether y is a function of x or x is a function of y doesn't really enter the picture in our explanation of the two basic formulae, (8.6), (8.7) above. We could just interchange the role of these two variables and write x whenever we see y and y whenever we see x. This then gives us the following result.

If $x = F(y)$ is a differentiable curve defined on the interval $[c, d]$, then its length is given by (8.8), This formula is good for finding the length of curves defined by the *inverse*, F, of the function, f, or any other function $x = F(y)$.

$$L(F; c, d) \quad = \quad \int_c^d \sqrt{1 + (F'(y))^2}\, dy. \tag{8.8}$$

Example 429.

Find the length of the line segment whose equation is given by $y = 3x + 2$ where $-2 \le x \le 4$.

Solution Here, $f(x) = 3x + 2$ is a differentiable function whose derivative is given by $f'(x) = 3$. Since y is given in terms of x we use (8.7) for the expression of its length. In this case, $a = -2$ and $b = 4$. This gives us (see Figure 193),

$$\begin{aligned} L(f;-2,4) &= \int_{-2}^{4} \sqrt{1+(f'(x))^2}\,dx, \\ &= \int_{-2}^{4} \sqrt{1+3^2}\,dx, \\ &= \sqrt{10}\int_{-2}^{4} dx, \\ &= 6\sqrt{10}. \end{aligned}$$

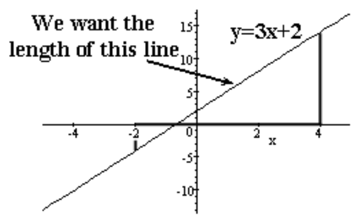


Figure 193.

Check

Let's check this answer using simple geometry. Look at Figure 194 in the margin. The triangles ABC and CDE are each right-angled triangles and so their hypotenuse is given by the Theorem of Pythagoras. Note that the point C coincides with $x = -\frac{2}{3}$. It follows that the length of the hypotenuse CD is given by

$$\begin{aligned} CD^2 &= DE^2 + CE^2 \\ &= 14^2 + \left(4 - (-\frac{2}{3})\right)^2 \\ &= 14^2 + \left(\frac{14}{3}\right)^2 = 14^2\left(1+\frac{1}{9}\right) \\ &= \frac{196\cdot 10}{9}. \end{aligned}$$

It follows that

$$\begin{aligned} CD &= \sqrt{\frac{196\cdot 10}{9}} \\ &= \frac{\sqrt{196}\cdot\sqrt{10}}{\sqrt{9}} \\ &= \frac{14\cdot\sqrt{10}}{3}. \end{aligned}$$

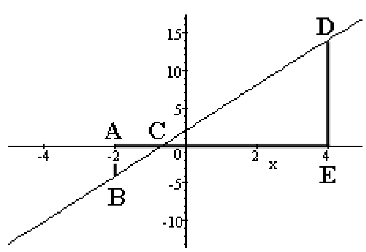


Figure 194.

A similar calculation shows that the length of the hypotenuse BC of triangle ABC is given by

$$\begin{aligned} BC^2 &= AB^2 + AC^2 \\ &= (4)^2 + \left(-\frac{2}{3} - (-2)\right)^2 \\ &= 4^2\left(1+\frac{1}{9}\right) \\ &= \frac{16\cdot 10}{9} \end{aligned}$$

and so BC has length $4\cdot\frac{\sqrt{10}}{3}$. Adding up these two lengths, BC and CD, we find $18\cdot\frac{\sqrt{10}}{3} = \mathbf{6\sqrt{10}}$, as above. Now, which method do you prefer?

Remark We could use the formula defined by (8.8) in order to find the arc length in Example 429. In this case we would need to find the inverse function, F, of f using the methods of Chapter 3.7. In this case we get that $x = F(y) = \frac{y-2}{3}$ (just solve for x in terms of y, remember?). Since the domain of f (which is $[-2, 4]$) must equal the range of F, and the range of f (which is $[-4, 14]$) must equal the domain of F, it follows that $c = -4,\ d = 14$ in equation (8.8). We see that the length of the line segment is also given by

$$\begin{aligned} L(F; -4, 14) &= \int_{-4}^{14} \sqrt{1 + \left(\frac{1}{3}\right)^2}\ dy \\ &= \frac{\sqrt{10}}{3} \int_{-4}^{14} dy \\ &= 6\sqrt{10} \end{aligned}$$

as before.

Example 430.

Find the arc length of the curve whose equation is given by $y = \frac{x^2}{2}$ where $0 \le x \le 1$, (i.e., the length of the shaded arc in Figure 195).

Solution In this example $f(x) = \frac{x^2}{2}$ and $f'(x) = x$, for every x. The length of the required arc is given by (8.7), or

$$\begin{aligned} L(f; 0, 1) &= \int_0^1 \sqrt{1 + x^2}\ dx \\ & \quad \left(\text{Now let } \ x = \tan\theta,\ dx = \sec^2\theta\ d\theta, \sqrt{1+x^2} = \sec\theta \ldots\right) \\ &= \int_0^{\frac{\pi}{4}} \sec^3\theta\ d\theta \\ & \quad \text{(and use Example 351,)} \\ &= \frac{1}{2}\left(\tan x \sec x + \ln|\sec x + \tan x|\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(1 \cdot \sqrt{2} + \ln|\sqrt{2} + 1|\right) - (0 \cdot 1 + \ln|1 + 0|) \\ &= \frac{1}{2}\left(\sqrt{2} + \ln|\sqrt{2} + 1|\right) \approx 1.1478. \end{aligned}$$

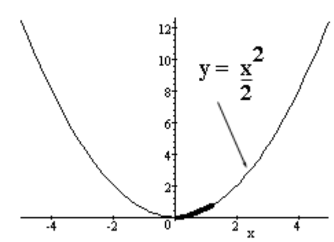


Figure 195.

Example 431.

Find the length of the curve whose equation is given by

$$y = \frac{2}{3}x^{\frac{3}{2}}$$

between $x = 0$ to $x = 3$.

Solution Since the curve is given in the form $y = f(x)$ we can use (8.7) with $a = 0$ and $b = 3$ to find its length.

In this case we set $f(x) = \frac{2}{3}x^{\frac{3}{2}}$ so that $f'(x) = \sqrt{x}$, for every x. The length of the required arc is given by (8.7), or

$$\begin{aligned}
L(f;0,3) &= \int_0^3 \sqrt{1+f'(x)^2}\,dx \\
&= \int_0^3 \sqrt{1+x}\,dx \\
&\quad \big(\text{and using the substitution } u=\sqrt{1+x}, \text{ etc.,}\big) \\
&= \int_1^4 \sqrt{u}\,du \\
&= \frac{14}{3}.
\end{aligned}$$

Example 432.

Find the length of the curve whose equation is given by

$$x = g(y) = \frac{y^4}{4} + \frac{1}{8y^2}$$

from $y = 1$ to $y = 2$.

Solution Here x is a given differentiable function of y, so we can use (8.8) with $c = 1$ and $d = 2$. Then

$$\begin{aligned}
L(g;1,2) &= \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_1^2 \sqrt{1+\left(y^3-\frac{1}{4y^3}\right)^2}\,dy \\
&= \int_1^2 \sqrt{\left(y^3+\frac{1}{4y^3}\right)^2}\,dy = \int_1^2 \left(y^3+\frac{1}{4y^3}\right)dy \\
&= \frac{123}{32}.
\end{aligned}$$

A **parameter** is simply another name for a *variable*. Then, why not just call it a variable? Basically, this is because we want to express our basic x, y variables in terms of it (the parameter). This means that when we see the phrase " t is a parameter ...", we think: "OK, this means that our basic variables (here x, y) are expressed in terms of "t".

For example, if we let t be a variable ($0 \le t < 2\pi$) related to x, y by

$$x = \cos t, \quad y = \sin t.$$

Then t is a parameter (by definition). Now, "eliminating" the parameter shows that x and y are related to each other too, and $x^2 + y^2 = 1$. So, the arc defined by this parametric representation is a *circle* of radius 1, centered at the origin (see figure 196). We call the previous display a **parametric representation** of an arc in the $x, y-plane$ defined by these points with coordinates (x, y) when $0 \le t < 2\pi$.

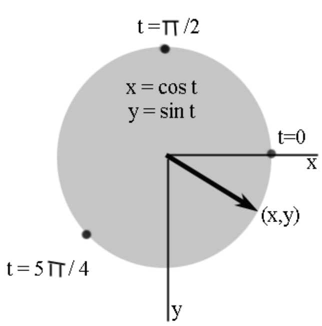


Figure 196.

Thus, the circle $x^2 + y^2 = 1$ may be represented *parametrically* by

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \quad 0 \le t \le 2\pi,$$

(because $\cos^2 t + \sin^2 t = 1$).

On the other hand, the straight line

$$y = mx + b$$

has the parametric form (representation)

$$\begin{array}{l} x = t \\ y = mt + b. \end{array}$$

So, the insertion of any number, t, actually gives a point on the curve, a point which changes with t.

Now, it turns out that when a curve C, has the parametric representation

$$(x, y) \text{ on } C: \quad \left.\begin{array}{l} x = g(t) \\ y = h(t) \end{array}\right\} \quad a \le t \le b,$$

then

$$\boxed{\text{Length of } C \text{ between } a \text{ and } b \quad = \quad \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (8.9)}$$

For a proof of this fact see Exercise 20 at the end of this section.

Example 433.

The position of a particle $P(x, y)$ at time t is given by the curve C described by

$$x = \frac{1}{3}(2t+3)^{3/2}, y = \frac{t^2}{2} + t.$$

Find the distance it travels from time $t = 0$ to $t = 3$.

Solution Note that the curve is described parametrically. We want to know how far the particle travels along the curve C starting from $t = 0$ (corresponding to the point $((x, y) = (\sqrt{3}, 0)))$ to $t = 3$, (corresponding to $(x, y) = (9, \frac{15}{2})$), i.e., we need to find the length of the curve C between the points corresponding to $t = 0$ and $t = 3$.

Since C is given parametrically by $x = g(t)$, $y = h(t)$ where g, h are differentiable over $(0, 3)$ we use equation (8.9), that is, the length of C between $t = 0$ and $t = 3$ is given by:

$$Length = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

where

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1}{3}(2t+3)^{3/2}\right) = \frac{1}{2}(2t+3)^{1/2} \cdot 2 = (2t+3)^{1/2}$$

and

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{t^2}{2} + t\right) = t + 1.$$

So,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t+3) + (t+1)^2} = t + 2.$$

It follows that the length of the curve between the required time values is given by

$$Length \ = \int_0^3 (t+2)\ dt = \frac{21}{2}.$$

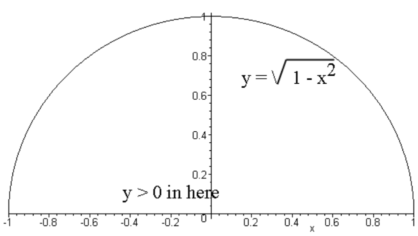


Figure 197.

Example 434.

Verify that the length of the arc of a semicircle of radius 1 (*i.e.,* one-half the circumference) is given by the value π.

Solution We'll do this one in *two* ways:

Method 1 First, we can describe this arc by means of the equation $x^2 + y^2 = 1$ where $y > 0$ (see Figure 197). Solving for y we find (note that *positive* square root is used here for the value of y):

$$y = \sqrt{1 - x^2},$$

where $-1 \leq x \leq 1$. Write $y = f(x)$ and use (8.7) with $a = -1$ and $b = 1$. We find

$$y'(x) = \frac{-x}{\sqrt{1-x^2}},$$

and after some simplification,

$$\sqrt{1 + (y'(x))^2} = \frac{1}{\sqrt{1-x^2}}.$$

The required length is given by

$$\begin{aligned} Length \ &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\ dx \\ &= \operatorname{Arcsin} x \, \Big|_{-1}^{1} = \frac{\pi}{2} - \left(\frac{-\pi}{2}\right) = \pi, \end{aligned}$$

where we have used the results in Table 6.7 in order to evaluate this integral.

Method 2 We *parametrize* this semicircular arc by setting $x = \cos t, y = \sin t$ where $0 \le t < \pi$ (Why π and not 2π here?). In this case we have that the length of the arc, given by (8.9),

$$\begin{aligned} \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_0^\pi \sqrt{(\sin t)^2 + (\cos t)^2}\, dt \\ &= \int_0^\pi \sqrt{1}\, dt \\ &= \pi, \end{aligned}$$

as before.

Example 435.

Find the length of the curve $y = f(x)$ where $y = \int_0^x \sqrt{\cos 2t}\, dt$ from $x = 0$ to $x = \frac{\pi}{4}$.

Solution

Since

$$y = \int_0^x \sqrt{\cos 2t}\, dt$$

it follows that

$$\begin{aligned} y'(x) &= \frac{d}{dx} \int_0^x \sqrt{\cos 2t}\, dt \\ &= \sqrt{\cos 2x}, \end{aligned}$$

by the Fundamental Theorem of Calculus (Section 6.4).

Hence, the length of the curve is given by

$$\int_0^{\frac{\pi}{4}} \sqrt{1 + (y'(x))^2} dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 2x}\, dx$$

Now, we need to recall some trigonometry: Note that since $\dfrac{1 + \cos 2x}{2} = \cos^2 x$,

$$1 + \cos 2x = 2\cos^2 x$$

which, when substituted into the previous integral, gives us

$$\begin{aligned} Length &= \int_0^{\frac{\pi}{4}} \sqrt{1+\cos 2x}\; dx \\ &= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 x}\; dx \\ &= \sqrt{2}\int_0^{\frac{\pi}{4}} \cos x\; dx \\ &= 1, \end{aligned}$$

(because $\sqrt{\cos^2 x} = |\cos x|$ by definition and $|\cos x| = \cos x$ on our range $[0, \frac{\pi}{4}]$).

Example 436.

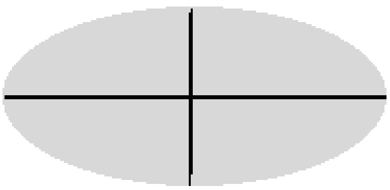


Figure 198. *The elliptical track $x^2 + 4y^2 = 1$.*

A racing track is in the form of an elliptical curve whose equation is given by $x^2 + 4y^2 = 1$, where x is in kilometers (so that it is twice as long as it is wide, see Figure 198). Estimate the distance travelled after completing a full 5 laps.

Solution Using symmetry we see that it is sufficient to estimate the length of "one-half" of the ellipse, then double it (this gives one lap), and finally multiply the last result by 5 in order to obtain the required answer. Solving for y as a function of x we get $y = \dfrac{\sqrt{1-x^2}}{2}$ from which

$$y'(x) = \frac{-x}{2\sqrt{1-x^2}}.$$

Now compare this example with Example 434, above. The additional multiplicative factor of 2 in the denominator of the last display really complicates things as you will see. We proceed as usual and find, after some simplification, that the length of one-half of the elliptical track is given by

$$\int_{-1}^{1} \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \int_{-1}^{1} \sqrt{\frac{4-3x^2}{4(1-x^2)}}\; dx.$$

Now, this is NOT an easy integral. We could try a *trigonometric substitution* such as $x = \cos\theta$ and see what happens or, we can estimate it using *Simpson's Rule* (see Section **??**). The last idea is probably the best one. Let's use $n = 6$, say, (larger n's are better but more work is required). In this case, (fill in the details), Simpson's Rule with $n = 6$ applied to the integral

$$\int_{-1}^{1} \sqrt{\frac{4-3x^2}{4(1-x^2)}}\; dx \qquad (8.10)$$

gives the approximate value, 2.65 km. Doubling this value and multiplying by 5 gives us the required answer which is approximately 26.50 km.

$\boxed{\text{Be careful!}}$ The denominator in (8.10) is zero at the endpoints $x \pm 1$. This means that we should replace these values by something like ± 0.99 and *then* use Simpson's Rule. Alternately, you can treat the integral as an *improper integral* first and find an exact answer (but this would be hard in this case).

Exercise Set 43.

Use the methods of this section to find the length of the arcs of the following curves between the specified points.

1. $y = 3, \quad 0 \le x \le 2$
2. $y = x - 1, \quad 0 \le x \le 4$
3. $y = 2x + 1, \quad -1 \le x \le 1$
4. $x = y + 1, \quad 0 \le y \le 2$
5. $2x - 2y + 6 = 0, \quad -2 \le x \le 1$
6. $y = \frac{2}{3}x^{3/2}, \quad 0 \le x \le 8$
7. $y = x^2, \quad 0 \le x \le 1$
   • Use the trigonometric substitution $x = \frac{\tan\theta}{2}$ and Example 351.
8. $y = 2x^2 + 1, \quad 0 \le x \le 1$
   • Use a trigonometric substitution.
9. $y = 2x^2, \quad 0 \le x \le 2$
   • Use a trigonometric substitution.
10. $y = \frac{1}{4}x^4 + \frac{1}{8x^2}, \quad 1 \le x \le 3$
11. $x = 2\cos t, \quad y = 2\sin t, \quad 0 \le t < 2\pi$
12. $x = 1 + \cos t, \quad y = 1 - \sin t, \quad 0 \le t < 2\pi$
13. $x = t, \quad y = 2 - t, \quad 0 \le t \le 1$
14. $y = \int_1^x \sqrt{t^2 - 1}\, dt, \quad 1 \le x \le 2$
15. $y = 2 + \int_0^x \sqrt{\cos 2t}\, dt, \quad 0 \le x \le \frac{\pi}{2}$
16. $4x^2 + y^2 = 1, \quad -1 \le y \le 1$
    • Use Simpson's Rule with $n = 6$.
17. $y = \ln x, \quad 1 \le x \le 4$
    • Use a trigonometric substitution.
18. $y = \ln(\sec x), \quad 0 \le x \le \frac{\pi}{4}$
19. Once a meteor penetrates the earth's gravitational field its flight-path (or trajectory, or orbit) is approximately parabolic. Assume that a meteor follows the flight-path given by the parabolic arc $x = 1 - y^2$, where $y > 0$ is its vertical distance from the surface of the earth (in a system of units where 1 unit = 100 km). Thus, when $y = 0$ units the meteor collides with the earth. Find the distance travelled by the meteor from the moment that its vertical distance is calculated to be $y = 1$ to the moment of collision.

    • You need to evaluate an integral of the form $\int_0^1 \sqrt{1 + 4y^2}\, dy$. See Exercise 7 above.
20. Prove formula (8.9) for the length of the arc of a curve represented parametrically over an interval $[c, d]$ in the following steps:

    Let $y = f(x)$ be a given differentiable curve and assume that $x = x(t)$ where x is differentiable, and *increasing* for $c \le t \le d$. Then y is also a function of t.

    a) Use the Chain Rule to show that

$$y'(t) = f'(x(t)) \cdot x'(t).$$

b) Next, use the substitution $x = x(t)$ to show that

$$\int_a^b \sqrt{1 + (f'(x))^2}\ dx = \int_{x^{-1}(a)}^{x^{-1}(b)} \sqrt{1 + (f'(x(t)))^2}\ x'(t)\ dt.$$

c) Finally, show that

$$\int_{x^{-1}(a)}^{x^{-1}(b)} \sqrt{1 + (f'(x(t)))^2}\ x'(t)\ dt = \int_c^d \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

provided we choose $c = x^{-1}(a), d = x^{-1}(b)$.

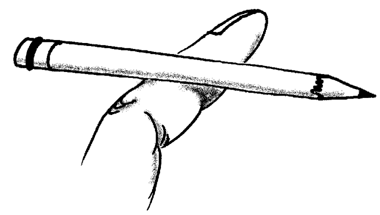
Suggested Homework Set 35. *Do problems 3, 7, 12, 15, 18*

NOTES:

8.5 Moments and Centers of Mass

The Big Picture

At some point in your life you must have balanced a pencil or a pen on one of your fingers. Usually it was a matter of *trial and error*. You didn't really know where the "balancing point" was until you actually found it by moving the instrument back and forth until it stopped wobbling. By doing so, you actually found out some things about nature! The first thing you found out is that *gravity is uniform*. Basically, this means that the way in which gravity manifests itself on the instrument is the same whether it is acting at the sharp end or at the other end. Furthermore, this balancing point, also called the **center of mass** or **centroid** of the pencil, is a property of every object (or collection of objects) on earth that have *mass*. In sum, for a collection of masses to *balance*, in some sense, they must share a common center of mass. So, yes, even a galaxy has a center of mass! Another example where this notion of a center of mass is used and usually taken for granted includes, but is certainly not restricted to, the teeter-totter or seesaw (found in some parks).



This section is about finding the center of mass of linear objects and (thin) planar regions (like an oddly shaped single sheet of paper, for example). We can assume a *uniform* distribution of mass inside the region whose center of mass we're trying to find, or we can even assume that the distribution of mass is not uniform. For example, the mass distribution along a pencil (with eraser) is not uniform because of the presence of the eraser. It would be approximately uniform if the eraser were absent. The measure of such uniformity of mass along or inside an object is called its **mass density** and denoted by either the symbol $\delta(x)$, or $\delta(x, y)$ (if we are dealing with a two-dimensional region).

> ### Review
>
> This section involves applying the definite integral to the solution of problems involving the center of mass. You should be familiar with all the techniques of integration from Chapter 7 as well as the method of *slices* introduced in Section 8.2.

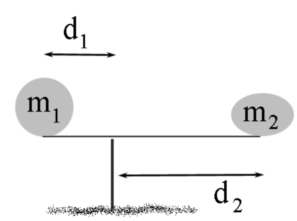


Figure 199.

Moments and Centers of Mass

One of the earliest observations about centers of mass is due to the Greek philosopher and mathematician, Archimedes (287-212). Of the many discoveries attributed to him we find the **Principle of the Lever**.

In its simplest form this principle states that if we take a homogeneous rod of negligible mass and we attach two masses, say m_1 and m_2 at opposite ends, then we can find a point along it that is its center of mass. This center of mass is found by looking for distances d_1 and d_2 from each end, such that

$$m_1 d_1 = m_2 d_2.$$

Experience tells us that there is only *one* such pair of distances, see Figure 199. Why?

Let's actually find the *position* (denoted by $\overline{x}$) of the center of mass given the masses m_1 and m_2 by placing the rod in Figure 199 along the $x-$axis so that m_1 lies at x_1 and m_2 lies at x_2, see Figure 200. Since $\overline{x}$ is the center of mass, the Principle of the Lever tells us that

$$m_1 \underbrace{(\overline{x} - x_1)}_{d_1} = m_2 \underbrace{(x_2 - \overline{x})}_{d_2}.$$

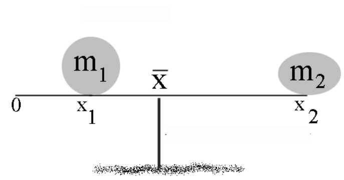


Figure 200.

Now we can isolate the $\overline{x}$ term in the preceding equation by grouping all such terms together . We find $m_1\overline{x} + m_2\overline{x} = m_1x_1 + m_2x_2$. From this we get

$$\overline{x} \quad = \quad \frac{m_1x_1 + m_2x_2}{m_1 + m_2}. \tag{8.11}$$

This result gives us the position of the center of mass, $\overline{x}$, in terms of the positions of the masses and the masses themselves. In other words, once we know *what* the masses are and *where* the masses are, then we know where the center of mass is too! So, there is only one such center of mass just like we expected.

$\boxed{\textbf{Example 437.}}$ Show that you can balance a dune-buggy of mass 500 kg using a 10 m (extremely light!) railroad track. Where is the center of mass? (Assume that your weight is 70 kg.)

Solution Now a railroad track has quite a lot of mass, and so the formulae we developed cannot be used as such as they only apply to the case where the rod has *negligible* mass. Still, we can use them to get an idea, or an approximation to the center of mass. We can refer to Figure 200 where $m_1 = 500$, $m_2 = 70$ and $x_1 = 0$ and $x_2 = 10$ and use equation (8.11) which gives us the location of the center of mass, $\overline{x}$. Thus,

$$\begin{aligned} \overline{x} &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \\ &= \frac{500 \cdot 0 + 70 \cdot 10}{500 + 70} \\ &= 1.23m. \end{aligned}$$

As you can see, the center of mass is close to the dune-buggy, at roughly 1.2 meters away from it. At this point $\overline{x}$ you place a *fulcrum* (that part of the track on which it rests) and the system should balance.

Remark Using the notion of the center of mass and Example 437, above, you quickly realize that you can balance just about anything (regardless of its weight!) so long as the rod is *long* enough (and strong enough to hold you both).

Now the products m_1x_1 and m_2x_2 appearing in the expression (8.11) are called the **moments** of the masses m_1, m_2 relative to the origin. So, the center of mass is obtained by adding the two moments and dividing their sum by the total mass of the system. It turns out that *this is true regardless of the number of masses*, that is, the center of mass of a system of n objects all lying on a common line is given by

$$\overline{x} \quad = \quad \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{m_1 + m_2 + m_3 + \cdots + m_n} \tag{8.12}$$

$$\quad = \quad \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i}, \tag{8.13}$$

where we have used the summation convention of Chapter 6.3.

Example 438. Three pearls of masses $m_1 = 10$g, $m_2 = 8$g, $m_3 = 4$g are fixed in place along a thin string of very small mass which is then stretched and tied down at both ends. Find the center of mass of this system given that the pearls are all 10cm apart, (see Figure 201).

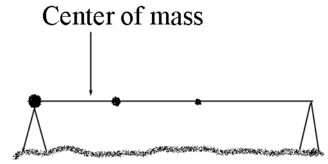


Figure 201.

Solution We place the first pearl at $x_1 = 0$. The second must then be at $x_2 = 10$ while the third must be at $x_3 = 20$, (see Figure 201). It doesn't matter how *long* the string is, since it is assumed to be very light and so its own weight won't displace the system's center of mass by much. Now,

$$\begin{aligned}\overline{x} &= \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \\ &= \frac{0 \cdot 10 + 10 \cdot 8 + 4 \cdot 20}{10 + 8 + 4} \\ &= \frac{160}{22} \\ &= 7.28cm,\end{aligned}$$

and so, once again, we notice that the center of mass is closer to the *heavier side* of the system of masses.

If the particles or masses are *not all* on a straight line (see Figure 202) the expressions for the center of mass change slightly but may still be written down using the idea of *moments*, above. It turns out that the center of mass, $(\overline{x}, \overline{y})$, of a system of n bodies with masses $m_1, m_2, \ldots, m_n$ in general position (meaning *anywhere*) on a plane is given by $(\overline{x}, \overline{y})$ where,

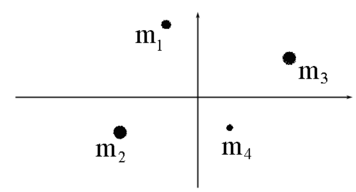


Figure 202.

$$\overline{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i} = \frac{M_y}{m} \tag{8.14}$$

and

$$\overline{y} = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i} = \frac{M_x}{m} \tag{8.15}$$

where the quantities defined by the symbols M_y, M_x are called **moments**, and m is the **total mass** of the system. Here, (x_i, y_i) denote the coordinates of the mass m_i. Note the similarity of the *form* of these moments to the same notion for the *one-dimensional case* (or the case where the bodies are all on a line, above).

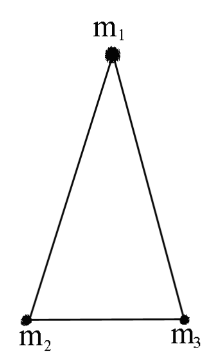


Figure 203.

Specifically, the **moment about the $y-$axis** is denoted by M_y and is defined by the sum of products of the form

$$M_y = \sum_{i=1}^n m_i x_i.$$

Similarly, the **moment about the $x-$axis** is denoted by M_x and is defined by the sum of products of the form

$$M_x = \sum_{i=1}^n m_i y_i.$$

The quantity M_y, reflects the tendency of a system to rotate about the $y-$axis, while the quantity M_x, reflects a tendency of the system to rotate about the $x-$axis.

Example 439. Find the center of mass of the isosceles triangle configuration (see Figure 203) of three bodies with $m_1 = 4, m_2 = 3, m_3 = 3$. Assume that the base has length 2 and that the equal sides have a common length equal to $\sqrt{17}$.

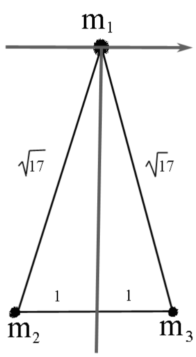


Figure 204.

Solution Note that it doesn't matter *where* we place the triangle. So, we position the triangle on a (rectangular) coordinate system with its vertex at the origin, and so that the bisector of the vertex lies along the $y-$axis, see Figure 204. Note that, by the Theorem of Pythagoras, the height of this triangle is 4 units. It follows that the vertices m_1, m_2, m_3 have coordinates $(x_1, y_1) = (0, 0), (x_2, y_2) = (-1, -4)$, and $(x_3, y_3) = (1, -4)$ respectively. The total mass is easily seen to be equal to 10 units. We only need to find the moments. Now there are three bodies, so $n = 3$. So,

$$M_x = \sum_{i=1}^{3} m_i y_i = 4 \cdot 0 + 3 \cdot (-4) + 3 \cdot (-4) = -24.$$

This means that $\overline{y} = -24/10 = -2.4$.

Similarly,

$$M_y = \sum_{i=1}^{3} m_i x_i = 4 \cdot 0 + 3 \cdot (-1) + 3 \cdot (1) = 0.$$

On the other hand, this implies that $\overline{x} = 0/10 = 0$. It follows that the center of mass of this system of three bodies is given by the equations (8.14) , (8.15) or

$$(\overline{x}, \overline{y}) = (0, -2.4).$$

Due to the symmetry of the configuration and the base masses, we see that the center of mass lies along the line of symmetry (namely, the $y-$axis). **Remark and Caution:** One can show that if the base masses in Example 439 are unequal, then the center of mass is **not necessarily** along the $y-$axis. Indeed, if we set $m_2 = 5$ (instead of $m_2 = 3$ as in said Example), then $M_y = -32, M_x = -2$ and the new center of mass is calculated to be at the point $(-0.2, -3.2)$. The point is that *even though the configuration of masses is symmetric about a line, this does not force the center of mass to lie along that line of symmetry.* There has to be complete symmetry, between the masses *and* the configuration!

The Center of Mass of a Region in the Plane

At this point we have enough information to derive the formula for the center of mass of a (thin) planar region with a mass density $\delta(x)$. Note the dependence of δ on x only! This means that that the mass density, δ, at a point (x, y) in the region is a function of its distance from the $y-$axis only. For example, each picket inside a picket fence around a garden has the same mass density, but here, this density is independent of the picket itself since every picket looks and weighs the same. In this case, $\delta(x)$ is a constant.

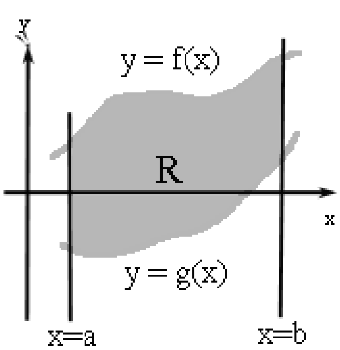


Figure 205.

We will derive a formula for the center of mass of the region $\mathcal{R}$ having a mass density $\delta(x)$.

Let $\mathcal{R}$ denote a region in the $xy-$plane bounded the graphs of two functions $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$, where in addition, we assume that $f(x) \geq g(x)$ (see Figure 205). At this point it would be helpful to review the material in Section 8.2.

Fix a point x where $a < x < b$. At x we build a vertical slice of the region $\mathcal{R}$ whose endpoints are given $(x, f(x)), (x, g(x))$ and whose width is dx, (see Figure 206). The

mass of this slice is given by the mass density along this slice multiplied by the area of the slice, that is,

$$m_{slice} = \underbrace{(f(x) - g(x))\ dx}_{Area\ of\ slice:\ dA} \cdot \underbrace{\delta(x)}_{mass\ density\ of\ slice}.$$

The moment of this slice about the $y-$axis is given by its mass (namely, $\delta \cdot (f(x) - g(x))\ dx$), multiplied by its distance from the $y-$axis (namely, x). So, the moment of this slice about the $y-$axis is given by

$$M_{y_{slice}} = x\ (f(x) - g(x))\ \delta(x)\ dx$$

Adding up the moments due to each such slice between a and b we obtain the general formula for the moment, M_y of the region $\mathcal{R}$:

$$\begin{aligned} M_y &= \int_a^b x\ (f(x) - g(x))\ \delta(x)\ dx \\ &= \int_a^b x\ \delta(x)\ \underbrace{(f(x) - g(x))\ dx} \\ &= \int_a^b \overline{x}_{slice}\quad \delta(x)\quad dA. \end{aligned}$$

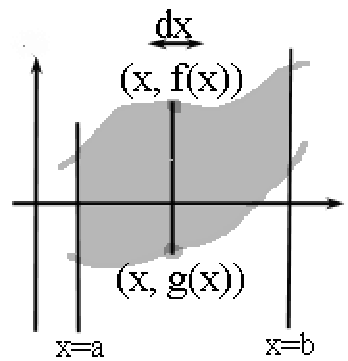


Figure 206.

where $\overline{x} = x$ is the $x-$coordinate of the center of mass of the slice itself.

The same method allows us to find the moment M_x of the slice about the $x-$axis. The $y-$coordinate, $\overline{y}$, of the center of mass of the slice is *half-way up* the slice since δ depends on x only! (not on y). This means that

$$\overline{y}_{slice} = \frac{f(x) + g(x)}{2}.$$

So, we find the expression

$$\begin{aligned} M_x &= \int_a^b \overline{y}_{slice}\quad \delta(x)\quad dA \\ &= \int_a^b \frac{(f(x) + g(x))}{2}\ \delta(x)\ (f(x) - g(x))\ dx \\ &= \int_a^b \frac{(f^2(x) - g^2(x))}{2} \delta(x)\ dx \end{aligned}$$

Combining these formulae for the moments and the total mass, we obtain that the center of mass $(\overline{x}, \overline{y})$ of the region $\mathcal{R}$, described in Figure 205, is given by

$$\overline{x} = \frac{\int_a^b x\ (f(x) - g(x))\ \delta(x)\ dx}{\int_a^b (f(x) - g(x))\ \delta(x)\ dx}. \tag{8.16}$$

$$\overline{y} = \frac{1}{2}\ \frac{\int_a^b (f^2(x) - g^2(x))\ \delta(x)\ dx}{\int_a^b (f(x) - g(x))\ \delta(x)\ dx} \tag{8.17}$$

The quantities, M_x, M_y have simple geometrical interpretations: **The system** *balances* **along the line $y = M_x$, if $m = 1$, and the system balances along the line $x = M_y$, if $m = 1$.**

Example 440. Find the approximate value of the center of mass of the earth-moon system at a point where their mutual distance is 3.8×10^5 km. (Data: the mass of the earth is given by 5.97×10^{24} kg, while that of the moon is 7.35×10^{22} kg.)

Solution This is a dynamic (in motion) system of two bodies so the center of mass actually moves around since these astronomical bodies are not generally equidistant from one another for all time. Furthermore, over 300 years ago Newton showed that celestial objects may be treated as *point masses.* This means that one could always assume that *all the mass was concentrated at the center of the object.*

Now, it doesn't matter which one we label m_1 or m_2. Let's say that we set $m_1 = 5.97 \times 10^{24}$ and $m_2 = 7.35 \times 10^{22}$. We can use (8.11) with $x_1 = 0$ and $x_2 = 3.8 \times 10^5$. Then their center of mass at the time where their distance is 3.8×10^5 km is given by

$$\begin{aligned} \overline{x} &= \frac{7.35 \times 10^{22} \times 3.8 \times 10^5}{5.9707 \times 10^{24}} \\ &= 4.678 \times 10^3 \\ &= 4678 \ km. \end{aligned}$$

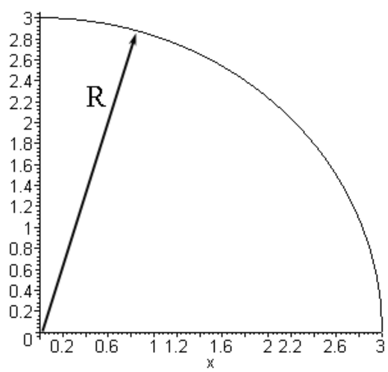


Figure 207.

The equatorial diameter of the earth is, however, 12,756 km! So, its equatorial radius is 6,378 km. This means that their center of mass is *inside the earth*!

Example 441. Find the center of mass of a quarter circle of radius R assuming that the density throughout is uniformly constant (*i.e.,* $\delta(x) = constant$, see Figure 207).

Solution We may always assume that the circle is centered at the origin. In this case, the boundary of the quarter circle is given by points (x, y) where $y = \sqrt{R^2 - x^2}$, and $0 \leq x \leq R$. We can use formulae (8.17, 8.16) for the coordinates of the center of mass. Here we will set

$$f(x) = \sqrt{R^2 - x^2}, \qquad g(x) = 0,$$

since it is easy to see that the region whose center of mass we seek is bounded by the curves $y = f(x)$, $y = g(x) = 0$, and the vertical lines $x = 0$ and $x = R$. Now $\overline{y}$ is given by (8.17) or

$$\begin{aligned}
\overline{y} &= \frac{1}{2}\,\frac{\int_a^b (f^2(x) - g^2(x))\ \delta(x)\ dx}{\int_a^b (f(x) - g(x))\ \delta(x)\ dx} \\
&= \frac{1}{2}\,\frac{(constant) \cdot \int_0^R ((R^2 - x^2) - 0)\ dx}{(constant) \cdot \int_0^R \sqrt{R^2 - x^2}\ \ dx} \\
&= \frac{1}{2}\,\frac{\int_0^R (R^2 - x^2)\ dx}{\int_0^R \sqrt{R^2 - x^2}\ dx} \\
&= \frac{1}{2}\,\frac{\frac{2R^3}{3}}{\frac{\pi R^2}{4}} \\
&= \frac{4R}{3\pi}.
\end{aligned}$$

The evaluation of the square-root integral in the denominator is accomplished using the trigonometric substitution $x = R\cos\theta$ and a combination of the methods in Section 7.6 and Section 7.5.

Similarly, the $x-$coordinate, $\overline{x}$, of the center of mass is given by

$$\begin{aligned}
\overline{x} &= \frac{\int_a^b x\ (f(x) - g(x))\ \delta(x)\ dx}{\int_a^b (f(x) - g(x))\ \delta(x)\ dx} \\
&= \frac{(constant) \cdot \int_0^R x\ \sqrt{R^2 - x^2}\ dx}{(constant) \cdot \int_0^R \sqrt{R^2 - x^2}\ \ dx} \\
& \quad \text{In the numerator use the substitution: } u = R^2 - x^2,\ du = -2x\ dx \\
&= \frac{\frac{R^3}{3}}{\frac{\pi R^2}{4}} \\
&= \frac{4R}{3\pi}.
\end{aligned}$$

So, the center of mass

$$(\overline{x}, \overline{y}) = \left(\frac{4R}{3\pi}, \frac{4R}{3\pi}\right)$$

of a quarter plate is located along the *line of symmetry* given by $y = x$ (since the mass density is uniform and the quarter circle is symmetric about this line).

$\boxed{\textbf{Shortcut}}$ It would be easier to evaluate the *total mass integral*

$$m_{quarter\ plate} = \int_0^R \sqrt{R^2 - x^2}\ \delta(x)\ dx$$

on *physical* grounds! For example, the mass of the circular plate is, by definition, its mass density times its surface area. But its mass density is *constant* while its surface area is πR^2. Thus, the mass of the circular plate is $(constant) \cdot \pi R^2$. It follows that the mass of the quarter-plate is given by

$$m_{quarter\ plate} = \frac{(constant) \cdot \pi R^2}{4}.$$

Example 442. Calculate the center of mass of a two-dimensional skateboarding ramp (the shaded region in Figure 208), under the assumption that the mass density, $\delta(x) = const.$

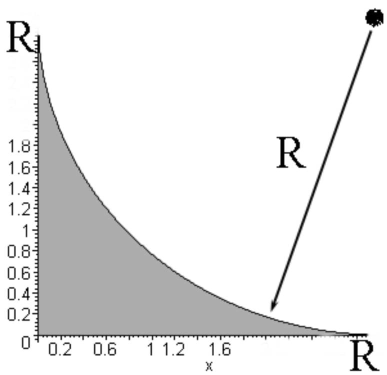


Figure 208.

Solution The region is bounded by an arc of the circle with center at (R, R) and radius R, the line segment from $x = 0$ to $x = R$, and the line segment from $y = 0$ to $y = R$. The equation of the circle under consideration is given by $(x-R)^2+(y-R)^2 = R^2$. It follows that, in (8.17),

$$f(x) = R - \sqrt{R^2 - (x-R)^2}, \quad g(x) = 0,$$

and $a = 0, b = R$.

Now the total mass, m, of this region can be found (without integrating!) by multiplying its area by the mass-density. In this case, the area of the shaded region can be easily found by noting that it is the difference between the area of a square with side R and vertex at $(0, 0)$, and the area of the quarter-circle of radius R, described above. So the total mass of the region is given by

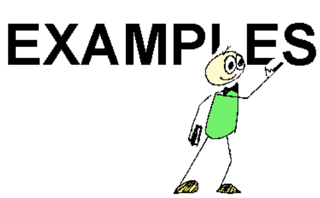
$$m = const.\left(R^2 - \frac{\pi R^2}{4}\right) = const.\left(1 - \frac{\pi}{4}\right)R^2.$$

On the other hand, the region is also symmetric with respect to the line $y = x$. It follows that the center of mass must occur along this line and so it must be the case that for the point $(\overline{x}, \overline{y})$ we must have $\overline{x} = \overline{y}$. It therefore suffices to find either $\overline{x}$ or $\overline{y}$. We choose to find $\overline{x}$ since the required integral for the moment is relatively easier to evaluate. Thus,

$$\begin{aligned}\overline{x} &= \frac{\int_a^b x\ (f(x) - g(x))\ \delta(x)\ dx}{\int_a^b (f(x) - g(x))\ \delta(x)\ dx} \\ &= \frac{(const.)\cdot \int_0^R x\ \left(R - \sqrt{R^2 - (x-R)^2}\right)\ dx}{(const.)\cdot\left(1 - \frac{\pi}{4}\right)R^2}.\end{aligned}$$

Now,

$$\begin{aligned}\int_0^R x\ \left(R - \sqrt{R^2-(x-R)^2}\right)\ dx &= R\cdot\left.\frac{x^2}{2}\right|_0^R - \int_0^R x\ \sqrt{R^2-(x-R)^2}\ dx \\ &\quad (\text{Let } x - R = R\sin\theta,\ dx = \ R\cos\theta\ d\theta) \\ &= \frac{R^3}{2} - R^3\cdot\int_{-\frac{\pi}{2}}^{0}(1+\sin\theta)\cos^2\theta\ d\theta \\ &= \frac{R^3}{2} - \left(\frac{R^3\pi}{4} - \frac{R^3}{3}\right) \\ &= \frac{(10-3\pi)R^3}{12}.\end{aligned}$$



It follows that

$$\overline{x} = \frac{\frac{(10-3\pi)R^3}{12}}{\left(1-\frac{\pi}{4}\right)R^2} = \frac{(10-3\pi)R}{(12-3\pi)},$$

and $\overline{x} = \overline{y}$. In this case, the system balances along the line $y = x$, as well as at the point $(\overline{x}, \overline{y})$.

Summary

- For a wire, thin rod or slice along the x-axis from $x = a$ to $x = b$.
  The *Moment about the origin*, $M_0 = \int_a^b x\delta(x)\ dx$, where δ is the mass density (along the wire). The density is said to be *uniform* if $\delta(x) = constant$ inside the region.

  The *total mass*, $m = \displaystyle\int_a^b \delta(x)\ dA$.

  The *Center of Mass*, (or "c.m."), $\overline{x}$, $\overline{x} = \frac{M_0}{M}$.

- For masses m_i at (x_i, y_i): $\overline{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}, \overline{y} = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i}$ for c.m.
- For a thin plate covering a region in the xy-plane where dA is the area of a thin slice:
  The *Moment about the* $x-$*axis*,

  $$M_x = \int \overline{y}_{slice}\ \delta\ dA.$$

  Geometrically, the system *balances* along the line $y = M_x$, if $m = 1$.

  The *Moment about the* $y-$*axis*,

  $$M_y = \int \overline{x}_{slice}\ \delta\ dA$$

  Geometrically, the system *balances* along the line $x = M_y$, if $m = 1$.

  The total mass, $m = \int_a^b \delta\ dA$.
  Its Center of Mass is at $(\overline{x}, \overline{y})$, where $\overline{x} = \frac{M_y}{m}, \overline{y} = \frac{M_x}{m}$.

  Here δ is the density per unit area, $(\overline{x}_{slice}, \overline{y}_{slice})$ is the center of mass of a typical thin slice of mass $\delta\ dA$, where dA is the area of our slice.

- The center of mass of a thin slice of uniform density lies halfway between the two ends of the slice (show this using first part: c.m.$= \frac{\delta \int_a^b x\ dx}{\delta(b-a)} = \frac{b+a}{2}$).

We'll work out the next few examples from *first principles* so that you get a feel for the concepts behind the formulae we derived above.

Example 443.

Find the center of mass of the region bounded by the parabola $y = h^2 - x^2$ and the $x-$axis assuming that its mass density is uniform. Here, h is a fixed real constant,

(see Figure 209).

Solution We lay out this problem out just like when we calculate areas: But first, note that since the parabola is symmetric about the $y-$axis (or, equivalently, the line $x = 0$), the center of mass must lie along this line (since δ is uniform) and this forces $\overline{x} = 0$ without having to calculate it directly. Nevertheless, we will calculate it as a check.

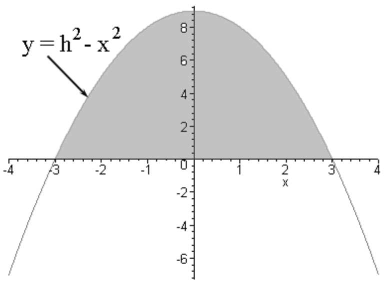


Figure 209.

- Make a thin slice (horizontal / vertical)
- Find its points of intersection with the curve and the axes
- Now find the center of mass $(\overline{x}_{slice}, \overline{y}_{slice})$ of the thin slice Since the density is uniform, $\delta = constant$ and so the center of mass is halfway down the slice. So its coordinates are

$$(\overline{x}_{slice}, \overline{y}_{slice}) = (x, \tfrac{h^2-x^2}{2}), \text{ or } \overline{x}_{slice} = x, \overline{y}_{slice} = \tfrac{h^2-x^2}{2}.$$

Next, the area of our slice,

$$dA = (base) \times (height) = dx\ ((h^2 - x^2) - 0) \qquad ,$$

or,
$$dA = (h^2 - x^2)\ dx.$$
Hence,

$$\begin{aligned} M_x &= \int \overline{y}_{slice}\delta\ dA \\ &= \int_{-h}^{h} (\frac{h^2 - x^2}{2})\delta\ (h^2 - x^2)\ dx \\ &= \frac{\delta}{2}\int_{-h}^{h} (h^2 - x^2)^2\ dx \\ &= 2\frac{\delta}{2}\int_{0}^{h} (h^2 - x^2)^2\ dx = \cdots \\ &= \frac{8h^5\delta}{15}. \end{aligned}$$

Remember that h is a <u>constant</u> here. On the other hand, since $\overline{x} = x$ we have

$$\textstyle\int_{-h}^{h} \overline{x}_{slice}\delta\ dA = \delta \int_{-h}^{h} x\ (h^2 - x^2)\ dx = 0, \leftarrow Why?$$

Finally, the mass of the region $m = \int_{-h}^{h} \delta dA$ or

$$\textstyle m = \int_{-h}^{h} \delta(h^2 - x^2)\ dx = 2\delta \int_{0}^{h} (h^2 - x^2)\ dx = \cdots = \frac{4}{3}\delta h^3 \qquad .$$

$$\textit{Thus the center of mass} \quad = (\overline{x}, \overline{y}) = (\tfrac{M_y}{M}, \tfrac{M_x}{M}) = (0, \tfrac{\frac{8h^5\delta}{15}}{\frac{4\delta h^3}{3}})$$

$$= (0, \tfrac{2h^2}{5})$$

Example 444.

Find the center of mass of a thin homogeneous (*i.e.*, uniform density) plate covering the region bounded by the curves $y = 2x^2 - 4x$ and $y = 2x - x^2$, (see Figure 210).

Solution Recall that the mid-point of the line joining two points (x_1, y_1) and (x_2, y_2) has coordinates $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

Make a thin slice (here, vertical) and find its center of mass. Since δ = constant, the center of mass of the slice is halfway down slice and so its coordinates are

$$\begin{aligned}(\overline{x}_{slice}, \overline{y}_{slice}) &= (\frac{x+x}{2}, \frac{(2x-x^2)+(2x^2-4x)}{2}) \\ &= (x, \frac{x^2-2x}{2}).\end{aligned}$$

We see that $\overline{x}_{slice} = x$ and $\overline{y}_{slice} = \frac{x^2-2x}{2}$. Now,

$$\begin{aligned}dA &= (base)\ (height) \\ &= (dx)\ ((2x-x^2)-(2x^2-4x)) \\ &= (6x-3x^2)\ dx.\end{aligned}$$

Thus

$$M_y = \int_0^2 \overline{x}_{slice}\delta\ dA = \int_0^2 x\delta\ (6x-3x^2)\ dx = \ldots = 4\delta.$$

Next,

$$\begin{aligned}M_x &= \int_0^2 \overline{y}_{slice}\ \delta\ dA = \delta\ \int_0^2 (\frac{x^2-2x}{2})\ (6x-3x^2)\ dx \\ &= \frac{\delta}{2}\cdot 3\int_0^2 (x^2-2x)\ (2x-x^2)\ dx \\ &= -\frac{3\delta}{2}\int_0^2 (x^2-2x)^2\ dx = -\frac{3\delta}{2}\int_0^2 (x^4-4x^3+4x^2)\ dx \\ &= \cdots \\ &= -\frac{8\delta}{5}.\end{aligned}$$

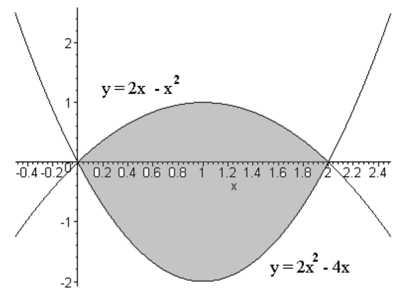


Figure 210.

Finally, the mass m of the region is given by $m = \int_0^2 \delta\ dA$, or

$$\begin{aligned}m &= \int_0^2 \delta\ (6x-3x^2)\ dx = \delta\int_0^2 (6x-3x^2)\ dx = \delta[3x^2-x^3]_0^2 \\ &= \cdots \\ &= 4\delta.\end{aligned}$$

Thus the coordinates of center of mass are given by

$$\begin{aligned}(\overline{x}, \overline{y}) &= (\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{4\delta}{4\delta}, \frac{-\frac{8\delta}{5}}{4\delta}) \\ &= (1, -\frac{2}{5}).\end{aligned}$$

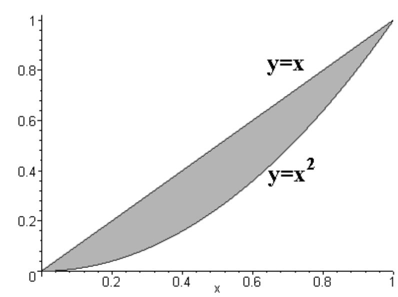


Figure 211.

Remarks: Note that $x = 1$ is a line of symmetry for the region. Since the region is assumed to be of uniform density ("homogeneous") the center of mass must be on this line. Furthermore, since the center of mass generally "*leans on the heavier side of a region*", in this case, the part of the region below the $x-$axis has more area and so it has more mass. So, the center of mass should be within it, which it is (since $\overline{y} < 0$).

Example 445.

The density of a thin plate (or "lamina") bounded by the curves $y = x^2$ and $y = x$ in the first quadrant is $\delta(x) = 12x$. Find the plate's center of mass, (see Figure 211).

Solution First we note that the plate is not of uniform density since $\delta(x) \neq$ *constant* throughout. From this expression for δ we see that the density increases with x, and is lowest when $x = 0$.

Now, the points of intersection are at $(0,0)$ and $(1,1)$. The center of mass of a thin (vertical) slice is

$$(\overline{x}_{slice}, \overline{y}_{slice}) = (x, \frac{x+x^2}{2}).$$

The density, δ of a slice is $\delta(x) = 12x$ while the area, dA, of a slice is $dA = (x-x^2)\ dx$. Next, the mass of the shaded region in Figure 211 is given by

$$m = \int_0^1 \delta(x)\ dA = \int_0^1 12x\ (x-x^2)\ dx = \int_0^1 (12x^2 - 12x^3)\ dx = 1.$$

Now,

$$\begin{aligned} M_y &= \int_0^1 \overline{x}_{slice}\delta(x)\ dA = \int_0^1 x\ (12x)\ (x-x^2)\ dx \\ &= \int_0^1 (12x^3 - 12x^4)\ dx = \cdots \\ &= \frac{3}{5}. \end{aligned}$$

Finally,

$$\begin{aligned} M_x &= \int_0^1 \overline{y}_{slice}\ \delta(x)\ dA = \int_0^1 (\frac{x+x^2}{2})\ (12x)\ (x-x^2)\ dx \\ &= 6 \int_0^1 x\ (x+x^2)\ (x-x^2)\ dx = 6 \int_0^1 x\ (x^2-x^4)\ dx \\ &= 6 \int_0^1 (x^3 - x^5)\ dx = 6 \left(\frac{x^4}{4} - \frac{x^6}{6} \right) \Big|_0^1 \\ &= 6\ [\frac{1}{4} - \frac{1}{6}] = \frac{3}{2} - 1 \end{aligned}$$

$$= \frac{1}{2}.$$

Combining these moments we get that the region's center of mass coordinates are

$$(\overline{x}, \overline{y}) = (\frac{M_y}{m}, \frac{M_x}{m}) = (\frac{\frac{3}{5}}{1}, \frac{\frac{1}{2}}{1}) = (\frac{3}{5}, \frac{1}{2}).$$

Exercise Set 44.

Find the center of mass of the following systems.

1. A thin string containing the masses $m_1 = 0.5, m_2 = 1.5$ (in *grams*) separated by a distance of 5 cm.
2. A thin string containing the masses $m_1 = 2, m_2 = 4, m_3 = 6$, each separated by a distance of 1 cm.
3. The system contains 3 masses placed at the points
$$m_1(0,0), m_2(0,1), m_3(1,0)$$
with $m_1 = 3,\ m_2 = 4,\ m_3 = 5$.
4. The system contains 3 masses placed at the points
$$m_1(0,0), m_2(2,0), m_3(1,\sqrt{3})$$
with $m_1 = 3,\ m_2 = 3,\ m_3 = 3$.

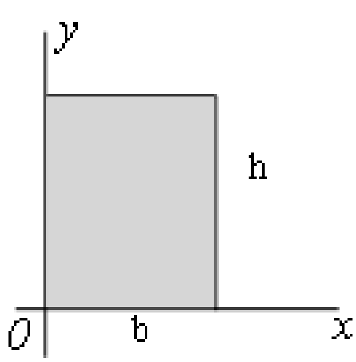


Figure 212.

5. A semicircle of radius R, centered at the origin and lying in the region $y \geq 0$, having a uniform density, $\delta = constant$.
6. A rectangle of base b and height h having a uniform density, see Figure 212.
7. The V-shaped homogeneous region bounded by the lines $y = x$, $y = -x$ and $y = 1$.
8. The V-shaped region bounded by the lines $y = x$, $y = -x$ and $y = 1$, where $\delta(x) = 1 - x$.
9. The homogeneous region bounded by the $x-$axis, the line $x = 1$ and $y = \sqrt{x}$.
10. The homogeneous region bounded by the $x-$axis, the line $x = 2$ and $y = \frac{x^2}{4}$.
11. The homogeneous region bounded by the $x-$axis and $y = 2\sin(\pi x)$ for $0 \leq x \leq 1$.
12. The region bounded by the $x-$axis and $y = e^x$ for $0 \leq x \leq 2$, where $\delta(x) = x$.
    - Use Integration by Parts here.
13. The region above the $x-$axis bounded by an arc of the circle $x^2 + y^2 = 4$ and the vertical lines $x = -1$ and $x = +1$, where $\delta(x) = 2$. This region looks like the doorway of an ancient Roman door (see Figure 213).
14. The region bounded by the curves $y = 2x^2 - 4x$ and $y = 2x - x^2$ where $\delta(x) = 2x$.

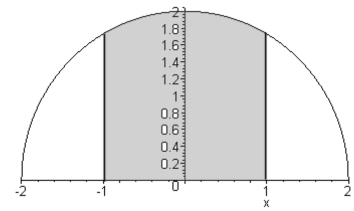


Figure 213.

Suggested Homework Set 36. *Problems 2, 4, 7, 13, 14*

NOTES:

8.6 Chapter Exercises

In each of the following exercises: a) Sketch the given region, **b)** Choose a typical slice, **c)** Determine the area of the region, and **d)** Find the volume of the solid of revolution obtained when the given region is rotated about the stated axis.

1. The region bounded by the curves $y = x$, $y = 1$, $x = 0$. Rotate this area about the $y-$axis.
2. The region bounded by the curves $y = x$, $x = 1$, $y = 0$. Rotate this area about the $x-$axis.
3. The region bounded by the curves $y = 2x$, $y = 1$, $x = 0$. Rotate this area about the $x-$axis.
4. The region bounded by the curves $y = 2x$, $x = 1$, $y = 0$. Rotate this area about the $y-$axis.
5. The region bounded by the curves $y = x$, $y = 2x$, $x = 2$. Rotate this area about the $y-$axis.
6. The region bounded by the curves $y = x^2$ and $y = 1$. Rotate this area about the $x-$axis.
7. The region bounded by the curve $y = \sin x$ between $x = 0$ and $x = \pi$. Rotate this area about the $x-$axis.
8. The region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$. Rotate this area about the $y-$axis.
9. The region bounded by the curve $y = xe^x$ between $x = 0$ and $x = 1$. Rotate this area about the $x-$axis.
   • The graph is that of a nice increasing function. Use a vertical slice and Integration by Parts.
10. The region bounded by the curve $y = xe^{-x}$ between $x = 0$ and $x = 2$. Rotate this area about the $y-$axis.
    • DON'T use a horizontal slice! Apply Integration by Parts.
11. The loop enclosed by the curves $y = x^2$ and $y^2 = x$. Rotate this area about the $x-$axis.
    • Find the two points of intersection and use a vertical slice!
12. The closed region bounded by the curves $y = x^3$, $y = 1$, and $y = x^3 - 3x + 1$. Rotate this area about the $x-$axis.
    • *Hint:* The region looks like a curvilinear triangle. Find all points of intersection. Divide the region into two pieces and then use vertical slices. The integrals are easy to evaluate.

Suggested Homework Set 37. *Do problems 1, 6, 11, 12*

8.7 Using Computer Algebra Systems

Use your favorite Computer Algebra System (CAS), like Maple, MatLab, etc., or even a graphing calculator to answer the following questions. In all the problems it is best to use some form of numerical integration.

1. Approximate the volume of a fruit bowl obtained by rotating the graph of the function $y = e^{-x^2}$ between $x = 0$ and $x = 0.4$ about the x-axis.

2. Find the surface area of an elliptical pool of water whose edge can be approximated by the equation $2x^2 + 9y^2 = 225$ where x and y are measured in meters.

3. A solid steel rivet is made by rotating the region defined by the graph of the function
$$f(x) = \begin{cases} 2, & \text{if } 0 \le x \le 2, \\ x - 1, & \text{if } 2 \le x \le 2.2. \end{cases}$$
about the x-axis, where x is in centimeters. Find the volume of the rivet.

4. Find the center of mass of the rivet in the preceding example assuming that its density is uniform.

5. Determine the amount of earth required to fill a flower pot to the brim if the pot is defined by rotating the graph of the function $2y = x + 1$ between $x = 10$ and $x = 30$ about the x-axis (where x is in centimeters).

6. Find the volume of the UFO defined by rotating the region described by (careful here!)
$$y = \begin{cases} x^2, & \text{if } 0 \le y \le 4, \\ 8 - x^2, & \text{if } 4 \le y \le 8. \end{cases}$$
when it is rotated about the y-axis, where y is in meters.

7. Find the center of mass of the saucer-shaped object in the preceding example assuming that its density is uniform.

8. A small glass paper-weight is to be made by revolving the triangular region defined by $y = 1 - 0.4x$, $0 \le x \le 2.4$, about the x-axis, where x is in centimeters. Determine the amount of glass required to build such a paper-weight.

9. A spool of thin black thread is wound tightly around a cylindrical drum so as to form a dark region that can be modelled by the set, $\mathcal{D}$, of points bounded by the curves $y = 1$, $y = 2$, $x = 4$ and $y = x + 1$ in the first quadrant.
   - Find the volume of the region $\mathcal{R}$, obtained by rotating $\mathcal{D}$ about the x-axis, where x is in centimeters.
   - If the thread itself is cylindrical and has radius 0.05 millimeters, determine the approximate *length* of thread that is wound up in $\mathcal{R}$.

10. **The jellybean jar problem** An ellipsoid is a three-dimensional analogue of an ordinary ellipse (squashed circle) except that it has three semi-axes instead of two. It is known that if an ellipsoid has semi-axes length given by $x = a$, $y = b$ and $z = c$ then its volume is given by $4\pi abc/3$. These ellipsoids can be used to model ordinary jellybeans. Let's assume that a typical jellybean has $a = 1$, $b = 0.5$ and $c = 0.3$, all units being centimeters. Approximate the number of (essentially identical) jellybeans that can fit in a cylindrical jar of volume 2 liters. (Of course there will be gaps whose volume is hard to estimate. Nevertheless, you can come up with a rough estimate that sometimes can actually clinch the first prize!).

NOTES:

Chapter 9

Appendix A: Review of Exponents and Radicals

In this section we review the basic laws governing exponents and radicals. This material is truly necessary for a manipulating fundamental expressions in Calculus. We recall that if $a > 0$ is any real number and r is a positive integer, the symbol a^r is shorthand for the product of a with itself r-times. That is, $a^r = a \cdot a \cdot a \cdots a$, where there appears r a's on the right. Thus, $a^3 = a \cdot a \cdot a$ while $a^5 = a \cdot a \cdot a \cdot a \cdot a$, etc. By definition we will always take it that $a^0 = 1$, regardless of the value of a, so long as it is not equal to zero, and $a^1 = a$ for any a.

Generally if $r, s \geq 0$ are any two non-negative real numbers and $a, b > 0$, then the Laws of Exponents say that

$$a^r \cdot a^s = a^{r+s} \tag{9.1}$$
$$(a^r)^s = a^{r \cdot s}, \qquad (a^r)^{-s} = a^{-r \cdot s} \tag{9.2}$$
$$(ab)^r = a^r \cdot b^r \tag{9.3}$$
$$\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \tag{9.4}$$
$$\frac{a^r}{a^s} = a^{r-s}. \tag{9.5}$$

The Laws of Radicals are similar. They differ only from the Laws of Exponents in their representation using *radical* symbols rather than powers. For example, if $p, q > 0$ are integers, and we interpret the symbol $a^{\frac{p}{q}}$ as the q-th root of the number a to the power of p, *i.e.*,

$$a^{\frac{p}{q}} = \sqrt[q]{\boldsymbol{a^p}} = \left(\sqrt[q]{\boldsymbol{a}}\right)^p, \tag{9.6}$$

we obtain the Laws of Radicals

$$\sqrt[p]{a^p} = a, \quad \left(\sqrt[p]{a}\right)^p = a$$
$$\sqrt[p]{ab} = \sqrt[p]{a} \cdot \sqrt[p]{b}$$
$$\sqrt[p]{\frac{a}{b}} = \frac{\sqrt[p]{a}}{\sqrt[p]{b}}$$
$$\sqrt[p]{\sqrt[r]{a}} = \sqrt[pr]{a}.$$

Note that we obtain the rule $\sqrt[p]{ab} = \sqrt[p]{a} \cdot \sqrt[p]{b}$ by setting $r = \frac{1}{p}$ in (9.3) above and using the symbol interpreter $a^{\frac{1}{p}} = \sqrt[p]{a}$. For example, by (9.6), we see that $9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = 3^3 = 27$, while $(27)^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$.

We emphasize that these two laws are *completely general* in the sense that the symbols a, b appearing in them need not be single numbers only (like 3 or 1.52) but can be any abstract combination of such numbers or even other symbols and numbers together! For example, it is the case that

$$\left(2x + y\sqrt{x}\right)^{-1} = \frac{1}{2x + y\sqrt{x}}$$

and this follows from the fact that

$$a^{-1} = \frac{1}{a}$$

for any non-zero number a. Incidentally, this latest identity follows from (9.1) with $r = 1, s = -1$ and the defintion $a^0 = 1$. In order to show the power of these formulae we use the Box Method of Section 1.2 to solidify their meaning. Thus, instead of writing the Laws of Exponents and Radicals as above, we rewrite them in the form

$$\begin{aligned}
\square^r \cdot \square^s &= \square^{r+s} && (9.7)\\
(\square^r)^s &= \square^{r\cdot s}, \qquad (\square^r)^{-s} = \square^{-r\cdot s} && (9.8)\\
(\square_1\ \square_2)^r &= (\square_1)^r \cdot (\square_2)^r && (9.9)\\
\left(\frac{\square_1}{\square_2}\right)^r &= \frac{(\square_1)^r}{(\square_2)^r} && (9.10)\\
\frac{\square^r}{\square^s} &= \square^{r-s}, && (9.11)
\end{aligned}$$

and remember that we can put any abstract combination of numbers or even other symbols and numbers together inside the Boxes in accordance with the techniques described in Section 1.2 for using the Box Method. So, for example, we can easily see that

$$\left(2x + y\sqrt{x}\right)^{-1} = \frac{1}{2x + y\sqrt{x}}$$

alluded to above since we know that

$$\square^{-1} = \frac{1}{\square},$$

and we can put the group of symbols $2x + y\sqrt{x}$ *inside* the Box so that we see

$$\left(\boxed{2x + y\sqrt{x}}\right)^{-1} = \frac{1}{\boxed{2x + y\sqrt{x}}}$$

and then remove the sides of the box to get the original identity.

Another example follows: Using (9.9) above, that is,

$$(\square_1\ \square_2)^r = (\square_1)^r \cdot (\square_2)^r$$

we can put the symbol $3y$ inside box 1, *i.e.,* $\square_1$, and $x + 1$ inside box 2, *i.e.,* $\square_2$, to find that if $r = 2$ then, once we remove the sides of the boxes,

$$(3y(x + 1))^2 = (3y)^2(x + 1)^2 = 9y^2(x + 1)^2.$$

The Box Method's strength lies in assimilating large masses of symbols into one symbol (the box) for ease of calculation!

Remark Using the same ideas and (9.8) we can show that $\left(3^2\right)^{-3} = 3^{-2\cdot 3} = 3^{-6}$ and NOT equal to 3^{2-3} as some might think! You should leave the parentheses alone and not drop them when they are present!

Of course we can put anything we want inside this box so that (if we put $\sqrt{2}x - 16xy^2 + 4.1$ inside) it is still true that (use (9.8))

$$\left(\square^2\right)^{-3} = \left(\square^{-3}\right)^2 = \square^{-3\cdot 2} = \square^{-6}$$

or

$$\left(\left(\sqrt{2}x - 16xy^2 + 4.1\right)^2\right)^{-3} = \left(\sqrt{2}x - 16xy^2 + 4.1\right)^{-6}.$$

Finally, don't forget that

$$\square^0 = 1, \quad \square^1 = \square, \quad \text{and,} \quad \square^{-1} = \frac{1}{\square}.$$

Example 446. Simplify the product $2^3 3^2 2^{-1}$, without using your calculator.

Solution We use the Laws of Exponents:

$$\begin{aligned} 2^3 3^2 2^{-1} &= 2^3 2^{-1} 3^2 \\ &= 2^{3-1} 3^2 && \text{by (9.7)} \\ &= 2^2 3^2 \\ &= (2\cdot 3)^2 && \text{by (9.9)} \\ &= 6^2 \\ &= 36. \end{aligned}$$

Example 447. Simplify the expression $(2xy)^{-2} 2^3 (yx)^3$.

Solution Use the Laws (9.7) to (9.11) in various combinations:

$$\begin{aligned} (2xy)^{-2} 2^3 (yx)^3 &= (2xy)^{-2} (2yx)^3 && \text{by (9.9)} \\ &= (2xy)^{-2+3} && \text{by (9.7)} \\ &= (2xy)^1 \\ &= 2xy. \end{aligned}$$

Example 448. Simplify $2^8 4^{-2} (2x)^{-4} x^5$.

Solution We use the Laws (9.7) to (9.11) once again.

$$\begin{aligned} 2^8 4^{-2} (2x)^{-4} x^5 &= 2^8 (2^2)^{-2} (2x)^{-4} x^5 \\ &= 2^8 2^{-4} (2x)^{-4} x^5 && \text{by (9.8)} \\ &= 2^{8-4} (2x)^{-4} x^5 && \text{by (9.7)} \\ &= 2^4 2^{-4} x^{-4} x^5 && \text{by (9.9)} \\ &= 2^{4-4} x^{-4+5} && \text{by (9.7)} \\ &= 2^0 x^1 \\ &= x. \end{aligned}$$

Example 449. Simplify $\dfrac{2^{2^3}\left(2^2\right)^{-3}}{4}$.

Solution Work out the highest powers first so that since $2^3 = 8$ it follows that $2^{2^3} = 2^8$. Thus,

$$\begin{aligned}\frac{2^{2^3}\left(2^2\right)^{-3}}{4} &= \frac{2^8 2^{-3\cdot 2}}{4} \qquad \text{by (9.8)}\\ &= \frac{2^8 2^{-6}}{2^2} = \frac{2^{8-6}}{2^2} = \frac{2^2}{2^2}\\ &= 1.\end{aligned}$$

Example 450. Write $\left((49)^{-\frac{1}{2}}\right)^3$ as a rational number (ordinary fraction).

Solution $\left((49)^{-\frac{1}{2}}\right)^3 = \left((49)^{\frac{1}{2}}\right)^{-3}$ by (9.8). Next, $(49)^{\frac{1}{2}} = \sqrt{49} = 7$, so, $\left((49)^{\frac{1}{2}}\right)^{-3} = 7^{-3} = \dfrac{1}{7^3} = \dfrac{1}{343}$.

Example 451. Simplify as much as possible: $(16)^{-\frac{1}{6}} 4^{\frac{7}{3}} (256)^{-\frac{1}{4}}$.

Solution The idea here is to rewrite the bases $4, 16, 256$, in lowest common terms, if possible. Thus,

$$\begin{aligned}(16)^{-\frac{1}{6}} 4^{\frac{7}{3}} (256)^{-\frac{1}{4}} &= (4^2)^{-\frac{1}{6}} 4^{\frac{7}{3}} (4^4)^{-\frac{1}{4}}\\ &= 4^{-\frac{1}{3}} 4^{\frac{7}{3}} 4^{-1} \qquad \text{by (9.8)}\\ &= 4^{-\frac{1}{3}+\frac{7}{3}-1} \qquad \text{by (9.7)}\\ &= 4^{\frac{6}{3}-1}\\ &= 4.\end{aligned}$$

Example 452. Simplify to an expression with positive exponents: $\dfrac{x^{-\frac{1}{6}} x^{\frac{2}{3}}}{x^{\frac{5}{12}}}$.

Solution Since the bases are all the same, namely, x, we only need to use a combination of (9.7) and (9.11). So,

$$\begin{aligned}\frac{x^{-\frac{1}{6}} x^{\frac{2}{3}}}{x^{\frac{5}{12}}} &= \frac{x^{-\frac{1}{6}+\frac{2}{3}}}{x^{\frac{5}{12}}} \qquad \text{by (9.7)}\\ &= \frac{x^{\frac{1}{2}}}{x^{\frac{5}{12}}}\\ &= x^{\frac{1}{2}-\frac{5}{12}} \qquad \text{by (9.11)}\\ &= x^{\frac{1}{12}}.\end{aligned}$$

Example 453. Show that if $r \neq 1$ then $1 + r + r^2 = \dfrac{1-r^3}{1-r}$.

Solution It suffices to show that $(1 + r + r^2)(1 - r) = 1 - r^3$ for any value of r.

Division by $1 - r$ (only valid when $r \neq 1$) then gives the required result. Now,

$$\begin{aligned}
(1 + r + r^2) \cdot (1 - r) &= (1 + r + r^2) \cdot (1) + (1 + r + r^2) \cdot (-r) \\
&= (1 + r + r^2) + (1) \cdot (-r) + r \cdot (-r) + r^2 \cdot (-r) \\
&= 1 + r + r^2 - r - r^2 - r^3 \qquad \text{by } (9.7) \\
&= 1 - r^3
\end{aligned}$$

and that's all.

Example 454. For what values of a is $x^4 + 1 = (x^2 + ax + 1) \cdot (x^2 - ax + 1)$?

Solution We simply multiply the right side together, compare the coefficients of like powers and then find a. Thus,

$$\begin{aligned}
x^4 + 1 &= (x^2 + ax + 1) \cdot (x^2 - ax + 1) \\
&= x^2 \cdot (x^2 - ax + 1) + ax \cdot (x^2 - ax + 1) + 1 \cdot (x^2 - ax + 1) \\
&= (x^4 - ax^3 + x^2) + (ax^3 - a^2x^2 + ax) + (x^2 - ax + 1) \\
&= x^4 - ax^3 + x^2 + ax^3 - a^2x^2 + ax + x^2 - ax + 1 \\
&= x^4 + (2 - a^2) \cdot x^2 + 1.
\end{aligned}$$

Comparing the coefficients on the left and right side of the last equation we see that $2 - a^2 = 0$ is necessary. This means that $a^2 = 2$ or $a = \pm\sqrt{2}$.

Note Either value of a in Example 454 gives the same factors of the polynomials $x^4 + 1$. More material on such factorization techniques can be found in Chapter 5.

Exercise Set 45.

Simplify as much as you can to an expression with positive exponents.

1. $16^2 \times 8 \div 4^3$
2. $(25^2)^{\frac{1}{2}}$
3. $2^4 4^2 2^{-2}$
4. $\frac{3^2 4^2}{12}$
5. $5^3 15^{-2} 3^4$
6. $(2x + y) \cdot (2x - y)$
7. $1 + (x - 1)(x + 1)$
8. $\left((25)^{-\frac{1}{2}}\right)^2 + 5^{-2}$
9. $(4x^2y)^2 2^{-4} x^{-2} y$
10. $(1 + r + r^2 + r^3) \cdot (1 - r)$
11. $(a^9 b^{15})^{\frac{1}{3}}$
12. $(16a^{12})^{\frac{3}{4}}$
13. $\frac{x^{\frac{1}{4}} x^{-\frac{2}{3}}}{x^{\frac{1}{6}}}$
14. $\left(\frac{1}{16}\right)^{-\frac{3}{2}}$

15. $\left(1+5^{\frac{1}{3}}\right)\cdot\left(1-5^{\frac{1}{3}}+(25)^{\frac{1}{3}}\right)$
16. $\left(9x^{-8}\right)^{-\frac{3}{2}}$
17. $9^{-\frac{1}{6}}3^{\frac{7}{3}}(81)^{-\frac{1}{4}}$
18. $\dfrac{(12)^{\frac{3}{2}}\,(16)^{\frac{1}{8}}}{(27)^{\frac{1}{6}}\,(18)^{\frac{1}{2}}}$
19. $\dfrac{3^{n+1}9^{n}}{(27)^{\frac{2n}{3}}}$
20. $\dfrac{\sqrt{xy}\;x^{\frac{1}{3}}\;2y^{\frac{1}{4}}}{(x^{10}y^{9})^{\frac{1}{12}}}$

21. Show that there is **no** real number a such that $(x^2+1)=(x-a)\cdot(x+a)$.
22. Show that $(1+x^2+x^4)\cdot(1-x^2)=1-x^6$.
23. Find an expression for the quotient $\dfrac{1-x^8}{1-x}$ as a sum of powers of x only.
24. Show that $(x-1)(x+1)(1+x^2)+1=x^4$.
25. Show that $3\left(x^2yz\right)^3\div x^4y^3-3x^2z^3=0$ for any choice of the variables x,y,z so long as $xy\neq 0$.
26. Using the identities (9.7) and $a^0=1$ only, show that $a^{-r}=\dfrac{1}{a^r}$ for any real number r.
27. Show that if r,s are any two integers and $a>0$, then $(a)^{-rs}=(a^r)^{-s}=(a^s)^{-r}$.
28. Give an example to show that

$$2^{x^y}\neq(2^x)^y\,.$$

In other words, find two numbers x,y that have this property.

29. Show that for any number $r\neq -1$ we have the identity $1-r+r^2=\dfrac{1+r^3}{1+r}$ and use this to deduce that for any value of $x\neq -2$,

$$1-\frac{x}{2}+\frac{x^2}{4}=\frac{x^3+8}{4(x+2)}.$$

30. If $a>0$ and $2x=a^{\frac{1}{2}}+a^{-\frac{1}{2}}$ show that

$$\frac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}}=\frac{a-1}{2}.$$

Suggested Homework Set 38. *Do all even-numbered problems from 2 - 28.*

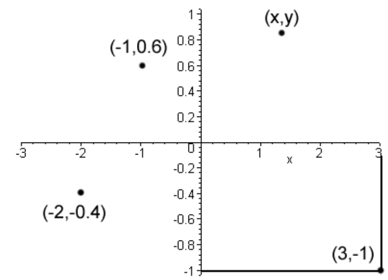
Web Links

Many more exercises may be found on the web site:

http://www.khanacademy.org/math/algebra/exponents-radicals

Chapter 10

Appendix B: The Straight Line

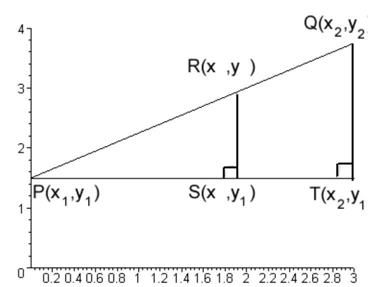


Points in the Euclidean plane

Figure 214.

In this section we review one of the most fundamental topics of analytic geometry, the representation of a **straight line** with respect to a given set of coordinate axes. We recall that a point in the Euclidean plane is denoted by its two coordinates (x, y) where x, y are real numbers either positive, negative or zero, see Figure 214.

Thus, the point $(3, -1)$ is found by moving three positive units to the right along the x-axis and one unit "down" (because of the negative sign) along a line parallel to the y-axis. From the theory of plane Euclidean geometry we know that two given points determine a unique (straight) line. Its *equation* is obtained by describing every point on the straight line in the form $(x, y) = (x, f(x))$ where $y = f(x)$ is the equation of the straight line defined by some function f. To find this equation we appeal to basic Euclidean geometry and, in particular, to the result that states that *any two similar triangles in the Euclidean plane have proportional sides*, see Figure 215. This result will be used to find the equation of a straight line as we'll see.



The triangles PRS and PQT have proportional sides as they are similar.

Figure 215.

We start off by considering two given points P and Q having coordinates (x_1, y_1) and (x_2, y_2) respectively. Normally, we'll write this briefly as $P(x_1, y_1)$ etc. Remember that the points P, Q are given ahead of time. Now, we join these two points by means of a straight line $\mathcal{L}$ and, on this line $\mathcal{L}$ we choose some point that we label as $R(x, y)$. For convenience we will assume that R is between P and Q.

Next, see Figure 215, we construct the two similar right-angled triangles $\triangle PQT$ and $\triangle PRS$. Since they are similar the length of their sides are proportional and so,

$$\frac{PS}{SR} = \frac{PT}{TQ}.$$

In terms of the coordinates of the points in question we note that $PS = x - x_1$, $SR = y - y_1$, $PT = x_2 - x_1$, $TQ = y_2 - y_1$. Rewriting the above proportionality relation in terms of these coordinates we get

$$\frac{x - x_1}{y - y_1} = \frac{x_2 - x_1}{y_2 - y_1},$$

or equivalently, solving for y and rewriting the equation, we see that

$$y = mx + b$$

where
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
is called the **slope of the straight line** and the number $b = y_1 - mx_1$ is called the ***y*-intercept** (*i.e.,* that value of y obtained by setting $x = 0$). The x-intercept is that value of x obtained by setting $y = 0$. In this case, the ***x*-intercept** is the complicated-looking expression

$$x = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1}.$$

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be any two points on a line $\mathcal{L}$. The equation of $\mathcal{L}$ is given by

$$y = mx + b \tag{10.1}$$

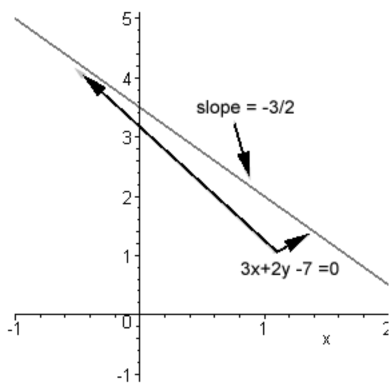
and will be called the **slope-intercept form of a line** where

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{10.2}$$

is called the **slope of the straight line** and the number

$$b = y_1 - mx_1 \tag{10.3}$$

is the ***y*-intercept**.



The graph of the line $3x+2y-7=0$ with a negative slope equal to $-3/2$.

Figure 216.

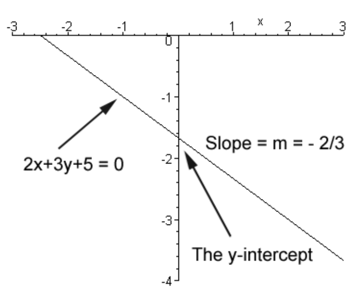
Example 455. Find the slope of the line whose equation is $3x + 2y - 7 = 0$.

Solution First, let's see if we can rewrite the given equation in "slope-intercept form". To do this, we solve for y and then isolate it (by itself) and then compare the new equation with the given one. So, subtracting $3x - 7$ from both sides of the equation gives $2y = 7 - 3x$. Dividing this by 2 (and so isolating y) gives us $y = \frac{7}{2} - \frac{3}{2}x$. Comparing this last equation with the form $y = mx + b$ shows that $m = -\frac{3}{2}$ and the y-intercept is $\frac{7}{2}$. Its graph is represented in Figure 216.

Example 456. Find the equation of the line passing through the points $(2, -3)$ and $(-1, -1)$.

Solution We use equations (10.1), (10.2) and (10.3). Thus, we label the points as follows: $(x_1, y_1) = (2, -3)$ and $(x_2, y_2) = (-1, -1)$. But the slope m is given by (10.2), *i.e.,*
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1+3}{-1-2} = -\frac{2}{3}.$$
On the other hand the y-intercept is given by
$$b = y_1 - mx_1 = -3 + \frac{2}{3}(2) = -\frac{5}{3}.$$

The equation of the line is therefore $y = -\frac{2}{3}x - \frac{5}{3}$ or, equivalently, $2x + 3y + 5 = 0$, (see Figure 217).



The line $2x + 3y + 5 = 0$ and its y-intercept.

Figure 217.

Remark: It doesn't matter *which* point you label with the coordinates (x_1, y_1), you'll still get the same slope value and y-intercept! In other words, if we interchange

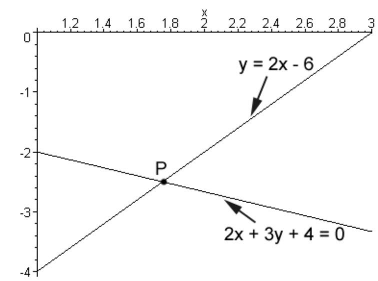
the roles of (x_1, y_1) and (x_2, y_2) we get the *same* value for the slope, etc. and the *same* equation for the line.

Example 457. Find the equation of the line through $(1, 4)$ having slope equal to 2.

Solution We are given that $m = 2$ in (10.1), so the equation of our line looks like $y = 2x + b$ where b is to be found. But we are given that this line goes thourhg the point $(1, 4)$. This means that we can set $x = 1$ and $y = 4$ in the equation $y = 2x + b$ and use this to find the value of b. In other words, $4 = 2 \cdot 1 + b$ and so $b = 2$. Finally, we see that $y = 2x + 2$ is the desired equation.

Example 458. Find the equation of the line whose x-intercept is equal to -1 and whose y-intercept is equal to -2.

Solution Once again we can use (10.1). Since $y = mx + b$ and the y-intercept is equal to -2 this means that $b = -2$ by definition. Our line now takes the form $y = mx - 2$. We still need to find m though. But by definition the fact that the x-intercept is equal to -1 means that when $y = 0$ then $x = -1$, *i.e.,* $0 = m \cdot (-1) - 2$ and this leads to $m = -2$. Thus, $y = -2x - 2$ is the equation of the line having the required intercepts.



The two lines $2x+3y+4 = 0$ and $y = 2x-6$ and their point of intersection $P(\frac{7}{4}, -\frac{5}{2})$

Figure 218.

Example 459. Find the point of intersection of the two lines $2x + 3y + 4 = 0$ and $y = 2x - 6$.

Solution The point of intersection is necessarily a point, let's call it (x, y) once again, that belongs to *both* the lines. This means that $2x + 3y + 4 = 0$ AND $y = 2x - 6$. This gives us a system of two equations in the two unknowns (x, y). There are two ways to proceed; (1): We can isolate the y-terms, then equate the two x-terms and finally solve for the x-term, or (2): Use the method of elimination. We use the first of these methods here.

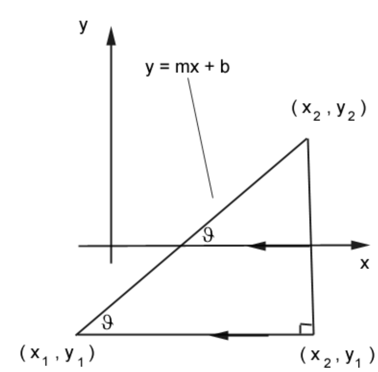
Equating the two y-terms means that we have to solve for y in each equation. But we know that $y = 2x - 6$ and we also know that $3y = -2x - 4$ or $y = -\dfrac{2}{3}x - \dfrac{4}{3}$. So, equating these two y's we get

$$2x - 6 = -\frac{2}{3}x - \frac{4}{3}$$

or, equivalently,

$$6x - 18 = -2x - 4.$$

Isolating the x, gives us $8x = 14$ or $x = \frac{7}{4}$. This says that the x-coordinate of the required point of intersection is given by $x = \frac{7}{4}$. To get the y-coordinate we simply use EITHER one of the two equations, plug in $x = \frac{7}{4}$ and then solve for y. In our case, we set $x = \frac{7}{4}$ in, say, $y = 2x - 6$. This gives us $y = 2 \cdot (\frac{7}{4}) - 6 = -\frac{5}{2}$. The required point has coordinates $(\frac{7}{4}, -\frac{5}{2})$, see Figure 218.



The angle θ between the line $y = mx + b$ and the x-axis is related to the slope m of this line via the relation $m = \tan\theta$.

Figure 219.

Prior to discussing the *angle between two lines* we need to recall some basic notions from Trigonometry, see Appendix 11. First we note that the slope m of a line whose equation is $y = mx + b$ is related to the *angle* that the line itself makes with the x-axis. A look at Figure 219 shows that, in fact,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta,$$

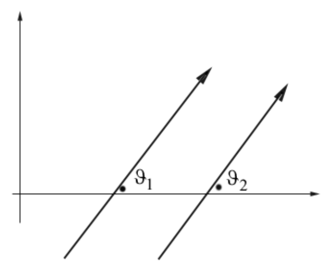
by definition of the tangent of this angle.

So,

$$\boxed{m = \text{Slope} = \tan\theta}$$

where the angle θ is usually expressed in **radians** in accordance with the conventions of Calculus.

Now, if two lines are parallel their corresponding angles are equal (this is from a really old result of Euclid - sometimes called the **corresponding angle theorem**, CAT, for short). This means that the angle that each one makes with the x-axis is the same for each line (see Figure 220), that is $\theta_2 = \theta_1$. But this means that the slopes are equal too, right? Okay, it follows that if two lines are parallel, then their slopes are equal and conversely, if two lines have equal slopes then they must be parallel. If $\theta_2 = \theta_1 = \frac{\pi}{2}$, the lines are still parallel but they are now perpendicular with respect to the x-axis. In this case we say they have no slope or their slope is infinite. Conversely, if two lines have no slopes they are parallel as well (just draw a picture).



Parallel lines have the same slope and, conversely, if two lines have the same slope they are parallel

Figure 220.

We now produce a relation that guarantees the *perpendicularity* of two given lines. For instance, a glance at Figure 221 shows that if θ_1, θ_2 are the angles of inclination of the two given lines and we assume that these two lines are perpendicular, then, by a classical result of Euclidean geometry, we know that

$$\begin{aligned}
\theta_2 &= \theta_1 + \frac{\pi}{2} \\
\tan\theta_2 &= \tan\left(\theta_1 + \frac{\pi}{2}\right) \\
&= -\cot\theta_1 \\
&= -\frac{1}{\tan\theta_1}.
\end{aligned}$$

Since $m_2 = \tan\theta_2, m_1 = \tan\theta_1$, it follows that $m_2 = -\frac{1}{m_1}$. We have just showed that two lines having slopes m_1, m_2 are perpendicular only when

$$\boxed{m_2 = -\frac{1}{m_1}}$$

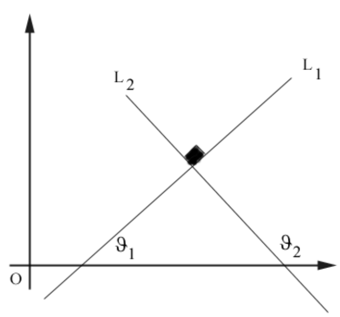


Figure 221.

that is, two lines are perpendicular only when the product of their slopes is the number -1. The converse is also true, that is, if two lines have the product of their slopes equal to -1 then they are perpendicular. This relates the geometrical notion of perpendicularity to the stated relation on the slopes of the lines. It follows from this that if a line has its slope equal to zero, then it must be parallel to the x-axis while if a line has no slope (or its slope is infinite) then it must be parallel to the y-axis.

Example 460. Find the slopes of the sides of the triangle whose vertices are $(6,2),(3,5)$ and $(5,7)$ and show that this is a right-triangle.

Solution Since three distinct points determine a unique triangle on the plane it suffices to find the slopes of the lines making up its sides and then showing that the product of the slopes of two of them is -1. This will prove that the triangle is a right-angled triangle.

Now the line, say $\mathcal{L}_1$, joining the points $(6,2),(3,5)$ has slope $m_1 = \frac{5-2}{3-6} = -1$ while the line, $\mathcal{L}_2$, joining the points $(3,5)$ and $(5,7)$ has slope $m_2 = \frac{7-5}{5-3} = 1$. Finally, the line, $\mathcal{L}_3$, joining $(6,2)$ to $(5,7)$ has slope $m_3 = \frac{7-2}{5-6} = -5$. Since $m_1 \cdot m_2 = (-1)\cdot(1) = -1$ it follows that those two lines are perpendicular, see

Figure 222. Note that we didn't actually have to calculate the *equations* of the lines themselves, just the *slopes*!

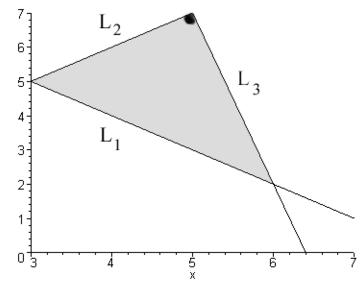


Figure 222.

Example 461. Find the equation of the straight line through the point $(6,-2)$ that is (a) parallel to the line $4x-3y-7=0$ and (b) perpendicular to the line $4x-3y-7=0$.

Solution (a) Since the line passes through $(x_1,y_1)=(6,-2)$ its equation has the form $y-y_1=m_1(x-x_1)$ or $y=m_1(x-6)-2$ where m_1 is its slope. On the other hand, since it is required to be parallel to the $4x-3y-7=0$ the two must have the *same* slope. But the slope of the given line is $m=\frac{4}{3}$. Thus, $m_1=\frac{4}{3}$ as well and so the line parallel to $4x-3y-7=0$ has the equation $y=\frac{4}{3}(x-6)-2$ or, equivalently (multiplying everything out by 3), $3y-4x+30=0$.

(b) In this case the required line must have its slope equal to the negative reciprocal of the first, that is $m_1=-\frac{3}{4}$ since the slope of the given line is $m=\frac{4}{3}$. Since $y=m_1(x-6)-2$, see above, it follows that its equation is $y=-\frac{3}{4}(x-6)-2$ or, equivalently, $4y+3x-10=0$.

Exercise Set 46.

1. Find the slope of the line whose equation is $2x-3y=8$
2. Find the slope of the line whose equation is $2x-3y=-8$
3. Find the slope of the line whose equation is $y-3x=2$
4. Find the equation of the line passing through the point $(2,-4)$ and $(6,7)$
5. Find the equation of the line passing through the point $(-4,-5)$ and $(-2,-3)$
6. Find the equation of the line passing through $(-1,-3)$ having slope -2
7. Find the equation of the line passing through $(6,-2)$ having slope $\frac{4}{3}$
8. Write the equation of the line whose $x-$intercept is 2 and whose $y-$intercept is 3
9. Write the equation of the line whose $x-$intercept is $\frac{1}{2}$ and whose $y-$intercept is $\frac{1}{3}$
10. Find the point of intersection of the two lines $y=x+1$ and $2y+x-1=0$
11. Find the points of intersection of the two lines $2y=2x+2$ and $3y-3x-3=0$. Explain your answer.
12. Find the point of intersection (if any) of the two lines $y-x+1=0$ and $y=x$
13. Recall that the distance between two points whose coordinates are $A(x_1,y_1)$, $B(x_2,y_2)$ is given by

$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

    Of course, this quantity AB is also equal to the length of the line segment joining A to B. Use this information to answer the following questions about the triangle formed by the points $A(2,0)$, $B(6,4)$ and $C(4,-6)$: (a) Find the equation of the line through AB; (b) Find the length of the altitude from C to AB (*i.e.,* the length of the perpendicular line through C meeting AB); (c) Find the area of this triangle ABC
14. Find the equation of the straight line through $(1,1)$ and perpendicular to the line $y=-x+2$
15. Find the equation of the straight line through $(1,1)$ and parallel to the line $y=-x+2$

Chapter 11

Appendix C: A Quick Review of Trigonometry

In this chapter we give a quick review of those concepts from Trigonometry that are essential for an understanding of the basics of Calculus. It will be assumed that the reader has some acquaintance with Trigonometry at the High-School level, although the lack of such knowledge should not deter the student from reading this Appendix. For basic notions regarding lines, their equations, distance formulae, etc. we refer the reader to the previous chapter on straight lines, Chapter 10. We also assume knowledge of the notions of an angle, and basic results from ordinary geometry dealing with triangles. If you don't remember any of this business just pick up any book on geometry from a used bookstore (or the university library) and do a few exercises to refresh your memory.

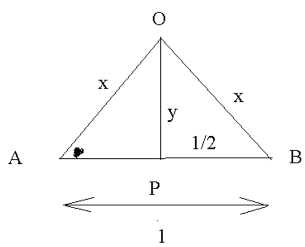
We recall that *plane trigonometry* (or just trigonometry) derives from the Greek and means something like *the study of the measure of triangles* and this, on the plane, or in two dimensions, as opposed to *spherical trigonometry* which deals with the same topic but where triangles are drawn on a sphere (like the earth).

The quick way to review trigonometry is by relying on the *unit circle* whose equation is given by $x^2 + y^2 = 1$ in plane (Cartesian) coordinates. This means that any point (x, y) on the circle has the coordinates related by the fundamental relation $x^2 + y^2 = 1$. For example, $(\sqrt{2}/2, -\sqrt{2}/2)$ is such a point, as is $(\sqrt{3}/2, 1/2)$ or even $(1, 0)$. However, $(-1, 1)$ is not on this circle (why?). In this chapter, as in Calculus, all angles will be measured in **RADIANS** (not degrees).

> **Don't forget** that, in Calculus, we always assume that angles are described in **radians** and not degrees. The conversion is given by
>
> $$Radians = \frac{(Degrees) \times (\pi)}{180}$$
>
> For example, 45 degrees $= 45\ \pi/180 = \pi/4 \approx 0.7853981633974$ radians.

So, for example, 360 degrees amounts to 2π radians, while 45 degrees is $\pi/4$ radians. Radian measure is so useful in trigonometry (and in Calculus) that we basically have to forget that "degrees" ever existed! So, from now on we talk about angular measure using radians only! (At first, if you find this confusing go back to the

box above and back substitute to get the measure in degrees). Okay, now let's review the properties of two really basic right-angled triangles, the **right-angled isosceles triangle** (that we refer to as RT45–abbreviation for a "right triangle with a 45 degree angle") because both its base angles must be equal to $\pi/4$ radians, and the right angled triangle one of whose angles measures $\pi/6$ radians (that we will refer to as RT30– why do you think we use the "30"?).

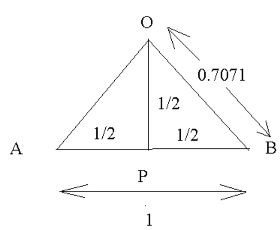


An RT45 Isosceles Triangle

Figure 223.

11.1 The right-angled isosceles triangle (RT45)

This triangle, reproduced in the margin as Fig. 223 has two equal angles at its base ($\angle$OAP = $\angle$OBP) of measure equal to $\pi/4$ and another angle (namely $\angle$AOB) of measure $\pi/2$ (the "right-angle"). Let's find the measure x and y of the side OA and the perpendicular OP in this triangle so that we can remember once and for all the various relative measures of the sides of such a triangle. We note that the line segment AB has length 1, and the segments AP and PB each have length 1/2 (since OP must bisect AB for such a triangle). Using the theorem of Pythagoras on $\triangle$OAB we see that $1^2 = x^2 + x^2$ (since the triangle is isosceles) from which we get that $2x^2 = 1$ or $x = \pm\sqrt{2}/2$. But we choose $x = \sqrt{2}/2$ since we are dealing with side-lengths. Okay, now have x, what about y?



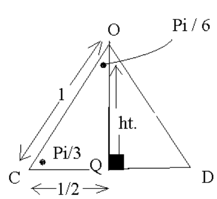
The final measures of the RT45 triangle

Figure 224.

To get at y we apply Pythagoras to the triangle $\triangle$OPB with hypotenuse OB. In this case we see that $x^2 = (1/2)^2 + y^2$ But we know that $x^2 = 1/2$, so we can solve for y^2, that is $y^2 = 1/2 - 1/4 = 1/4$ from which we derive that $y = \pm 1/2$. Since we are dealing with side-lengths we get $y = 1/2$, that's all. Summarizing this we get Fig. 224, the RT45 triangle, as it will be called in later sections. Note that it has two equal base angles equal to $\pi/4$ radians, two equal sides of length $\sqrt{2}/2$ and the hypotenuse of length 1.

Summary of the RT45: Referring to Fig. 227, we see that our RT45 triangle (well, half of it) has a hypotenuse of length 1, two sides of length 1/2, and two equal base angles of measure $\pi/4$ radians.

11.2 The RT30 triangle



The right-angled triangle: RT30

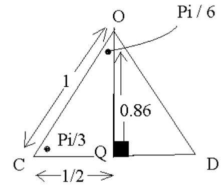
Figure 225.

Now, this triangle, reproduced in the margin (see Fig. 225) as $\triangle$OCQ, derives from the equilateral triangle $\triangle$OCD all of whose sides have length 1. In this triangle $\angle$OCQ = $\angle$ODQ) have radian measure equal to $\pi/3$ while angle $\angle$COQ has measure $\pi/6$. Let's find the length of the altitude $h = OQ$, given that we know that OC has length 1, and CQ has length equal to 1/2. Using the theorem of Pythagoras on $\triangle$OCQ we see that $1^2 = (1/2)^2 + h^2$ from which we get that $h^2 = 3/4$ or $h = \pm\sqrt{3}/2$. But we choose $h = \sqrt{3}/2$ since we are dealing with side-lengths, just like before.

Summarizing this we get Fig. 226, the RT30 triangle. Note that it has a base angle equal to $\pi/3$ radians, one side of length 1/2 and the hypotenuse of length 1.

Summary of the RT30: Referring to Fig. 226, we see that our RT30 triangle has a hypotenuse of length 1, one side of length $\sqrt{3}/2$, one side of length 1/2, and a hypotenuse equal to 1 unit. Its angles are $\pi/6, \pi/3, \pi/2$ in radians (or 30-60-90 in degrees).

The final “mental images” should resemble Fig. 227 and Fig. 228.



The final measures of the RT30 triangle

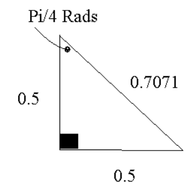
Figure 226.

11.3 The basic trigonometric functions

Now we return to the unit circle whose equation that consists of all points $P(x, y)$ such that $x^2+y^2 = 1$, see Fig. 229. The center (or origin) of our cartesian coordinate system is denoted by O and a point on the circle is denoted by P.

The positive x-axis consists of the set of all points x such that $x > 0$. So, for example, the points $(1, 0), (0.342, 0), (6, 0)$ etc. are all on the positive x-axis. We now proceed to define the trigonometric functions of a given angle θ (read this as “thay-ta”) whose measure is given in *radians*. The angle is placed in *standard position* as follows:

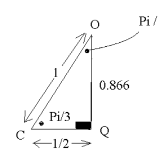
> If $\theta > 0$, its vertex is placed at O and one of the legs of θ is placed along the positive x-axis. The other leg, the terminal side, is positioned **counterclockwise** along a ray OP until the desired measure is attained. For instance, the angle $\pi/2$ (or 90 degrees) is obtained by placing a leg along the positive x-axis and another along the y-axis. The angle is measured counterclockwise. The point P on the unit circle corresponding to this angle has coordinates $(0, 1)$.



The RT45 triangle

Figure 227.

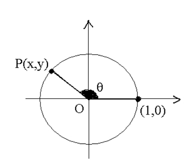
If $\theta < 0$, its vertex is placed at O and one of the legs of θ is placed along the positive x-axis. The other leg, the terminal side, is positioned **clockwise** along a ray OP until the desired measure is attained. For instance, the angle $-\pi/2$ (or -90 degrees) is obtained by placing a leg along the positive x-axis and another along the negative y-axis. The angle is measured clockwise. The point P on the unit circle corresponding to this angle has coordinates $(0, -1)$.



The final measures of the RT30 triangle

Figure 228.

Given a right-angled triangle $\triangle$ PQO, see Fig. 230, we recall the definitions of “opposite” and ”adjacent” sides: Okay, we all remember that the hypotenuse of $\triangle$ PQO is the side opposite to the right-angle, in this case, the side PO. The other two definitions depend on which vertex (or angle) that we distinguish. In the case of Fig. 230 the vertex O is distinguished. It coincides with the vertex of the angle $\angle$ QOP whose measure will be denoted by θ. The side OQ is called the *adjacent* side or simply the adjacent because it is *next* to O. The other remaining side is called the *opposite* because it is *opposite* to O (or not next to it anyhow). Now, **in trigonometry these three sides each have a length which, all except for the hypotenuse, can be either positive or negative** (or even zero, which occurs when the triangle collapses to a line segment). The hypotenuse, however, always has a positive length.



The unit circle

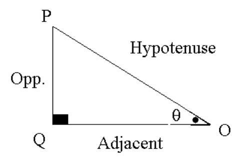
Figure 229.

We are now in a position to define the basic trigonometric functions. Let’s say we are given an angle (whose radian measure is) θ. We place this angle in standard position as in Fig. 229 and denote by P the point on the terminal side of this angle that intersects the unit circle (see Fig. 231). Referring to this same figure, at P we produce an altitude (or perpendicular) to the x-axis which meets this axis at Q, say. Then $\triangle$ PQO is a right-angled triangle with side PQ being the opposite side, and OQ being the adjacent side (of course, OP is the hypotenuse). The **trigonometric functions of this angle** θ are given as follows:

$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \cos\theta = \frac{adjacent}{hypotenuse} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opposite}{adjacent} \quad (11.1)$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}. \tag{11.2}$$

Example 462. Calculate the following trigonometric functions for the various given angles:



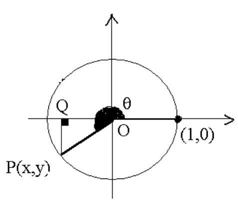
A typical right-angled triangle

Figure 230.

1. $\sin(\pi/4)$
2. $\cos(2\pi/3)$
3. $\tan(-\pi/6)$
4. $\sec(-\pi/3)$
5. $\csc(5\pi/4)$

Solution 1) For this one use the RT45 triangle, Fig. 227. According to (11.1), $\sin(\pi/4) = (1/2)/(1/\sqrt{2}) = \sqrt{2}/2$.

2) For the next angle, namely $2\pi/3$ (or 120 degrees) it is best to draw a picture such as the one in Fig. 232. Note that this angle gives the internal angle $\angle$ POQ the value of $\pi - (2\pi/3) = \pi/3$ radians. So $\angle$POQ $= \pi/3$. But $\triangle$ PQO is a RT30 triangle (see Fig. 228). Comparing Fig. 228 with the present triangle PQO we see that $\cos(2\pi/3) =$ adj./hyp. $= (-1/2)/1 = -1/2$.

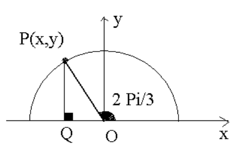


The unit circle

Figure 231.

3) In this case, we need to remember that the negative sign means that the angle is measured in a *clockwise direction*, see Fig. 233. Note that the opposite side QP has a negative value (since it is below the x-axis). The resulting triangle $\triangle$ PQO is once again a RT30 triangle (see Fig. 228). As before we compare Fig. 228 with the present triangle PQO. Since $\tan(-\pi/6) =$ opp./adj. $= (-1/2)/(\sqrt{3}/2) = -1/\sqrt{3}$, since the opposite side has value $-1/2$.

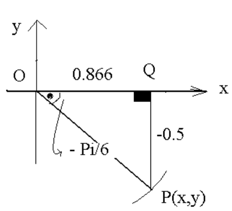
4) First we note that $\sec(-\pi/3) = 1/\cos(-\pi/3)$ so we need only find $\cos(-\pi/3)$. Proceeding as in 3) above, the angle is drawn in a clockwise direction, starting from the positive x-axis, an amount equal to $\pi/3$ radians (or 60 degrees). This produces $\triangle$ PQO whose central angle $\angle$ POQ has a value $\pi/3$ radians (see Fig. 234). Note that in this case the opposite side QP is negative, having a value equal to $-\sqrt{3}/2$. The adjacent side, however, has a positive value equal to $1/2$. Since $\cos(-\pi/3) =$ adj./hyp. $= (1/2)/1 = 1/2$, we conclude that $\sec(-\pi/3) = 1/(1/2) = 2$.



A 120 degree angle in standard position

Figure 232.

5) In this last example, we note that $5\pi/4$ (or 225 degrees) falls in the 3rd quadrant (see Fig. 235) where the point $P(x, y)$ on the unit circle will have $x < 0, y < 0$. We just need to find out the $\sin(5\pi/4)$, since the cosecant is simply the reciprocal of the sine value. Note that central angle $\angle POQ = \pi/4$ so that $\triangle$ PQO is a RT45 triangle (cf., Fig. 227). But $\sin(5\pi/4) =$ opp./hyp. and since the opposite side has a negative value, we see that $\sin(5\pi/4) = (-1/2)/(1/\sqrt{2}) = -1/\sqrt{2}$.



Remark: Angles whose radian measure exceeds 2π radians (or more than 360 degrees) are handled by reducing the problem to one where the angle is less than 2π radians by removing an appropriate number of multiples of 2π. For example, the angle whose measure is $13\pi/6$ radians, when placed in standard position, will look like $2\pi + \pi/6$, or just like a $\pi/6$ angle (because we already have gone around the unit circle once). So, $\sin(13\pi/6) = \sin(\pi/6) = 1/2$.

Figure 233.

Degs	0	30	45	60	90	120	135	150	180
Rads	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	**und.**	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0

Degs	210	225	240	270	300	315	330	360
Rads	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0
cos	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
tan	$\sqrt{3}/3$	1	$\sqrt{3}$	**und.**	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0

Table 11.1: Basic trigonometric functions and their values

The charts in Table 11.1 should be memorized (it's sort of like a "multiplication table" but for trigonometry). The first row gives the angular measure in degrees while the second has the corresponding measure in radians. In some cases the values are *undefined*, (for example, $\tan(\pi/2)$) because the result involves division by zero (an invalid operation in the real numbers). In this case we denote the result by **und.**

NOTES:

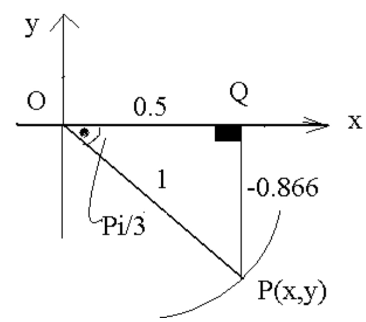


Figure 234.

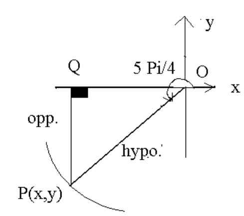


Figure 235.

11.4 Identities

This section involves mostly algebraic manipulations of symbols and not much geometry. We use the idea that the trigonometric functions are defined using the unit circle in order to derive the basic identities of trigonometry.

For example, we know that if θ is an an angle with vertex at the origin O of the plane, then the coordinates of the point $P(x, y)$ at its terminal side (where it meets

the unit circle) must be given by $(\cos\theta, \sin\theta)$. Why? By definition .... Think about it! If you want to find the $\cos\theta$, you need to divide the adjacent by the hypotenuse, but this means that the adjacent is divided by the number 1 (which is the radius of the unit circle). But since this number is equal to $\cos\theta$ this means that the adjacent is equal to $\cos\theta$. But the "adjacent side length" is also equal to the x-coordinate of the point P. So, $\cos\theta = x$. A similar argument applies to the y-coordinate and so we get $y = \sin\theta$. So, the coordinates of P are given by $(\cos\theta, \sin\theta)$. But P is on the unit circle, and so $x^2 + y^2 = 1$. It follows that

$$\sin^2\theta + \cos^2\theta = 1. \tag{11.3}$$

for any angle θ in radians (positive or negative, small or large). Okay, now we divide both sides of equation (11.3) by the number $\cos^2\theta$, provided $\cos^2\theta \neq 0$.. Using the definitions of the basic trigonometric functions we find that $\tan^2\theta + 1 = \sec^2\theta$. From this we get the second fundamental identity, namely that

$$\sec^2\theta - \tan^2\theta = 1. \tag{11.4}$$

provided all the quantities are defined. The thrid fundamental identity is obtained similarly. We divide both sides of equation (11.3) by the number $\sin^2\theta$, provided $\sin^2\theta \neq 0$.. Using the definitions of the basic trigonometric functions again we find that $1 + \cot^2\theta = \csc^2\theta$. This gives the third fundamental identity, i.e.,

$$\csc^2\theta - \cot^2\theta = 1. \tag{11.5}$$

once again, provided all the quantities are defined.

Next, there are two basic "Laws" in this business of trigonometry, that is the **Law of Sines** and the **Law of Cosines**, each of which is very useful in applications of trigonometry to the real world.

11.4.1 The Law of Sines

Before we proceed to recall this Law, remember that every angle in this book is to be measured in radians (and not degrees). This is particularly important for the Law of Sines where we will be relating the side length of a plane triangle with the angle opposite the side (when measured in radians). In order to set the scene for what follows we begin by referring to Fig. 237. Here we have a triangle OPR in standard position (and we can assume that R, P are on the unit circle with O at its center, since any other triangle would be similar to this one and so the sides would be proportional). Denote ∠ POR by A, ∠ORP by B and ∠ RPO by C, for brevity. Also, (cf., Fig. 237) denote the side lengths RP, PO, OR by a, b, c (all assumed positive in this result).

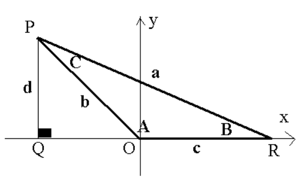


Figure 236.

Now comes the proof of the sine law, given by equation (11.6) below. Referring to Fig. 237 once again and using the definition of the sine function, we see that

$$\frac{d}{b} = \sin A \implies d = b\sin A.$$

In addition, since PQR is a right-angled triangle,

$$\frac{d}{a} = \sin B \implies d = a\sin B.$$

Combining these last two equations and eliminating d we find that $b\sin A = a\sin B$ and so provided that we can divide both sides by the product $\sin A\sin B$ we get

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Proceeding in exactly the same way for the other two angles we can deduce that $b \sin C = c \sin B$ from which we get the Sine Law:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{11.6}$$

Example 463. *Estimating the height of a building without asking an engineer:* Okay, but I'll assume you have a protractor! So, you're standing 10 m. away from the base of a very tall building and you pull out your protractor and measure the angle subtended by the point where you are standing (just pick a point 10 m away from the door, say) and the highest point on the building (of course, you will look absolutely nuts when you pull this thing out and start measuring by holding this instrument before your eyes). So, you measure this angle to be 72 degrees. Anyhow, given this simple scenario, how do you find the height of the building?

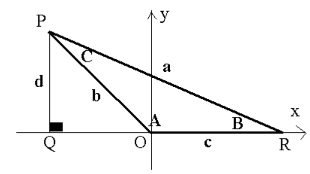


Figure 237.

Solution: See Fig. 238. You can guess that there is a right-angled triangle whose vertices are at your eyeball, the top point on the door and the top of the building. You also have two angles and one side of this triangle. So, the Sine Law tells you that you can always find the exact shape of the triangle (and so all the sides, including the height of the building). How? Well, first you need to convert 72 degrees to radians...so,

$$\frac{72 \times \pi}{180} = 1.2566 \text{ rads.}$$

Then, we need to find the "third" angle which is given by $180 - 72 - 90 = 18$ degrees, one that we must also convert to radians ... In this case we get 0.314159 rads.We put all this info. together using the Sine Law to find that

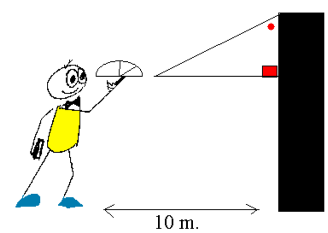


Figure 238.

$$\frac{height}{\sin 1.2566} = \frac{10}{\sin 0.314159}$$

and solving for the (approximate) height we get $height \approx 30.8$ m. Of course, you need to add your approximate height of, say, 1.8m to this to get that the building height is approximately $30.8 + 1.8 = 32.6$ meters.

11.4.2 The Law of Cosines

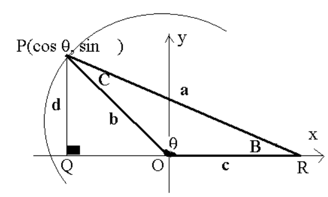


Figure 239.

You can think of this Law as a generalization of the Theorem of Pythagoras, the one you all know about, you know, about the square of the hypotenuse of a right-angled triangle etc. Such a "generalization" means that this result of Pythagoras is a *special case* of the Law of Cosines. So, once again a picture helps to set the scene. Look at Fig. 239. It is similar to Fig. 237 but there is additional information. Next, you will need the formula for the distance between two points on a plane (see Chapter 10).

Referring to Fig. 239 we asume that our triangle has been placed in standard position with its central angle at O (This simplifies the discussion). **We want a relationship between the central angle $\theta = \angle$ POR and the sides a, b, c of the triangle** so that when $\theta = \pi/2$ we get the classical Theorem of Pythagoras.

Now, by definition of the trigonometric functions we see that the point P has coordinates $P(b \cos \theta, b \sin \theta)$. The coordinates of R are easily seen to be $R(c, 0)$. By the *distance formula* we see that the square of the length of PR (or equivalently, the square of the distance between the points P and R) is given by

$$\begin{aligned} a^2 &= (b \cos \theta - c)^2 + (b \sin \theta - 0)^2, \\ &= b^2 \cos^2 \theta - 2bc \cos \theta + c^2 + b^2 \sin^2 \theta, \\ &= b^2 (\cos^2 \theta + \sin^2 \theta) + c^2 - 2bc \cos \theta, \\ &= b^2 + c^2 - 2bc \cos \theta, \quad \text{by (11.3)}. \end{aligned}$$

This last expression is the Cosine Law. That is, for any triangle with side lengths a, b, c and contained angle θ, (i.e., θ is the angle at the vertex where the sides of length b and c meet), we have

$$a^2 \quad = \quad b^2 + c^2 - 2bc\cos\theta. \tag{11.7}$$

Note that when $\theta = \pi/2$, or the triangle is right-angled, then a^2 is simply the square of the hypotenuse (because $\cos(\pi/2) = 0$) and so we recover Pythagoras' theorem.

11.4.3 Identities for the sum and difference of angles

Now, we can use these two laws to derive some really neat identities that relate the cosine of the sum or difference of two given angles with various trigonometric functions of the individual angles. Let's see how this is done. Consider Fig. 240 where ψ (pronounced "p-see") and ϕ (pronounced "fee") are the two given angles and the points P and R are on the unit circle. Let's say **we want a formula for $\cos(\psi - \phi)$**. First we find the coordinates of P and R in the figure and see that, by definition, we must have $P(\cos\psi, \sin\psi)$. Similarly, the coordinates of R are given by $R(\cos\phi, \sin\phi)$.

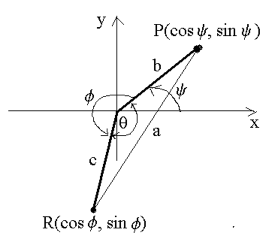


Figure 240.

Now, look at △ OPR in Fig. 240. By the Cosine Law (11.7), we have that $a^2 = b^2 + c^2 - 2bc\cos\theta$, where the central angle θ is related to the given angles via $\theta + \phi - \psi = 2\pi$ radians. Furthermore, $b = c = 1$ here because P and R are on the unit circle. Solving for θ in the previous equation we get

$$\theta = 2\pi - \phi + \psi.$$

But a^2 is just the square of the distance between P and R, b^2 is just the square of the distance between O and P and finally, c^2 is just the square of the distance between O and R. Using the distance formula applied to each of the lengths a, b, c above, we find that

$$\begin{aligned}(\cos\psi - \cos\phi)^2 + (\sin\psi - \sin\phi)^2 \quad &= \quad (\cos^2\psi + \sin^2\psi) + \\ &\quad\quad (\cos^2\phi + \sin^2\phi) - 2\cos(\theta), \\ 2 - 2\cos\psi\cos\phi - 2\sin\psi\sin\phi \quad &= \quad 2 - 2\cos(2\pi + \psi - \phi),\end{aligned}$$

where we have used (11.3) repeatedly with the angle θ there replaced by ψ and ϕ, respectively. Now note that $\cos(2\pi + \psi - \phi) = \cos(\psi - \phi)$. Simplifying the last display gives the identity,

$$\boxed{\cos(\psi - \phi) \quad = \quad \cos\psi\cos\phi + \sin\psi\sin\phi. \qquad (11.8)}$$

valid for any angles ψ, ϕ whatsoever. As a consequence, we can replace ψ in (11.8) by $\psi = \pi/2$, leaving ϕ as arbitrary. Since $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$, (11.8) gives us a new relation,

$$\cos\left(\frac{\pi}{2} - \phi\right) \quad = \quad \sin\phi. \tag{11.9}$$

But ϕ is arbitrary, so we can replace ϕ in (11.9) by $\pi/2 - \phi$ and get another new identity, that is,

$$\sin\left(\frac{\pi}{2} - \phi\right) \quad = \quad \cos\phi. \tag{11.10}$$

also valid for any angle ϕ whatsoever.

Now let ϕ, ψ be arbitrary angles once again. Replacing ϕ by $\psi + \phi$ in (11.9) we get

$$\sin(\psi + \phi) = \cos\left(\left(\frac{\pi}{2} - \psi\right) - \phi\right).$$

Using the *cos-difference-formula* (11.8) and combining this with (11.9) and (11.10) we obtain,

$$\begin{aligned} \sin(\psi + \phi) &= \cos\left(\left(\frac{\pi}{2} - \psi\right) - \phi\right) \\ &= \cos\left(\frac{\pi}{2} - \psi\right)\cos\phi + \sin\left(\frac{\pi}{2} - \psi\right)\sin\phi \\ &= \sin\psi\cos\phi + \cos\psi\sin\phi. \end{aligned}$$

We distinguish this formula for future use as the "sin-sum-formula" given by

$$\boxed{\sin(\psi + \phi) = \sin\psi\cos\phi + \cos\psi\sin\phi, \qquad (11.11)}$$

and valid for any angles ψ, ϕ as usual. Now, in (11.9) we replace ϕ by $-\psi$ and rearrange terms to find:

$$\begin{aligned} \sin(-\psi) &= \cos\left(\psi + \frac{\pi}{2}\right) \\ &= \cos\left(\psi - \left(-\frac{\pi}{2}\right)\right) \\ &= \cos(\psi)\cos(-\pi/2) + \sin(\psi)\sin(-\pi/2) \\ &= -\sin(\psi), \end{aligned}$$

since $\cos(-\pi/2) = 0$ and $\sin(-\pi/2) = -1$. This gives us the following identity,

$$\sin(-\psi) = -\sin(\psi) \qquad (11.12)$$

valid for any angle ψ.

In addition, this identity (11.8) is really interesting because the angles ϕ, ψ can really be anything at all! For example, if we set $\psi = 0$ and note that $\sin 0 = 0$, $\cos 0 = 1$, we get another important identity, similar to the one above, namely that

$$\cos(-\phi) = \cos(\phi) \qquad (11.13)$$

for any angle ϕ. We sometimes call (11.8) a "cos-angle-difference" identity. To get a "'cos-angle-sum" identity we write $\psi + \phi$ as $\psi + \phi = \psi - (-\phi)$ and then apply (11.8) once again with ϕ replaced by $-\phi$. This gives

$$\begin{aligned} \cos(\psi + \phi) &= \cos\psi\cos(-\phi) + \sin\psi\sin(-\phi). \\ &= \cos\psi\cos\phi - \sin\psi\sin\phi. \end{aligned}$$

where we used (11.13) and (11.12) respectively to eliminate the minus signs. We display this last identity as

$$\boxed{\cos(\psi + \phi) = \cos\psi\cos\phi - \sin\psi\sin\phi. \qquad (11.14)}$$

The final "sin-angle-difference" identity should come as no surprise. We replace ϕ by $-\phi$ in (11.11), then use (11.13) and (11.12) with ψ replaced by ϕ. This gives

$$\boxed{\sin(\psi - \phi) = \sin\psi\cos\phi - \cos\psi\sin\phi,} \quad (11.15)$$

Now we can derive a whole bunch of other identities! For example, the identity

$$\boxed{\cos(2\phi) = \cos^2\phi - \sin^2\phi,} \quad (11.16)$$

is obtained by setting $\phi = \psi$ in (11.14). Similarly, setting $\phi = \psi$ in (11.11) gives the new identity

$$\boxed{\sin(2\phi) = 2\sin\phi\cos\phi,} \quad (11.17)$$

Returning to (11.16) and combining this with (11.3) we find that

$$\begin{aligned}\cos(2\phi) &= \cos^2\phi - \sin^2\phi \\ &= \cos^2\phi - (1 - \cos^2\phi) \\ &= 2\cos^2\phi - 1.\end{aligned}$$

Isolating the square-term in the preceding formula we get a very important identity, namely,

$$\boxed{\cos^2(\phi) = \frac{1 + \cos(2\phi)}{2}.} \quad (11.18)$$

On the other hand, combining (11.16) with (11.3) we again find that

$$\begin{aligned}\cos(2\phi) &= \cos^2\phi - \sin^2\phi \\ &= (1 - \sin^2\phi) - \sin^2\phi \\ &= 1 - 2\sin^2\phi.\end{aligned}$$

Isolating the square-term in the preceding formula just like before we get the complementary identity to (11.17)

$$\boxed{\sin^2(\phi) = \frac{1 - \cos(2\phi)}{2}.} \quad (11.19)$$

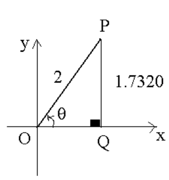


Figure 241.

The next example shows that you don't really have to know the value of an angle, but just the value of one of the trigonometric functions of that angle, in order to determine the other trigonometric functions.

Example 464. *Given that θ is an acute angle such that $\sin\theta = \sqrt{3}/2$, find $\cos\theta$ and $\cot\theta$.*

Solution: With problems like this it is best to draw a picture, see Fig. 241. The neat thing about trigonometry is you don't always have to put your triangles inside the unit circle, it helps, but you don't have to. This is one example where it is better

if you don't! For instance, note that $\sin\theta = \sqrt{3}/2$ means we can choose the side PQ to have length $\sqrt{3}$ and the hypotenuse OP to have length 2. So, using the Theorem of Pythagoras we get that the length of OQ is 1 unit. We still don't know what θ is, right? But we DO know all the sides of this triangle, and so we can determine all the other trig. functions of this angle, θ.

For example, a glance at Fig. 241 shows that $\cos\theta = 1/2$ and so $\cot\theta = 1/(\tan\theta) = 1/\sqrt{3}$.

N.B.: There is something curious here, isn't there? We drew our picture, Fig. 241 so that θ is an acute angle, because we were asked to do so! What if the original angle were obtuse? Would we get the same answers?

Answer: No. For example, the obtuse angle $\theta = 2\pi/3$ also has the property that $\sin\theta = \sqrt{3}/2$ (Check this!) However, you can verify that $\cos\theta = -1/2$ and $\cot\theta = -1/\sqrt{3}$. The moral is, the more information you have, the better. If we weren't given that θ was acute to begin with, we wouldn't have been able to calculate the other quantities uniquely.

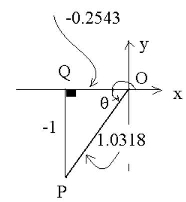


Figure 242.

Example 465. *If θ is an obtuse angle such that $\cot\theta = 0.2543$, find $\cos\theta$ and $\csc\theta$.*

Solution: Once again we draw a picture, see Fig. 242. This time we make the angle obtuse and put it in a quadrant where the cotangent is positive! Note that $\cot\theta = 0.2543 > 0$ and θ obtuse means that θ is in Quadrant III (cf., Fig. 243). So, this means we can choose the side OQ to have length -0.2543 and the opposite side to have length -1. Using the Theorem of Pythagoras once again we get that the length of the hypotenuse OP is 1.0318 units. Just as before , we DO now know all the sides of this triangle, and so we can determine all the other trig. functions of this angle, θ.

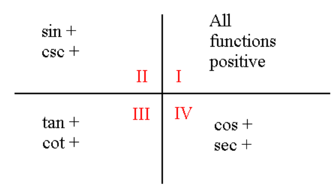


Figure 243.

So, a glance at Fig. 242 shows that $\cos\theta = -0.2543/1.0318 = -0.2465$ and $\csc\theta = 1/(\sin\theta) = -1.0318$.

Example 466. *Given that θ is an angle in Quadrant II such that $\cos^2\theta = \cos\theta + 1$, find the value of $\sin\theta$.*

Solution: No picture is required here, but it's okay if you draw one. Note that $\cos\theta$ is not given explicitly at the outset so you have to find it buried in the information provided. Observe that if we write $x = \cos\theta$ then we are really given that $x^2 = x + 1$ which is a quadratic equation. Solving for x using the *quadratic formula* we get $\cos\theta = (1 \pm \sqrt{5})/2$. However, one of these roots is greater than 1 and so it cannot be the cosine of an angle. It follows that the other root, whose value is $(1 - \sqrt{5})/2 = -0.618034$, is the one we need to use.

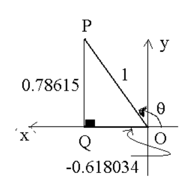


Figure 244.

Thus, we are actually given that $\cos\theta = -0.618034$, θ is in Quadrant II, and we need to find $\sin\theta$. So, now we can proceed as in the examples above. Note that in Quadrant II, $\sin\theta > 0$. In addition, if we do decide to draw a picture it would look like Fig. 244.

Since $\cos\theta = -0.618034$, we can set the adjacent side to have the value -0.618034 and the hypotenuse the value 1. From Pythagoras, we get the opposite side with a value of 0.78615. It follows that $\sin\theta = 0.78615/1 = 0.78615$.

Example 467. *Prove the following identity by transforming the expression on the left into the one on the right using the identities (11.3-11.5) above and the definitions of the various trig. functions used:*

$$\sin^2 x + \frac{1-\tan^2 x}{\sec^2 x} = \cos^2 x.$$

Solution: We leave the first term alone and split the fraction in the middle so that it looks like

$$\begin{aligned}
\sin^2 x + \frac{1-\tan^2 x}{\sec^2 x} &= \sin^2 x + \frac{1}{\sec^2 x} - \frac{\tan^2 x}{\sec^2 x}, \\
&= \sin^2 x + \cos^2 x - \frac{\sin^2 x}{\cos^2 x} \cdot \frac{cos^2 x}{1}, \quad \text{(by definition)} \\
&= \sin^2 x + \cos^2 x - \sin^2 x, \\
&= \cos^2 x,
\end{aligned}$$

which is what we needed to show.

All of the above identities (11.3-11.5) and (11.8-11.19) are used in this book (and in Calculus, in general) and you should strive to **remember all the boxed ones**, at the very least. Remembering how to get from one to another is also very useful, because it helps you to remember the actual identity by deriving it!

Exercise Set 47.

Evaluate the following trigomometric functions at the indicated angles using any of the methods or identities in this chapter (use your calculator only to CHECK your answers). Convert degrees to radians where necessary.

1. $\cos\frac{\pi}{3}$	6. $\sin\frac{5\pi}{4}$	11. $\cos\frac{3\pi}{2}$	16. $\cos\frac{-7\pi}{4}$	21. $\cos 225^o$
2. $\sin\frac{2\pi}{3}$	7. $\cos\frac{7\pi}{6}$	12. $\sin\frac{3\pi}{2}$	17. $\cos\frac{17\pi}{4}$	22. $\cos 405^o$
3. $\tan\frac{\pi}{6}$	8. $\sin\frac{-3\pi}{4}$	13. $\tan\frac{3\pi}{2}$	18. $\cos\frac{5\pi}{2}$	23. $\cos 960^o$
4. $\cos\frac{-5\pi}{4}$	9. $\cos\frac{3\pi}{4}$	14. $\tan\frac{7\pi}{4}$	19. $\cos\frac{11\pi}{6}$	24. $\sin(-210^o)$
5. $\cos\frac{7\pi}{4}$	10. $\sin\frac{5\pi}{3}$	15. $\sin\frac{7\pi}{6}$	20. $\cos\frac{-13\pi}{6}$	25. $\tan(-1125^o)$

26. If $\cot\phi = 3/4$ and ϕ is an acute angle, find $\sin\phi$ and $\sec\phi$.
27. If $\cos u = -1/4$ and u is in Quadrant II, find $\csc u$ and $\tan u$.
28. If $\sin\phi = 1/3$ and ϕ is an acute angle, find $\cos\phi$ and $\tan\phi$.
29. If $\tan v = -3/4$ and v is in Quadrant IV, find $\sin v$ and $\cos v$.
30. If $\sec\phi = 2$ and ϕ is an acute angle, find $\sin\phi$ and $\tan\phi$.
31. If $\csc w = -3$ and w is in Quadrant III, find $\cos w$ and $\cot w$.

Prove the following identities using the basic identities in the text by converting the left hand side into the right hand side.

32. $(\tan x + \cot x)^2 = \sec^2 x \csc^2 x.$ $\quad(= \sec^2 x + \csc^2 x$, too!)

33. $\sin\theta + \cot\theta\cos\theta = \csc\theta$.
34. $\dfrac{\cos x}{1+\sin x} + \tan x = \sec x$.
35. $\tan^2 y - \sin^2 y = \tan^2 y \sin^2 y$.
36. $\dfrac{1+\cot x}{1+\tan x} = \cot x$.
37. $\dfrac{1}{\tan\phi + \cot\phi} = \sin\phi\cos\phi$.
38. $\sin^2 x\cot^2 x + \cos^2 x\tan^2 x = 1$.
39. $\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \dfrac{1}{\sin^2 x - \sin^4 x}$.
40. $\sin^4\phi - \cos^4\phi = 2\sin^2\phi - 1$.
41. $(1+\tan^2 u)(1-\sin^2 u) = 1$.

Prove the following relations using the angle-sum/difference formulae, (11.8-11.15) and/or the others.

42. $\cos(\frac{\pi}{2}+\theta) = -\sin\theta$.
43. $\sin(\pi + x) = -\sin x$.
44. $\cos(\frac{3\pi}{2}+\theta) = \sin\theta$.
45. $\sin(\pi - x) = \sin x$.
46. $\cos(\pi - x) = -\cos x$.
47. $\sin(\frac{3\pi}{2}+\theta) = -\cos\theta$.
48. $\cos(\frac{3\pi}{2}-\theta) = -\sin\theta$.
49. $\tan(\pi + x) = \tan x$.
50. $\tan(\pi - x) = -\tan x$.

NOTES:

Chapter 12

Appendix D: The Natural Domain of a Function

You must have noticed that we didn't put too much emphasis on the actual *domain* of a function in Section 1.2. This is because, most of the time, the domain of the given function is sort of "clear" from the rule which defines that function. Remember that we always want the value of the function to be a real number, because we are dealing with real-valued functions in this book. This condition defines the natural domain of the function. In other words, If the value $f(x)$ of a function f at x is an actual, *trueblue, real number* ... something like $-1,\ 1.035246946,\ 1.6,\ 1/4,\ -355/113,\ \pi, \ldots$ then **we say that f is defined at** x and, as a result, x belongs to the **natural domain** of f (See the text in Figure 245). Remember that the natural domain of a function f is *actually a set of real numbers*, indeed, it is the largest set of numbers for which the function values $f(x)$ are defined. In Table 12.1 below you'll find the natural domain of real valued functions which occur frequently in Calculus.

Check this out!

The values of a function must be real numbers. When this happens we say that $f(x)$ **is defined or x is in the natural domain of f**. This means that, for a given real number x,

- You can't divide by 0 in the expression for $f(x)$,
- You can't take the square root of a negative number in the expression for $f(x)$,
- You can't multiply by infinity in the expression for $f(x)$,
- That sort of thing ...

Figure 245.

You should remember this table ... In fact, if you refer to this table you'll see that

Example 468. *The function defined by $f(x) = x^3 \cos x$, made up of the product of the two rules x^3 and $\cos x$, is defined for each x in the interval $(-\infty, +\infty)$ and so this is its natural domain (this is a consequence of the fact that the natural domain of the product of two or more functions is the (set-theoretic) intersection of the natural domains of all the functions under consideration).*

Of course, this function can have any subset of $(-\infty, +\infty)$ as its **domain**, *in which case that set would be given ahead of time.* **The thing to remember here is that the domain and the natural domain of a function may be different!**

For example, we could be asking a question about the function $f(x) = (2x^2+1)\sin x$ for x in the interval $0 < x < 1$ only. This interval, $0 < x < 1$, then becomes its domain (but its natural domain is still $(-\infty, +\infty)$).

Example 469. *The function f defined by*

$$f(x) = \frac{\sin x}{x}$$

$f(x)$	Natural domain of $f(x)$	Remarks (See Figure 245)
$x, x^2, x^3, \ldots$	$(-\infty, +\infty)$	Power functions are defined for any real x
$\frac{1}{x}, \frac{1}{x^2}, \ldots$	Every real $x \neq 0$	Division by 0 NOT allowed!
$\sqrt{x}$	$[0, +\infty)$	Square roots of negatives not allowed !
$\sqrt[n]{x}$	If n is **odd**: $(-\infty, +\infty)$	If $x < 0$ then $\sqrt[n]{x} < 0$
$\sqrt[n]{x}$	If n is **even**: $[0, +\infty)$	Even roots of negatives NOT allowed !
$\sqrt{g(x)}$	x with $g(x) \geq 0$	Can't allow $g(x) < 0$
$\sin x, \cos x$	$(-\infty, +\infty)$	x always in RADIANS not degrees
$\tan x, \sec x$	$x \neq \pm\pi/2, \pm 3\pi/2, \ldots$	Equality gives infinite values
$\cot x, \csc x$	$x \neq 0, \pm\pi, \pm 2\pi, \ldots$	Equality gives infinite values

Table 12.1: The Natural Domain of Some Basic Functions

is defined for every real number x except when $x = 0$ (because we are dividing by 0). Hence its natural domain is the set of all real numbers excluding the number $x = 0$.

TYPICAL DIFFERENTIAL EQUATIONS

- Schrödinger's equation

$$-\frac{h^2}{2\pi m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

where x is in $(-\infty, +\infty)$. It appears in Quantum Mechanics.

- The Logistic equation

$$\frac{dN}{dt} = \alpha N(1 - \beta N)$$

where t is in $(0, +\infty)$, and α, β are two fixed real numbers. This one shows up in Chaos Theory.

Figure 246.

Example 470. *The function g defined by the rule $g(x) = x \cot x$ is not defined at zero and at every integer multiple of π (see Table 12.1 above) but it is defined everywhere else! Hence its natural domain is the set of all numbers $\{x : x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$.*

Example 471. *Let's say that we define a function f by the rule by $f(x) = \sin x$ and say nothing about its domain. Then we take it that x can be any angle whatsoever right? And this angle is in radians, right? OK, good. The point is that this function f* **is** *defined for any angle x whatsoever (by trigonometry) and so x can be any real number. This means that the natural domain of f is given by the open interval $(-\infty, +\infty)$.*

Note that it is very common that when we study a field called **differential equations** *we are asked to consider functions on a subset of the natural domain of the unknown function. In our case, for instance, we may be asked to consider the rule $f(x) = x^3 \cos x$ on the interval $(0, \pi)$, or $(-1.2\ ,\ \infty)$, and so on, instead of its natural domain which is $(-\infty, +\infty)$. We'll be learning about differential equations later on so* **don't worry** *if the symbols in Figure 246 don't make any sense to you right now. Just remember that such equations are very useful, for instance, the motion of a satellite around the earth is given by a solution to a differential equation.*

Example 472. *Find the natural domain of the function defined by the rule*

$$f(x) = \frac{1}{x^2 - 4}.$$

Solution *Okay, first notice that this function is of the type "one over something". The only time there can be a problem is if the something is zero (since we would be dividing by zero, see Figure 245), because if it isn't zero, the quotient is defined as a real number and this is good! Now, the denominator is zero only when $x^2 - 4 = 0$. Factoring this expression gives, $x^2 - 4 = (x-2)(x+2)$ and for this to be zero it is necessary that either $x = 2$ or $x = -2$. But these points are NOT ALLOWED in the natural domain of f! Thus the natural domain of our function is the set of all real numbers except the numbers ± 2.*

Example 473. *What is the natural domain of the function f defined by*

$$f(x) = \sqrt{1 - x^2}\ ?$$

Solution *We refer to Table 12.1 where we put $g(x) = 1 - x^2$ in the expression for $\sqrt{g(x)}$. Using the result there (and Figure 245) we see that we must be looking for x such that $g(x) \geq 0$, right? OK, this means that $1 - x^2 \geq 0$ or, adding x^2 to both sides, we find $1 \geq x^2$. Now, solving for x using the* **square root of the square rule** *(see Figures 247 and 6), we get $1 \geq |x|$ where $|x|$ represents the function which associates to each symbol x its* **absolute value** *(see Definition 2 above). So, the natural domain of this function is the closed interval $[-1, +1]$ or, the set of all points x for which $|x| \leq 1$, which is the same thing.*

Example 474. *Find the natural domain of the function f defined by*

$$f(x) = \frac{x-2}{2x^2 - 1}.$$

Solution *OK, this function is a quotient of two functions, namely, "$(x-2)$" and "$2x^2 - 1$". It's alright for the numerator to be zero but it's not alright for the denominator to be zero (Figure 245), because f is not defined at those points where $2x^2 - 1 = 0$. There are no "square roots" to worry about and any "infinities" can only arise when the denominator is actually equal to zero. So the natural domain consists of the set of all real numbers x where $2x^2 - 1 \neq 0$ or, what comes to the same thing, those x for which $x^2 \neq 1/2$. Finally, taking the square root of both sides, we get that if $x \neq \pm\sqrt{1/2}$ then such an x is in the natural domain of f. That is, the natural domain of our f is the set of all real numbers x such that $x \neq \pm\sqrt{1/2} = \pm 1/\sqrt{2} \approx \pm\ 0.707\ldots$.*

Example 475. *What is the natural domain of the function g defined by $g(t) = \sqrt{(t+2)^{-1}}$.*

Solution *This one has a square root in it so we can expect some trouble ... (remember Figure 245). But don't worry, it's not that bad. Let's recall that we can't take the square root of a negative number, so, if there are t's such that $(t+2)^{-1} < 0$ then these t's are NOT in the natural domain, right? But $(t+2)^{-1} = \frac{1}{t+2}$, so, if $t+2 < 0$*

The Square Root of the Square Rule!!

For any real number x,

$$\sqrt{x^2} = |\ x\ |.$$

More generally,

$$\sqrt{\square^{\ 2}} = |\ \square\ |$$

where $\square$ is any symbolic expression involving x or any other variable.

Figure 247.

then t is NOT in the natural domain, i.e., if $t < -2$ then t is not in the natural domain. Are there any other points NOT in the natural domain ? We can't divide by zero either, ... but if $t = -2$ then we ARE dividing by zero, because when $t = -2$, $(t+2)^{-1} = \frac{1}{t+2} = 1/0$. So this value of t is NOT in the natural domain either! After this there are no other numbers NOT in the natural domain other than the ones we found. Thus, the natural domain consists of all real numbers t with the property that $t > -2$ (because all the others, those with $t \leq -2$, are not in the natural domain).

Exercise Set 48.

Use the methods of this section and Table 12.1 to find the natural domain of the functions defined by the following rules.

1. $f(x) = 3x^2 + 2x - 1.$
2. $g(t) = t^3 \ \tan t.$
3. $h(z) = \sqrt{3z+2}.$
4. $k(x) = \dfrac{-2}{\sqrt{4-x^2}}.$
5. $f(x) = x \ \cot x.$
6. $f(x) = 16.345$
7. $g(x) = \sqrt{|\ \cos x\ |}$
8. $f(t) = 3\ t^{4/5} \ \sin t$
9. $h(x) = \dfrac{x^2+2x+1}{x^3-1}$ *Hint:* $x^3 - 1 = (x-1)\ (x^2+x+1)$
10. $h(z+3)$ *where* h *is given above by* $h(z) = \sqrt{3z+2}$.
11. $f(x) = \dfrac{1}{x\sqrt{x^2-1}}$
12. $g(\square) = -\sqrt{1-\square^2}$
13. $f(\triangle) = \dfrac{1}{1+\triangle^2}$
14. $f(\heartsuit) = 3\heartsuit^{1/4} - 2$
15. $k(x) = \dfrac{1}{\sqrt{1-x^2}}$

NOTES:

Acknowledgments

This manual was the combined effort of the author and about twenty volunteer typesetters who undertook the task of writing it in Latex code over 10 years ago. Most of them were my former students in first-year or upper-level Calculus courses that I taught. They spent many, many hours of long arduous work learning the new environment which LaTeX brings with it and submitted their work during the summer of 1998. My most sincere thanks goes to each one of them as I remember them here:

Among those who went *beyond the call of duty* and who have made an outstanding contribution, I cite: Genevieve Hay, M.Sc., M.B.A., (LaTeX guru *extraordinaire*), David Willis, Fu Keong Gary (Bomberman) Chan, Stephen Knox, Rosemary Takac, and Juliet Howland. I am also very grateful to Jean-Luc Cooke, Caroline Lam, Carson Tsang, Marsha Bissessarsingh, and Alan Speer for typesetting many parts of some of the chapters in the text. I also wish to thank Elzbieta Beres, Pedram Sodouri, and Selvakumar Krishnathasan for their contributions. I am particularly indebted to my colleague Dr. C. K. Fong, of the School of Mathematics and Statistics, for his careful proofreading and his suggestions during the final reading of the original manuscript. As well, I must acknowledge with thanks the availability of the excellent original course notes by Kristy Townshend, notes which were derived from a Pre-Calculus course that I taught in 1997 at Carleton. Many thanks are also due to Ms. Ann Woodside for her assistance in providing solutions to many of the exercises herein.

These dedicated individuals made the production of this manual possible. As well, I am grateful to Carleton University, Ottawa, for their generous support in the form of a Teaching Achievement Award for 1998-1999, thereby providing me with teaching release during the Winter of 1999 so as to complete the original draft.

On a more personal note, I am most grateful to my children and to my wife, Karen, for believing in me and for believing in this project, and for their unusual patience and understanding during my long absences in the Summer of 1998 and every time I undertook an update of the original ms. This work owes much to them: I may not have embarked upon so great an enterprise had I not been given all this moral support.

The author

Credits

The design, graphics, and caricatures in this book were created and manipulated by the author. Layout and front cover design and photograph by the author.

My most sincere thanks are extended to my colleague, Dr. Che Kao Fong (ret.), and to Genevieve Hay, M.Sc., M.B.A., for proofreading the manuscript and for their invaluable assistance in the arduous task of compiling the solutions to many of the exercises. Many thanks are also extended to my colleague Dr. Lazer Resnick for finding a number of typographical and other errors, and to Rosemary Takac and Genevieve Hay for their assistance in editing some of the graphics in Chapter 4.

The present e-book edition contains typographical revisions and corrections to earlier printings. Many thanks to the following individuals who have found errors and or typographical corrections in all past editions: Dr. Lazer Resnick, Physics (ret.), Stephen Burchill, Ian Rosso, Heather Leaman, Reza Rahimy, Michelle Bamji, Amy Cameron, Mina Boghdady, Jan Hiscock, Omar Nada, Glen Robitaille, Hamid Boland. Many thanks to Ann Woodside for her help in compiling material for business applications in the August, 2000 edition.

The 2001 edition owes most of its (typographical) revisions to my colleagues: Dr. Sam Melkonian, Dr. Sam Dubé, Dr. Elena Devdariani, and to the following engineering students: Michael Lombardi, Mr. Mortezavi, Derek Voisin, Shane McDowell, Anderson Wong, Javed Bagha and Nick Argent among others whose names may have been omitted inadvertently.

The 2002 edition included corrections to typographical errors found by my student, Alexei Pivkine.

The 2009 edition also includes corrections to typographical errors brought to my attention by my students: Bethany Mitchell, Stephanie Seemel, Kun Zhuang, Ricky Podrebarac, Andrew MacMillan, Andrew Kiwai, Myriam Younan, Robyn Chatwin-Davies, Luke Linseman, Chris Habets.

The 2011 revision includes corrections and suggestions by the following students: B. Weber, Abdullah Al-Kaf, Stephan Dumais, Trang Pham, and Spojmai Isaqzoy.

The 2012 revision includes corrections by Juliet Howland and Mark Blenkinsop.

The 2013 edition includes a new section on limits that are infinite, more solved exercises, and corrections of misprints.

The 2014 edition includes typographical fixes and I am grateful to my students Ashley Hooker, Josh Frey.

The 2015 (Nelson) edition includes typographical fixes by Mohammed Zidan and a few revisions to images:

Image page 1. Copyright: Nicku
Image page 3. Copyright: Bragin Alexey
Image page 9. Copyright: hkeita
Image page 14. Editorial Credit: Nicku / Shutterstock.com, Copyright: Nicku
Image page 17. Editorial Credit: bissig / Shutterstock.com, Copyright: bissig
Image page 35, Table 2.1. Courtesy of NASA,
http://hesperia.gsfc.nasa.gov/hessi/images/diagrams.gif
Image page 87, Copyright: Jakub Krechowicz
Image page 117, Copyright: Volodymyr Krasyuk
Image page 295, Copyright: Nicku
Image pages 12 etc., STOP signs: Copyright: tkemot
Image pages 29, etc., COFFEE cups: Copyright, Angelo Mingarelli.

The abridged 2016 edition is restricted only to material taught in the course Math 1004 at Carleton University. As a cost cutting measure, the comprehensive solution set is available online rather than in the bvody of the text itself.

I am grateful to all of you for taking the time to write, explain and compile your concerns and suggestions. To all of the above and to those tens of thousands of students that heard my lectures over the past 30 years, to you who inspired me to go on and write this book, an astronomical thank you!

The author.

Index